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Linear Optimization over Efficient Sets

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Abstract

In this paper, we consider an optimization problem of maximizing a single linear function over the set of efficient solutions to a multicriteria linear optimization problem. We show that this problem belongs to the class $\mathcal{NP}$-hard from the viewpoint of computational complexity. We also show that it can still be solved in finite steps using a simple algorithm.

Key words: Global optimization, d.c. optimization, multicriteria optimization, efficient set, $\mathcal{NP}$-hardness.

1 Introduction

In real world applications of optimization, it is often difficult to select an objective to be optimized among various conflicting ones. A typical example is the portfolio optimization, where many investors puzzle over whether to optimize return or risk. The best resolution of this sticky issue is to optimize all possible objectives simultaneously. The multicriteria optimization came from such a rose-colored idea. However, except in very rare cases, it is impossible to optimize multiple conflicting objectives simultaneously. As a compromise, an optimality concept newly introduced is efficiency, or Pareto optimality. Efficient solutions are those for which any change that makes some objective better off must necessarily make others worse off. Unfortunately, this concept is not the perfect answer to resolving the issue, because the number of efficient solutions is in general infinite. Decision makers, e.g., investors, have to again puzzle over which efficient solution to select. Then, how should we select the best one from those infinitely many efficient solutions? We can do it with further help of optimization, i.e., optimization over the set of efficient solutions.

In this paper, we discuss an optimization problem of maximizing a single linear function over the efficient set associated with a multicriteria linear optimization problem. Since the

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efficient set is usually not a convex set, but a difference of two convex sets (d.c. set), the problem is classified as multiextremal global optimization and has multiple locally optimal solutions. After reviewing some properties of this problem in the succeeding two sections, we show from the viewpoint of computational complexity that the problem belongs to the class $\mathcal{NP}$-hard in Section 4. Although this result is rather negative against the prospect of efficient algorithms, we show in Section 5 that a simple algorithm can generate a globally optimal solution in finite steps. Lastly, we refer to some future work on this class of problems in Section 6.

2 Multicriteria linear optimization

Let us consider the multicriteria linear optimization problem

$$\left| \begin{array}{ll} \text{maximize} & \mathbf{c}^1\mathbf{x} \\ \text{maximize} & \mathbf{c}^2\mathbf{x} \\ \quad\vdots & \ \vdots \\ \text{maximize} & \mathbf{c}^p\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} \leq \mathbf{b}, \end{array} \right. \tag{1}$$

where $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$ and $\mathbf{c}^i \in \mathbb{R}^n$, $i = 1,\ldots,p$. Let us denote the feasible set with

$$F = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} \leq \mathbf{b}\},$$

and assume for simplicity that F is nonempty and bounded. If $p = 1$, then (1) is just a linear programming problem

$$\left| \begin{array}{ll} \text{maximize} & \mathbf{c}\mathbf{x} \\ \text{subject to} & \mathbf{x} \leq F, \end{array} \right. \tag{2}$$

where $\mathbf{c} \in \mathbb{R}^n$. Before going into the main subject, we will first describe how much (1) with $p \geq 2$ is different from (2).

As long as the feasible set F is nonempty and bounded, there is a feasible solution $\mathbf{x}^* \in F$ which optimizes the objective function $\mathbf{c}\mathbf{x}$ of (2). However, in general, there is no feasible solution optimizing all objective functions $\mathbf{c}^1\mathbf{x}, \ldots, \mathbf{c}^p\mathbf{x}$ of (1) simultaneously. Therefore, we need a different optimality concept for (1) than the usual one for (2).

Definition 2.1. A feasible solution $\mathbf{x} \in F$ is said to be *efficient* for (1) if there is no $\mathbf{y} \in F \setminus \{\mathbf{x}\}$ such that

$$\mathbf{C}\mathbf{y} \geq \mathbf{C}\mathbf{x} \ \text{ and } \ \mathbf{C}\mathbf{y} \neq \mathbf{C}\mathbf{x},$$

where $\mathbf{C} \in \mathbb{R}^{p\times n}$ is the matrix whose rows are $\mathbf{c}^1,\ldots,\mathbf{c}^p$.

An efficient solution is also called a Pareto optimal solution, and is often interpreted as follows, in economics: "It is not possible to change the allocation of resources in such a way as to make some people better off without making others worse off." The following gives a Pareto optimality condition, which will be used for the later analysis.

Theorem 2.1. *Let* $\mathbf{x}' \in F$ *and* I *denote the index set of active constraints, i.e.,*

$$\mathbf{a}^i\mathbf{x}' = b_i, \quad i \in I; \quad \mathbf{a}^i\mathbf{x}' < b_i, \quad i \notin I,$$

where $\mathbf{a}^i \in \mathbb{R}^n$ *and* $b_i \in \mathbb{R}$ *denote the ith rows of* $\mathbf{A}$ *and* $\mathbf{b}$*, respectively. Then* $\mathbf{x}'$ *is efficient if and only if the following system has no solution* $\mathbf{u} \in \mathbb{R}^n$*:*

$$\mathbf{A}^I\mathbf{u} \leq \mathbf{0}, \quad \mathbf{C}\mathbf{u} \geq \mathbf{0}, \quad \mathbf{C}\mathbf{u} \neq \mathbf{0}, \tag{3}$$

where $\mathbf{A}^I \in \mathbb{R}^{|I| \times n}$ *and its rows are* $\mathbf{a}^i$*,* $i \in I$*.*

Proof. Suppose (3) has a solution $\mathbf{u}$. Let $\mathbf{x} = \mathbf{x}' + \alpha\mathbf{u}$. Then there is a positive number α' such that $\mathbf{x} \in F$ for all $\alpha \in (0, \alpha']$. However, $\mathbf{Cx} \geq \mathbf{Cx}'$ and $\mathbf{Cx} \neq \mathbf{Cx}'$ hold because $\mathbf{C}(\mathbf{x} - \mathbf{x}') = \alpha\mathbf{Cu}$. Hence, $\mathbf{x}'$ is not efficient. Suppose in turn (3) has no solution. Let $\mathbf{u} = \mathbf{x} - \mathbf{x}'$ for any $\mathbf{x} \in F$. Since $\mathbf{A}^I\mathbf{u} \leq \mathbf{b}_I$, we have $\mathbf{A}^I\mathbf{u} \leq \mathbf{0}$. Therefore, $\mathbf{u}$ does not satisfy $\mathbf{Cu} \geq \mathbf{0}$ and/or $\mathbf{Cu} \neq \mathbf{0}$. This implies that $\mathbf{x}'$ is efficient because $\mathbf{Cx} \geq \mathbf{Cx}'$ and/or $\mathbf{Cx} \neq \mathbf{Cx}'$ do not hold. □

Let us denote the set of feasible but inefficient solutions of (1) with

$$\overline{E} = \{\mathbf{x} \in F \mid \exists \mathbf{y} \in F, \mathbf{Cy} \geq \mathbf{Cx} \text{ and } \mathbf{Cy} \neq \mathbf{Cx}\}.$$

Then the set of efficient solutions is obviously given as

$$E = F \setminus \overline{E}.$$

It is easy to see that $\overline{E}$ is a convex set. This implies that the efficient set E is not a convex set. Since it can be represented as the difference of two convex sets, we refer to this kind of set as a *d.c. set* (difference of convex sets) [8, 16].

3 Target problem and its properties

Our target problem is not (1) itself but a single criterion optimization problem associated with (1):

$$\left| \begin{array}{ll} \text{maximize} & \mathbf{dx} \\ \text{subject to} & \mathbf{x} \in E, \end{array} \right. \tag{4}$$

where $\mathbf{d} \in \mathbb{R}^n$ and $E \subset \mathbb{R}^n$ is the efficient set of (1). This is a special class of d.c. optimization problems, and usually involves multiple locally optimal solutions different from the global one because the feasible set E is not convex. The first appearance of (4) in the literature is due to Philip in 72 [11]. In an interactive approach to (1), he considered the following instance of (4) to show the decision maker an approximate size of the efficient set E,

$$\left|\begin{array}{ll} \text{maximize} & -\mathbf{c}^i\mathbf{x} \\ \text{subject to} & \mathbf{x} \in E. \end{array}\right.$$

The most common application of (4) is a problem

$$\left|\begin{array}{ll} \text{maximize} & \lambda^{\mathsf{T}}\mathbf{C}\mathbf{x} \\ \text{subject to} & \mathbf{x} \in E, \end{array}\right.$$

where $\lambda \in \mathbb{R}^p$ is a positive vector. This problem is, however, equivalent to a linear programming problem

$$\left|\begin{array}{ll} \text{maximize} & \lambda^{\mathsf{T}}\mathbf{C}\mathbf{x} \\ \text{subject to} & \mathbf{x} \in F. \end{array}\right. \tag{5}$$

Actually, we can prove the following from Definition 2.1 via theorems of the alternative (see e.g., [14] for proof).

Theorem 3.1. *A vector* $\mathbf{x}' \in \mathbb{R}^n$ *is an efficient solution of (1) if and only if there exists a vector* $\lambda > \mathbf{0}$ *such that* $\mathbf{x}'$ *solves (5).*

For more general cases of (4), there are a variety of algorithms developed so far, i.e., adjacent vertex search algorithms [4, 6, 11], nonadjacent vertex search algorithms [1], face search algorithms [12, 13], branch-and-bound algorithms [7, 15], and so on. For a comprehensive survey of algorithms, the reader is referred to Yamamoto [17].

4 $\mathcal{NP}$-hardness of (4)

In [3, 5], it is pointed out that (4) is equivalent to the bilevel linear programming problem

$$\left|\begin{array}{lll} \text{maximize}_{\mathbf{x}} & \mathbf{c}^1\mathbf{x}+\mathbf{d}^1\mathbf{y} & \\ \text{subject to} & \mathbf{y} \text{ solves} & \left|\begin{array}{ll} \text{maximize}_{\mathbf{y}} & \mathbf{d}^2\mathbf{y} \\ \text{subject to} & \mathbf{A}\mathbf{x}+\mathbf{B}\mathbf{y} \leq \mathbf{b}, \end{array}\right. \end{array}\right. \tag{6}$$

which is known to be $\mathcal{NP}$-hard [2]. This naturally implies that (4) is also an $\mathcal{NP}$-hard problem, but strangely enough [5] has not yet been published anywhere, and [3] provides only an excerpt from this unpublished paper.

We will try here to prove the $\mathcal{NP}$- hardness of (4) in a different way, without using the equivalence between (4) and (6). For this purpose, let us consider a well-known $\mathcal{NP}$-complete problem:

0-1 knapsack.

Input: $a_j \in \mathbb{Z}_+$, $c_j \in \mathbb{Z}_+$ $(j = 1, \ldots, n)$, $b \in \mathbb{Z}_+$ and $z \in \mathbb{Z}_+$.

Question: Is there a subset $N \subset \{1, \ldots, n\}$ satisfying the following?

$$\sum_{j \in N} a_j \leq b \text{ and } \sum_{j \in N} c_j \geq z.$$

Note that 0-1 knapsack can be solved by checking if the following set is empty or not:

$$\{\mathbf{x} \in \{0,1\}^n \mid \mathbf{ax} \leq \mathbf{b}, \mathbf{cx} \geq z\},$$

where $\mathbf{a} = [a_1, \ldots, a_n]$, and $\mathbf{c} = [c_1, \ldots, c_n]$. Associated with 0-1 knapsack, consider a multi-criteria optimization problem:

$$\left|\begin{array}{ll} \text{“maximize”} & \begin{bmatrix} \mathbf{x}+\mathbf{y} \\ -\mathbf{x}+\mathbf{y} \end{bmatrix} \\ \text{subject to} & \mathbf{ax} \leq b, \quad \mathbf{x}+2\mathbf{y} \leq \mathbf{e} \\ & \mathbf{cx} \leq z, \quad \mathbf{x}-2\mathbf{y} \geq \mathbf{0} \\ & \mathbf{0} \leq \mathbf{x} \leq \mathbf{e}, \quad \mathbf{y} \geq \mathbf{0}. \end{array}\right. \tag{7}$$

where “maximize” means vector maximize, and $\mathbf{e} \in \mathbb{R}^n$ is the all-ones vector.

Lemma 4.1. *If* $(\mathbf{x}', \mathbf{y}')$ *is an efficient solution of (7), then*

$$y'_j = \min\{x'_j,\ 1 - x'_j\}/2, \quad j = 1, \ldots, n.$$

Proof. If $y'_j < \min\{x'_j, 1 - x'_j\}/2$ for some j, then we can improve the jth and $(n+j)$th objectives by replacing y'_j with $\min\{x'_j, 1 - x'_j\}/2$, without reducing the values of the other objectives. □

Lemma 4.2. *Assume that* $\mathbf{x}' \in \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{ax} \leq b, \mathbf{cx} \geq z, \mathbf{0} \leq \mathbf{x} \leq \mathbf{e}\}$. *Let*

$$y'_j = \min\{x'_j,\ 1 - x'_j\}/2, \quad j = 1, \ldots, n.$$

Then $(\mathbf{x}', \mathbf{y}')$ *is an efficient solution of (7).*

Proof. It is obvious that $(\mathbf{x}', \mathbf{y}')$ is feasible for (7). Let J be a subset of $\{1, \ldots, n\}$ such that $x'_j + 2y'_j = 1$. Then we see from Theorem 2.1 that $(\mathbf{x}', \mathbf{y}')$ is efficient if there is no $(\mathbf{u}, \mathbf{v})$

satisfying

$$\begin{array}{llll} u_j+2v_j\le 0, & u_j\le 0, & v_j\ge 0, & j\in J \\ u_j-2v_j\ge 0, & u_j\ge 0, & v_j\ge 0, & j\notin J \\ \mathbf{u}+\mathbf{v}\ge \mathbf{0}, & -\mathbf{u}+\mathbf{v}\ge \mathbf{0} & & \\ \mathbf{u}+\mathbf{v}\ne \mathbf{0}, & -\mathbf{u}+\mathbf{v}\ne \mathbf{0}. & & \end{array}$$

Select an arbitrary $j\in J$. Then we have $u_j+v_j\le -v_j\le 0$, and hence $u_j+v_j=0$. Also $v_j=u_j+2v_j\le 0$, and hence we have $u_j=v_j=0$. Similarly, $u_j=v_j=0$ hold for any $j\notin J$. Therefore, no $(\mathbf{u},\mathbf{v})$ satisfies the system. □

Theorem 4.3. *Let $E\subset\mathbb{R}^n\times\mathbb{R}^n$ and denote by E the efficient set of (7). The answer of 0-1 knapsack is 'yes' if and only if the optimal value of the following problem is zero,*

$$\left|\begin{array}{ll} \text{maximize} & -\mathbf{e}^{\mathsf{T}}\mathbf{y} \\ \text{subject to} & (\mathbf{x},\mathbf{y})\in E. \end{array}\right. \tag{8}$$

Proof. If $\mathbf{x}'\in\{\mathbf{x}\in\{0,1\}^n \mid \mathbf{ax}\le b,\ \mathbf{cx}\ge z\}$, then $(\mathbf{x}',\mathbf{0})$ is an efficient solution of (7) by Lemma 4.2. Since the upper bound of the objective function is zero, $(\mathbf{x}',\mathbf{0})$ is an optimal solution of (8). Conversely, suppose $(\mathbf{x}',\mathbf{0})$ is an optimal solution of (8). Since $(\mathbf{x}',\mathbf{0})$ is an efficient solution of (7), we see from Lemma 4.2 that

$$\min\{x'_j,\ 1-x'_j\}=0,\quad j=1,\dots,n.$$

Therefore, $\mathbf{x}'$ belongs to $\{\mathbf{x}\in\{0,1\}^n \mid \mathbf{ax}\le b,\ \mathbf{cx}\ge z\}$. □

The $\mathcal{NP}$-hardness of (4) follows immediately from this theorem.

Corollary 4.4. *The problem (4) is $\mathcal{NP}$-hard.*

5 Face enumerating algorithm for (4)

In this section, we briefly illustrate an algorithm for solving the problem (4) in a finite steps. Let us denote the constraint inequalities defining F as

$$\begin{gathered} \mathbf{a}^1\mathbf{x}\le b_1 \\ \vdots \\ \mathbf{a}^m\mathbf{x}\le b_m. \end{gathered}$$

The algorithm proposed here is a simple one which enumerates all faces of F. Namely, we label each constraint inequality $\mathbf{a}^i\mathbf{x}\le b_i$ with either 'active' or 'inactive', in the order of indices from $i=1$ to m. Then we have an enumeration tree with $\binom{m}{n}$ leaves, each corresponding to a face of F. However, some of those are not efficient, nor even feasible, and so we need some procedures for avoiding such an unnecessary enumeration.

Suppose at some intermediate node in the enumeration tree that $\mathbf{a}^i\mathbf{x} \leq b_i, i \in I \subset \{1,\dots,m\}$, are labeled 'active'. Also let

$$F_I = F \cap \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}^i\mathbf{x} = b_i,\ i \in I\}.$$

Efficiency test. We can check whether F_I is an efficient face or not, by solving a linear programming problem.

Proposition 5.1. *The following problem is either unbounded or has an optimal solution with optimal value zero,*

$$\left|\begin{array}{ll} \text{maximize} & \mathbf{e}^{\mathrm{T}}\mathbf{C}\mathbf{u} \\ \text{subject to} & \mathbf{A}^I\mathbf{u} \leq \mathbf{0}, \quad \mathbf{C}\mathbf{u} \geq \mathbf{0}, \end{array}\right. \tag{9}$$

where $\mathbf{A}^I \in \mathbb{R}^{|I|\times n}$ and its rows are $\mathbf{a}^i$, $i \in I$. In the latter case, any $\mathbf{x} \in F_I$ is an efficient solution of (1).

Proof. Since the feasible set of (9) is a polyhedral cone, it is either unbounded or a singleton $\{\mathbf{0}\}$. In the latter case, the system (3) has no solution. Hence, from Theorem 2.1 we see that any $\mathbf{x} \in F_I$ is an efficient solution of (1). □

Solution update. If F_I has proven to be an efficient face, we next solve the following to update the incumbent,

$$\left|\begin{array}{ll} \text{maximize} & \mathbf{d}\mathbf{x} \\ \text{subject to} & \mathbf{x} \in F_I. \end{array}\right. \tag{10}$$

Since F_I is a face of the polytope F, this problem is also a linear programming problem. We may backtrack the enumeration tree from the current node after solving (10).

A globally optimal solution of (4) can be generated using the above two basic procedures in the framework of branch-and-bound. A detailed description of the algorithm will be given in a future paper, together with numerical results.

6 Future work

In this paper, we have discussed a typical global optimization problem (4), which optimizes a linear function over an efficient set associated with a multicriteria linear optimization problem (1). Then we have proven $\mathcal{NP}$-hardness of this class and outlined a finite algorithm for generating a globally optimal solution.

Similar to the problem (4) is the linear multiplicative programming problem

$$\left|\begin{array}{ll} \text{maximize} & \prod_{i=1}^{p}(\mathbf{c}^i\mathbf{x} + \gamma_i) \\ \text{subject to} & \mathbf{x} \in F, \end{array}\right. \tag{11}$$

where γ_i is a scalar. This problem can also be thought of as a model for optimizing p objectives simultaneously. The only difference between (4) and (11) is in the function for evaluating p objectives. In fact, an optimal solution of (11) lies in the efficient set of the multicriteria linear optimization problem (1). The simplest subclass is

$$\left|\begin{array}{ll} \text{maximize} & (\mathbf{c}^1\mathbf{x}+\gamma_1)(\mathbf{c}^2\mathbf{x}+\gamma_2) \\ \text{subject to} & \mathbf{x}\in F, \end{array}\right. \tag{12}$$

which is known to be solved in less than twice the time required to solve two linear programming problems of the same size [9]. Nevertheless, (12) is an $\mathcal{NP}$-hard problem [10]. In the light of the resemblance between (4) and (11), we can predict that (4) is still $\mathcal{NP}$-hard even if it is associated with a bicriteria linear optimization problem

$$\left|\begin{array}{ll} \text{maximize} & \mathbf{c}^1\mathbf{x} \\ \text{maximize} & \mathbf{c}^2\mathbf{x} \\ \text{subject to} & \mathbf{x}\in F. \end{array}\right.$$

We leave proof to this prediction open for the future work.

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