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Convertible Subordinated Debt Financing and Optimal Investment Timing*

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1 Introduction

Issuance of convertible debt is an important means of debt financing for firms. In recent years, firms issuing the subordinated convertible debt have been common. The senior-sub structure gives preference to straight debt over convertible debt and to convertible debt over equity when the default occurs. Black and Cox [1] and Sundaresan and Wang [8] suppose that the holder of junior security will not get paid at all until the holders of senior security are completely paid off at default. The straight debt holders which get the most preference over the other security holders either are guaranteed the par price or receive the total recovery value of the firm at default time. Next, the convertible debt holders are guaranteed the payments at default. They receive either the par price or the remaining value calculated by subtracting the par price of straight debt from the total recovery value of the firm. If the remaining value is negative, the convertible debt holders cannot receive nothing at default. Under the senior-sub structure equity holders cannot receive nothing if the remaining value calculated by subtracting the par prices of straight debt and convertible debt from the total recovery value of the firm is negative at default.

There are some previous works for the optimal problem for the investment which is financed with equity, straight debt and convertible debt. Lyandres and Zhdanov [4] have demonstrated the feature of outstanding convertible debt to induce shareholders to accelerate the issuance of new equity to finance investment thus alleviating the underinvestment problem of Myers [6]. Yagi et al. [7] have examined two different investment policies that maximize the equity value and the firm value and show the agency cost as the difference between policy values. They have investigated how the issuance of convertible debt affects investment. Egami [3] has analyzed optimal strategies involved in convertible debt financed expansion problem and clarified how the conversion feature affects the equity holders' investment decisions. However, these studies assume the same priority of straight debt and convertible debt.

Our study examines the investment option exercising decision by equity when the investment is partially financed with convertible and straight debt with the senior-sub structure. We

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investigate how the senior-sub structure affects the optimal policies for default, conversion and investment the values of equity, straight debt, convertible debt and investment.

The remainder of this paper is organized as follows. Sec. 2 describes the model. In Sec. 3, we derive the solution by numerical calculations and provide some results of numerical analysis. Finally, Sec. 4 summarizes this paper.

2 The Model

We consider a firm with an option to invest at any time by paying a fixed investment cost I. The firm partially finances the cost of investment with straight debt with the instantaneous contractual coupon payment of s and infinite maturity, and convertible debt with coupon payment of c and infinite maturity. These coupon payments are tax-deductible at a constant corporate tax rate τ. We suppose that the firm observes the demand shock X_t for its product, where X_t is given by a geometric Brownian motion

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \quad X_0 = x, \tag{2.1}$$

μ and σ are the risk-adjusted expected growth rate and the volatility of X_t, respectively, and W_t is a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. After the investment option is exercised, we assume that the firm can receive the instantaneous profit

$$\pi(X_t, s, c) = (1-\tau)(QX_t - s - c) \tag{2.2}$$

where $Q > 0$ is the quantity produced from the asset in place.

In order to examine the effects of senior-sub structure for straight debt and convertible debt, we consider several settings. First, we present a benchmark model in which the investment is financed with all equity. Second, we examine the case in which the investment is financed with equity, straight debt and convertible debt with the same priority. Finally, we consider a firm with an option for investment to be financed with equity, straight debt and convertible debt with the senior-sub structure.

2.1 All-Equity Financing

In this section, we assume that the investment is financed entirely with equity. This case has been investigated for real options (e.g., Dixit and Pindyck [2] and McDonald and Siegel [5]). The optimal investment rule is to exercise the investment option at the first passage time of the stochastic shock to an upper threshold x^*. The value of the investment option $F(x)$ and the investment threshold x^* are given by

$$F(x) = \left(\frac{x}{x^*}\right)^{\beta_1} (\epsilon(x^*) - I), \quad x < x^*, \tag{2.3}$$

$$x^* = \frac{1}{1-\tau} \frac{\beta_1}{\beta_1 - 1} \frac{r-\mu}{Q} I, \tag{2.4}$$

where $\beta_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 1$ and $\epsilon(x)$ is the total post-investment profit in which the investment is financed entirely with equity,

$$\epsilon(x) = \frac{1-\tau}{r-\mu} Qx. \tag{2.5}$$

2.2 Equity, Straight Debt, and Convertible Debt Financing

Next, we consider the investment which is financed with equity, straight debt and convertible debt.

2.2.1 Optimal Investment Policy

The equity holders of the firm which invests selects the optimal investment timing, observing the demand shock X_t. Let $E(x)$, $D_s(x)$ and $D_c(x)$ be the values of equity, straight debt and convertible debt issued at the investment, respectively. Then, the firm value $V(x)$ is represented by

$$V(x) = E(x) + D_s(x) + D_c(x). \tag{2.6}$$

Letting $\mathcal{T}_{t_1,t_2}$ be the set of stopping times with respect to the filtration as $\{\mathcal{F}_s; t_1 \leq s \leq t_2\}$ and $T > 0$ be an investment time, the value of the investment partially financed with straight debt and convertible debt $F(x)$ is formulated as

$$\begin{aligned} F(x) &= \sup_{T \in \mathcal{T}_{0,\infty}} \mathbb{E}_0^x \left[e^{-rT} \left(E(X_T) - (I - D_s(X_T) - D_c(X_T)) \right) \right] \\ &= \sup_{T \in \mathcal{T}_{0,\infty}} \mathbb{E}_0^x \left[e^{-rT} \left(V(X_T) - I \right) \right] \end{aligned} \tag{2.7}$$

where $\mathbb{E}_t^x$ is the conditional expectation operator upon $X_t = x$ and r is the risk-free interest rate. The optimal investment time T^* is given by

$$T^* = \inf\{T \in [0,\infty) \mid X_T \geq x^*\}. \tag{2.8}$$

From Eq. (2.7), the value of the investment option is given by

$$F(x) = a_0 x^{\beta_1} \tag{2.9}$$

for $x < x^*$. A constant a_0 and the investment threshold x^* must be determined using the following value matching and smooth-pasting conditions:

$$F(x^*) = V(x^*) - I, \tag{2.10}$$

$$\frac{dF}{dx}(x^*) = \frac{dV}{dx}(x^*). \tag{2.11}$$

2.2.2 The Values of Equity, Straight Debt and Convertible Debt with the Same Priority

In this section we examine the values of equity, straight debt and convertible debt with the same priority, introduced in Egami [3], Lyandres and Zhdanov [4] and Yagi et al. [7]. We assume that the holders of convertible debt can convert the debt into a fraction η of the original equity, where $\eta = \alpha c$ and α is a constant.

Once the investment option has been exercised, the optimal default policy is established from the issue of debt. The optimal default strategy of the equity holders maximizes the equity value, selecting the default threshold x_d. In this section we assume that equity holders cannot receive nothing at default. The holders of convertible debt receive the continuous coupon payment of c and choose the optimal conversion threshold x_c, maximizing the value of convertible debt. Since straight debt and convertible debt have the same priority, the holders of straight debt are entitled to $\frac{s}{s+c}(1-\theta)\epsilon(x)$ at pre-conversion bankruptcy and $(1-\theta)\epsilon(x)$ at post-conversion bankruptcy. Similarly, the convertible debt holders are entitled to $\frac{c}{s+c}(1-\theta)\epsilon(x)$ at pre-conversion bankruptcy, where θ is the proportional bankruptcy cost, $0 \leq \theta \leq 1$.

We now model the values of equity, straight debt and convertible debt after the investment option has been exercised. Let T_d be the pre-conversion default time. The total value of equity at investment time t is given by

$$E(x) = \sup_{T_d \in \mathcal{T}_{t,\infty}} \mathbb{E}_t^x \left[\int_t^{T_c^* \wedge T_d} e^{-r(u-t)}(1-\tau)(QX_u - s - c)du + 1_{\{T_c^* < T_d\}} e^{-r(T_c^*-t)} \frac{1}{1+\eta} E_a(X_{T_c^*}) \right] \tag{2.12}$$

and the value of convertible debt at investment time t is given by

$$\begin{aligned} D_c(x) &= \sup_{T_c \in \mathcal{T}_{t,\infty}} \mathbb{E}_t^x \left[\int_t^{T_c \wedge T_d^*} e^{-r(u-t)} c du + 1_{\{T_c < T_d^*\}} e^{-r(T_c^*-t)} \frac{\eta}{1+\eta} E_a(X_{T_c}) \right. \\ &\quad \left. + 1_{\{T_d^* < T_c\}} e^{-r(T_d^*-t)} \frac{c}{s+c}(1-\theta)\epsilon(X_{T_d^*}) \right], \end{aligned} \tag{2.13}$$

where the optimal default and conversion times T_d^* and T_c^*, respectively, are given by

$$T_d^* = \inf\{T_d \in [t,\infty) \mid X_{T_d} \leq x_d\}, \tag{2.14}$$

$$T_c^* = \inf\{T_c \in [t,\infty) \mid X_{T_c} \geq x_c\} \tag{2.15}$$

and $1_{\{T_c^* < T_d\}}$ is an indicator function that is equal to one if $T_c^* < T_d$ and is equal to zero otherwise. $E_a(x)$ is the total value of equity after conversion. Eq. (2.12) implies that the equity holders can receive the tax-deductible earning after paying coupon payments until conversion or default and that the equity value is diluted by converting, that is, $\frac{1}{1+\eta}$ represents the dilution factor. Eq. (2.13) means that the convertible debt holders can receive the coupon payment c until conversion or default and a fraction η of the original equity on conversion, and are entitled to the unleveraged value of the net proportional bankruptcy cost, $(1-\theta)\epsilon(x)$, of the firm at bankruptcy.

Next we consider the value of straight debt. Since the holders of straight debt can receive the continuous coupon payment of s, the value of the straight debt at investment time t is given by

$$\begin{aligned} D_s(x) &= \mathbb{E}_t^x\Bigg[\int_t^{T_c^*\wedge T_d^*} e^{-r(u-t)} s du + 1_{\{T_c^*<T_d^*\}} e^{-r(T_c^*-t)} D_{s,a}(X_{T_c^*}) \\ &\quad +1_{\{T_d^*<T_c^*\}} e^{-r(T_d^*-t)} \frac{s}{s+c}(1-\theta)\epsilon(X_{T_d^*})\Bigg], \end{aligned} \tag{2.16}$$

where $D_{s,a}(x)$ is the total value of straight debt after conversion. Eq. (2.16) means that the straight debt holders can receive the coupon payment until default and are entitled to a fraction of te unlevered value of the firm at bankruptcy.

Once the convertible debt has been converted, the firm becomes an entity that issues equity and straight debt, and the new optimal default policy is established. Using standard arguments of real options framework, the post-conversion default threshold $x_{d,c}$, the equity value $E_a(x)$ and the value of straight debt $D_{s,a}(x)$ are given by

$$x_{d,c} = \frac{\beta_2}{\beta_2-1}\frac{s(r-\mu)}{r}, \tag{2.17}$$

$$E_a(x) = \epsilon(x) - \frac{(1-\tau)s}{r} - \left(\frac{x}{x_{d,c}}\right)^{\beta_2}\left(\epsilon(x_{d,c}) - \frac{(1-\tau)s}{r}\right), \tag{2.18}$$

$$D_{s,a}(x) = \frac{s}{r} - \left(\frac{x}{x_{d,c}}\right)^{\beta_2}\left(\frac{s}{r} - (1-\theta)\epsilon(x_{d,c})\right), \tag{2.19}$$

where $\beta_2 = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{1}{2}-\frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} < 0$.

Next, we consider the values prior to conversion. It follows from the optimal problems of the equity holders, straight debt holders, and convertible debt holders in (2.12), (2.16), and (2.13), respectively, that the values of equity, straight debt, and convertible debt prior to default and conversion are given by

$$E(x) = a_1x^{\beta_1} + a_2x^{\beta_2} + (1-\tau)\left(\frac{Qx}{r-\mu} - \frac{s+c}{r}\right), \tag{2.20}$$

$$D_s(x) = a_3x^{\beta_1} + a_4x^{\beta_2} + \frac{s}{r}, \tag{2.21}$$

$$D_c(x) = a_5x^{\beta_1} + a_6x^{\beta_2} + \frac{c}{r}. \tag{2.22}$$

Constants a_i, $i = 1,\cdots,6$, the default threshold x_d, and the conversion threshold x_c must be determined using the following value matching and smooth-pasting conditions:

$$\begin{aligned}
E(x_d) &= 0, && (2.23)\\
\frac{dE}{dx}(x_d) &= 0, && (2.24)\\
E(x_c) &= \frac{1}{1+\eta}E_a(x_c), && (2.25)\\
D_s(x_d) &= \frac{s}{s+c}(1-\theta)\frac{(1-\tau)}{r-\mu}Qx_d, && (2.26)\\
D_s(x_c) &= D_{s,a}(x_c), && (2.27)\\
D_c(x_d) &= \frac{c}{s+c}(1-\theta)\frac{(1-\tau)}{r-\mu}Qx_d, && (2.28)\\
D_c(x_c) &= \frac{\eta}{1+\eta}E_a(x_c), && (2.29)\\
\frac{dD_c}{dx}(x_c) &= \frac{\eta}{1+\eta}\frac{dE_a}{dx}(x_c). && (2.30)
\end{aligned}$$

Eqs. (2.23), (2.24), (2.26) and (2.28) are conditions in default. On the other hand, Eqs. (2.25), (2.27), (2.29) and (2.30) are conditions in conversion. Eq. (2.23) is the value matching condition that ensures that the equity value at the default threshold equals zero. Eq. (2.24) is the smooth-pasting condition that ensures the optimality of the default threshold x_d. Eqs. (2.26) and (2.28) are the value matching conditions that ensure that the values of debt are equal to respective fractions of the unlevered value of the firm net of proportional bankruptcy cost. Eq. (2.25) is the value matching condition requiring that the value of equity at the conversion threshold equal a proportion of the post-conversion value of equity given in Eq. (2.18). Eq. (2.27) is the value matching condition that ensures that the pre-conversion value of straight debt equals the post-conversion value of straight debt in Eq. (2.19). Equation (2.29) is the value matching condition that ensures that the value of the convertible debt at the conversion threshold is equal to a proportion of the post-conversion value of equity. Eq. (2.30) is the smooth-pasting condition that ensures the optimality of the conversion threshold x_c. Eqs. (2.23)-(2.30) have eight unknown variables $(a_i, i = 1, \cdots, 6, x_d, x_c)$. These equations can be solved numerically.

2.2.3 The Values of Equity, Straight Debt and Convertible Debt with the Senior-Sub Structure

In this section we examine the values of equity, straight debt and convertible debt with the senior-sub structure. The senior-sub structure gives preference to straight debt over convertible debt and to convertible debt over equity when the default occurs. As in Black and Cox [1] and Sundaresan and Wang [8], we suppose that the holder of junior security will not get paid at all until the holders of senior security are completely paid off at default.

At the pre-conversion default time T_d, the straight debt holders receive

$$D_s(X_{T_d}) = \min\left(F_s, (1-\theta)\epsilon(X_{T_d})\right), \tag{2.31}$$

where F_s is the par value of the straight debt and is equal to $F_s = D_s(x^*)$. The payoff function (2.31) means that the straight debt holders receive either F_s or the total recovery value of the firm $(1-\theta)\epsilon(x_d)$ at default time T_d.

On the other hand, at the pre-conversion default time T_d, the convertible debt holders receive

$$D_c(X_{T_d}) = \min(F_c, \max((1-\theta)\epsilon(X_{T_d}) - F_s, 0)). \tag{2.32}$$

Let F_c denote the par value of convertible debt issued at investment time. The convertible debt is also issued at par, and thus we have $F_c = D_c(x^*)$. The payoff function (2.32) implies that the convertible debt holders receive either F_c or the remaining value calculated by subtracting the par price of straight debt from the total recovery value of the firm. If the remaining value is negative, the convertible debt holders cannot receive nothing at default.

Furthermore, under the senior-sub structure equity holders can receive

$$E(X_{T_d}) = \max((1-\theta)\epsilon(X_{T_d}) - F_s - F_c, 0) \tag{2.33}$$

at default. The payoff function (2.33) represents the equity holders cannot receive nothing if the remaining value calculated by subtracting the par prices of straight debt and convertible debt from the total recovery value of the firm is negative at default.

When the equity, the straight debt and the convertible debt have the senior-sub structure, the formulation of the equity value at investment time t is given by

$$\begin{aligned} E(x) &= \sup_{T_d \in \mathcal{T}_{t,\infty}} \mathbb{E}_t^x \Bigg[\int_t^{T_c^* \wedge T_d} e^{-r(u-t)}(1-\tau)(QX_u - s - c)du + 1_{\{T_c^* < T_d\}} e^{-r(T_c^*-t)} \frac{1}{1+\eta} E_a(X_{T_c^*}) \\ &\quad + 1_{\{T_d < T_c^*\}} e^{-r(T_d-t)} \max\Big((1-\theta)\epsilon(X_{T_d}) - D_s(x^*) - D_c(x^*), 0\Big) \Bigg], \end{aligned} \tag{2.34}$$

where $E_a(x)$ is the post-conversion value of equity. The values of convertible debt and straight debt at investment time t are formulated as

$$\begin{aligned} D_c(x) &= \sup_{T_c \in \mathcal{T}_{t,\infty}} \mathbb{E}_t^x \Bigg[\int_t^{T_c \wedge T_d^*} e^{-r(u-t)} c du + 1_{\{T_c < T_d^*\}} e^{-r(T_c-t)} \frac{\eta}{1+\eta} E_a(X_{T_c^*}) \\ &\quad + 1_{\{T_d^* < T_c\}} e^{-r(T_d^*-t)} \min\Big(D_c(x^*), \max\Big((1-\theta)\epsilon(X_{T_d^*}) - D_s(x^*), 0\Big)\Big) \Bigg] \end{aligned} \tag{2.35}$$

and

$$\begin{aligned} D_s(x) &= \mathbb{E}_t^x \Bigg[\int_t^{T_c^* \wedge T_d^*} e^{-r(u-t)} s du + 1_{\{T_c^* < T_d^*\}} e^{-r(T_c^*-t)} D_{s,a}(X_{T_c^*}) \\ &\quad + 1_{\{T_d^* < T_c^*\}} e^{-r(T_d^*-t)} \min\Big(D_s(x^*), (1-\theta)\epsilon(X_{T_d^*})\Big) \Bigg], \end{aligned} \tag{2.36}$$

where $D_{s,a}(x)$ is the post-conversion value of straight debt.

Also, the post-conversion value of equity is formulated as

$$\begin{aligned}E_a(x) &= \sup_{T_{d,c}\in\mathcal{T}_{t,\infty}} \mathbb{E}_t^x\Bigg[\int_t^{T_{d,c}} e^{-r(u-t)}(1-\tau)(QX_u-s)du \\ &\quad +e^{-r(T_{d,c}-t)}\max\Big((1-\theta)\epsilon(X_{T_{d,c}})-D_s(x^*),0\Big)\Bigg]. \qquad (2.37)\end{aligned}$$

This equation means that the equity holders cannot receive nothing if the remaining value calculated by subtracting the par price of straight debt from the total recovery value of the firm is negative at default. The optimal post-conversion default time is given by

$$T_{d,c}^* = \inf\{T_{d,c} \in [t,\infty) \mid X_{T_{d,c}} \leq x_{d,c}\}. \qquad (2.38)$$

Then, the value of straight debt after conversion is formulated as

$$D_{s,a}(x) = \mathbb{E}_t^x\left[\int_t^{T_{d,c}^*} e^{-r(u-t)}sdu + e^{-r(T_{d,c}^*-t)}\min\Big(D_s(x^*),(1-\theta)\epsilon(X_{T_{d,c}^*})\Big)\right]. \qquad (2.39)$$

This equation represents that the straight debt holders can receive the coupon payment until default and are entitled to either the par price or a fraction of the unlevered value of the firm at bankruptcy. From Eqs. (2.37) and (2.39), the post-conversion values of equity and straight debt are given by

$$E_a(x) = a_7x^{\beta_2} + (1-\tau)\left(\frac{Qx}{r-\mu} - \frac{s}{r}\right), \qquad (2.40)$$

$$D_{s,a}(x) = a_8x^{\beta_2} + \frac{s}{r}. \qquad (2.41)$$

Constants a_i, $i = 7, 8$ and the post-conversion default threshold $x_{d,c}$ are determined from the following value matching and smooth-pasting conditions:

$$E_a(x_{d,c}) = \max((1-\theta)\epsilon(x_{d,c}) - D_s(x^*), 0), \qquad (2.42)$$

$$\frac{dE_a}{dx}(x_{d,c}) = \begin{cases} (1-\theta)\dfrac{(1-\tau)Q}{r-\mu}, & x_{d,c} > \dfrac{r-\mu}{(1-\theta)(1-\tau)Q}D_s(x^*), \\ 0, & \text{otherwise}, \end{cases} \qquad (2.43)$$

$$D_{s,a}(x_{d,c}) = \min\left(D_s(x^*), (1-\theta)\frac{(1-\tau)}{r-\mu}Qx_{d,c}\right). \qquad (2.44)$$

The pre-conversion values of equity, straight debt and convertible debt are given by Eqs. (2.20), (2.21) and (2.22), respectively. Hence, we must determine constants a_i, $i = 1, \cdots, 6$ and the pre-conversion default and conversion thresholds. The value matching condition and the smooth-pasting conditions in default and conversion are given by

$$
\begin{aligned}
E(x_d) &= \max\left((1-\theta)\epsilon(x_d) - D_s(x^*) - D_c(x^*), 0\right), & (2.45)\\
\frac{dE}{dx}(x_d) &= \begin{cases} (1-\theta)\dfrac{(1-\tau)Q}{r-\mu}, & x_d > \dfrac{r-\mu}{(1-\theta)(1-\tau)Q}(D_s(x^*)+D_c(x^*)) \\ 0, & \text{otherwise} \end{cases} & (2.46)\\
E(x_c) &= \frac{1}{1+\eta}E_a(x_c), & (2.47)\\
D_s(x_d) &= \min\left(D_s(x^*), (1-\theta)\epsilon(x_d)\right), & (2.48)\\
D_s(x_c) &= D_{s,a}(x_c), & (2.49)\\
D_c(x_d) &= \min\left(D_c(x^*), \max\left((1-\theta)\epsilon(x_d) - D_s(x^*), 0\right)\right). & (2.50)\\
D_c(x_c) &= \frac{\eta}{1+\eta}E_a(x_c), & (2.51)\\
\frac{dD_c}{dx}(x_c) &= \frac{\eta}{1+\eta}\frac{dE_a}{dx}(x_c). & (2.52)
\end{aligned}
$$

Since Eqs. (2.42)–(2.43), (2.45)–(2.46), (2.48) and (2.50) include the optimal investment threshold x^*, Eqs. (2.7)-(2.8) which determine the value of investment option, the pre-conversion values of equity and straight debt, and the post-conversion values of equity, straight debt and convertible debt must be also solved, simultaneously. Since Eqs. (2.7)–(2.8), (2.42)–(2.52) have 13 unknown variables $(a_i, i = 0, \cdots, 8, x^*, x_{d,c}, x_d, x_c)$, these equations are solved numerically.

3 Numerical Analysis

In this section, the calculation results of the value of equity, each debt, and the investment option are presented in order to investigate how the senior-sub structure affects the optimal policies for default, conversion and investment. We use the following base case parameters: $Q = 1$, $\mu = 0$, $\sigma = 0.2$, $r = 0.05$, $I = 20$, $c = 1$, $\alpha = 1.5$, $\theta = 0.3$, and $\tau = 0.3$.

Figs. 1 and 2 show the values of equity, straight debt, and convertible debt, as functions of the demand shock x in the case of Sec. 2.2.3, that is, the investment is financed with equity, straight debt, and convertible debt with the senior-sub structure. The solid line depicts the equity value $E(x)$, the dotted line represents that of straight debt, and the dash-dotted line gives that of convertible debt, respectively. The coupon payment for straight debt in Fig. 1 is equal to $s = 0.2$ and that in Fig. 2 is equal to $s = 1$. The coupon payment for convertible debt in both figures is equal to $c = 1$. In both these parameters the equity holders cannot receive nothing at default. As shown in these figures, the threshold values for the pre-conversion default and the conversion are 0.69 and 2.15 in Fig. 1, and 1.18 and 2.32 in Fig. 2, respectively. When the coupon payment for straight debt s is relatively low, the issued price (the par value) of straight debt $D_s(x^*)$ is also low. Then, since the straight debt holders are completely paid off at default, the convertible debt holders can receive the firm value minus the par value of straight debt $(1-\theta)\epsilon(x_d) - D_s(x^*) > 0$, that is, the value of convertible debt has the non-negative value at the default threshold x_d in Fig. 1. On the other hand, when the coupon payment for straight debt is relatively high, the par value of straight debt is also high. Then, since the payoff to straight debt holders is not completely guaranteed, the convertible debt holders cannot receive nothing at default, that is, the value of convertible debt is equal to zero in Fig 2.

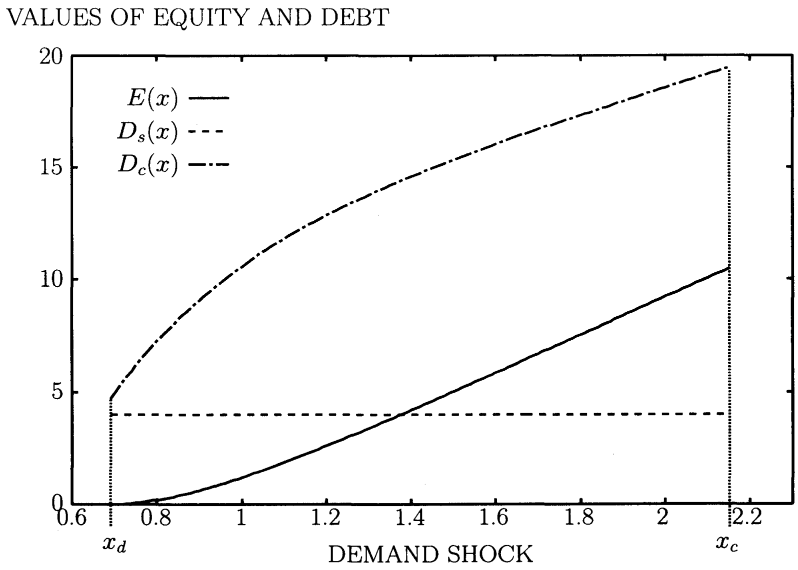

Figure 1: Equity, straight debt and convertible debt ($s = 0.2$, $c = 1$)

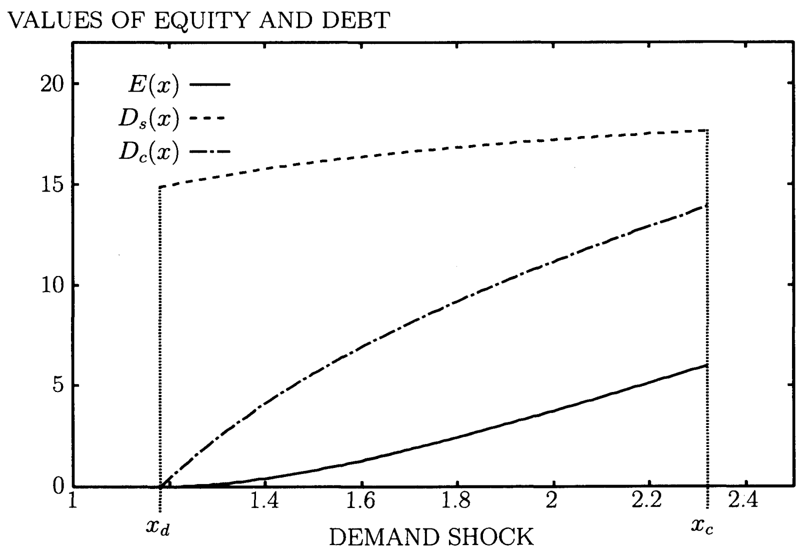

Figure 2: Equity, straight debt and convertible debt ($s = 1$, $c = 1$)

Tab. 1 shows the optimal default threshold values x_d for the fraction of coupon payment for straight debt $\frac{s}{s+c}$ when the coupon payment for convertible debt is equal to $c = 1$. The upper row represents the default threshold in the case of the senior-sub structure in Sec. 2.2.3 and the lower row represents that in the case of the same priority in Sec. 2.2.2. In both the cases, we may recognize that the default occurs earlier when the coupon payment for straight debt is higher. Furthermore, we can confirm that the senior-sub structure for straight debt and convertible debt does not really affect the default threshold because the value of equity which is used in the selection of the default is unaffected directly.

Fig. 3 presents th optimal conversion threshold x_c for the fraction of coupon payment for straight debt $\frac{s}{s+c}$. The solid line represents the threshold in the case of the senior-sub structure and the dotted line gives that in the case of the same priority, respectively. When the coupon payment for straight debt is high, the conversion occurs earlier because the convertible debt holders cannot receive nothing at default. Meanwhile, as the coupon payment for straight debt becomes lower, the effect of accelerating the conversion is weakening because the convertible debt holders can receive the positive value. Furthermore, in Tab. 1, we can see that the default occurs earlier because the conversion occurs earlier when straight debt and convertible debt have the senior-sub structure.

Table 1: Optimal default threshold x_d for the fraction of coupon payment for straight debt $\frac{s}{s+c}$

$\frac{s}{s+c}$	0.1	0.2	0.3	0.4	0.5	0.6
senior-sub structure	0.6363	0.7211	0.8310	0.9776	1.1806	1.4906
same priority	0.6359	0.7197	0.8283	0.9740	1.1759	1.4843

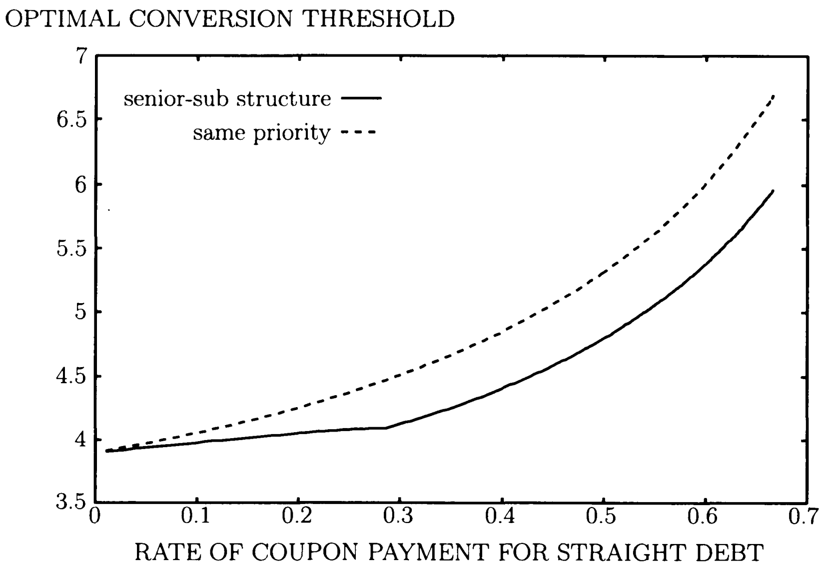

Figure 3: Optimal conversion threshold x_c for the fraction of coupon payment for straight debt $\frac{s}{s+c}$

Tab. 2 shows the optimal investment threshold values x^* for $\frac{s}{s+c}$. In both the cases, we may recognize that the firm invests at the high investment threshold in order to prevent the default from occurring because the default threshold is higher when the coupon payment for straight debt is higher. We can make sure that the senior-sub structure does not really affect the investment threshold similarly to the default threshold because the firm value which is used in the selection of the investment is unaffected directly. However, we can see that the investment delays because the default occurs earlier when the equity, the straight debt and the convertible debt have the senior-sub structure.

Finally, in Fig. 4 we show the value of investment $F(x)$ for the fraction of the coupon payment for straight debt when $x = 2$. In both the senior-sub structure and the same priority, when the coupon payment for straight debt is low, the value of investment is low because the firm value is also low. On the other hand, as the coupon payment for straight debt becomes higher, the value of investment is lower because the default threshold is lower for $x = 2$. We can see in this figure that the value of convertible debt decreases and then the value of investment is lower since the conversion occurs earlier when the equity, the straight debt and the convertible debt have the senior-sub structure.

Table 2: Optimal investment threshold x^* for the fraction of coupon payment for straight debt $\frac{s}{s+c}$

$\frac{s}{s+c}$	0.1	0.2	0.3	0.4	0.5	0.6
senior-sub structure	2.2199	2.2376	2.2788	2.3588	2.5032	2.7703
same priority	2.2197	2.2368	2.2771	2.3562	2.4993	2.7641

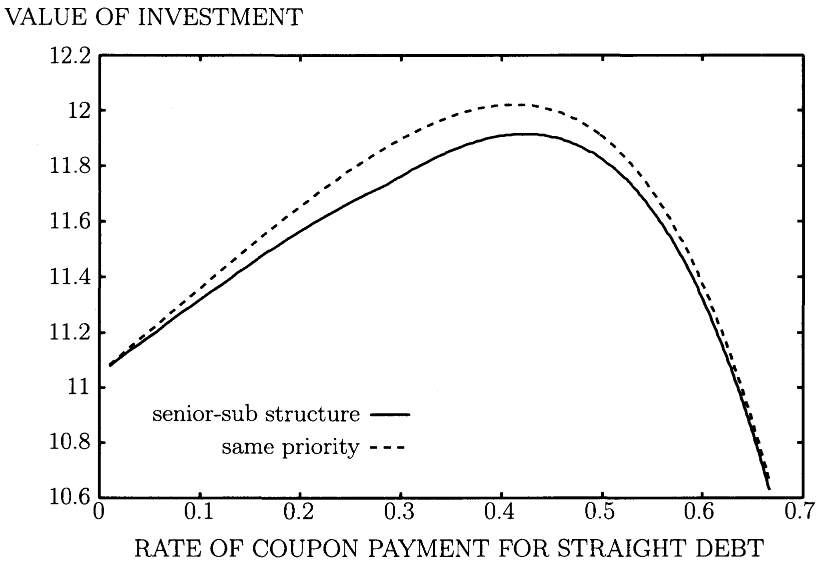

Figure 4: The value of investment $F(x)$ for the fraction of coupon payment for straight debt $\frac{s}{s+c}$ $(x = 2.0)$

4 Conclusions

In this paper, we have investigated the optimal investment policy of the firm financed by issuing equity, straight debt, and convertible debt with the senior-sub structure. The values of equity, straight debt, and convertible debt after exercising the investment option were shown. We also showed the threshold value for default, conversion and investment, and the investment option value. In particular, we found that the senior-sub structure for straight debt and convertible debt leads to the accelerating conversion, decreases the values of convertible debt and investment, and does not really affect the default and the investment.

The trade-off in debt financing is between the expected costs of bankruptcy and the tax benefit of debt. Straight debt is associated with both higher expected tax shields and larger expected bankruptcy costs. Thus, it is conceivable that there is an interior optimal capital structure, involving equity, straight debt, and convertible debt. In the future, we would like to try to solve for this equilibrium which is subject to the debt issued at par which is constant and analyze the sensitivity of its properties to various model parameters.

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