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EXTENSION OF OPERATORS WITH SEPARABLE RANGE

MANUEL GONZÁLEZ

ABSTRACT. A Banach space E is *injective* if it satisfies the following *extension property:* for every space X and every subspace Y of X, each operator $T : Y \longrightarrow E$ admits an extension $\widehat{T} : X \longrightarrow E$. Many people have investigated these spaces, but it remains unknown whether every injective Banach space is isomorphic to a space of continuous functions $C(K)$ with K a Stonian compact.

We consider two weaker forms of injectivity: E is *separably injective* if it satisfies the extension property when X is separable; it is *universally separably injective* if it satisfies the extension property when Y is separable. Obviously, injective $\Rightarrow$ universally separably injective $\Rightarrow$ separably injective, but the converse implications fail. We show that the corresponding classes of Banach spaces are much richer in examples and structural properties than injectivity.

1. INTRODUCTION

A Banach space E is said to be λ-*injective* ($\lambda \geq 1$) if it satisfies the following *extension property:*

for every Banach space X and every subspace Y of X, each operator $T : Y \rightarrow E$ admits an extension $\widehat{T} : X \rightarrow E$ satisfying $\|\widehat{T}\| \leq \lambda\|T\|$.

The space E is *injective* if it is λ-injective for some $\lambda \geq 1$.

Nachbin, Goodner, Kelley and Hasumi [21, 10, 15, 12] characterized the 1-injective spaces as those Banach spaces linearly isometrically isomorphic to a $C(K)$ space, with K a Stonian compact. However, despite the deep investigations of Argyros [1, 2, 3, 4], Haydon [13], Rosenthal [23, 24] and other authors, finding a description of the class of injective Banach spaces seems to be an unmanageable problem. It is not even known if every injective space is isomorphic to a 1-injective space or to a $C(K)$ space.

We deal with two weaker forms of injectivity which admit a similar definition. Namely, we say that E is λ-*separably injective* (and write $E \in \Upsilon_\lambda$) if it satisfies the previously described extension property, but only for X separable. Also, we say that E is λ-*universally separably injective* (and write $E \in \Upsilon_\lambda^{univ}$) if it satisfies the extension property when Y is separable. Obviously, $\Upsilon_\lambda^{univ} \subset \Upsilon_\lambda$.

The space E is *separably injective* if $E \in \Upsilon := \bigcup_{\lambda\geq 1} \Upsilon_\lambda$; and it is *universally separably injective* if $E \in \Upsilon^{univ} := \bigcup_{\lambda\geq 1} \Upsilon_\lambda^{univ}$.

Our aim is to show that the classes Υ and Υ^{univ} are much richer in examples and structure than the class of injective spaces. Here we include only a few proofs. For a detailed account, we refer to [5].

Among other results, we give several characterizations of spaces in Υ_λ and prove some basic properties of these spaces: they are $\mathcal{L}_{\infty,\lambda+}$-spaces and have Pełczyński's property (V). The space c_0 is in Υ_2 and every (infinite dimensional) separable Banach space in Υ is isomorphic to c_0. A $C(K)$ space belongs to Υ_1 if and only if the compact

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K is an F-space; we also give some other characterizations. We also show some stability properties of the classes Υ and Υ^{univ}: both have the three-space property; and if $(E_n) \subset \Upsilon_\lambda$, then $c_0(E_n) \in \Upsilon$. Moreover, if Y is a subspace of X, then $Y, X \in \Upsilon$ implies $X/Y \in \Upsilon$, and $X \in \Upsilon^{univ}$ and $Y \in \Upsilon$ imply $X/Y \in \Upsilon^{univ}$; thus $\ell_\infty/c_0 \in \Upsilon^{univ}$. The stability properties allow us to construct many new examples of spaces in Υ or Υ^{univ}. Among them, we show spaces in Υ^{univ} which are not isomorphic to any complemented subspace of any $C(K)$ space. We also show that an ultraproduct of Banach spaces (following a countably incomplete ultrafilter) is injective only in the trivial case in which it is finite dimensional. However, if an ultraproduct is a $\mathcal{L}_\infty$ space, then it is universally separably injective.

Notations and Conventions. Throughout the paper the ground field is $\mathbb{R}$. Of course, most of our results can be adapted to the complex setting. The Banach-Mazur distance between the Banach spaces X and Y is

$$\mathrm{dist}_{BM}(X,Y) = \inf\{\|T\| \cdot \|T^{-1}\| : T \text{ is an isomorphism between } X \text{ and } Y\}.$$

A Banach space X is a $\mathcal{L}_{\infty,\lambda}$-space (with $1 \leq \lambda < \infty$) if every finite dimensional subspace F of X is contained in another finite dimensional subspace of X whose Banach-Mazur distance to the corresponding ℓ_∞^n is at most λ. A space X is a $\mathcal{L}_\infty$-space if it is a $\mathcal{L}_{\infty,\lambda}$-space for some $\lambda \geq 1$; and it is a $\mathcal{L}_{\infty,\lambda+}$-space when it is a $\mathcal{L}_{\infty,\lambda'}$-space for all $\lambda' > \lambda$.

We write $C(K)$ for the Banach space of all continuous functions on the compact space K, with the sup norm. Topological spaces are assumed to be Hausdorff. We write $|S|$ for the cardinality of a set S.

Let Γ be a set. We denote by $\ell_\infty(\Gamma)$ the space of all bounded scalar functions on Γ, endowed with the sup norm. Moreover, $c_0(\Gamma)$ is the closed subspace spanned by the characteristic functions of the singletons of Γ.

The *density character* $\mathrm{dens}(X)$ of a Banach space X is the least cardinal $\mathfrak{m}$ for which X has a dense subset of cardinality $\mathfrak{m}$. Observe that $\mathrm{dens}(\ell_\infty(\Gamma)) = 2^{|\Gamma|}$.

This paper describes joint work with Antonio Avilés, Félix Cabello, Jesús M.F. Castillo and Yolanda Moreno [5]. It was presented during the R.I.M.S. Conference *Prospects of non-commutative analysis in operator theory* at Kyoto University, October 28-30, 2009. The author thanks Profesors Muneo Chō and Kotaro Tanahashi for their attentions during this Conference.

2. Injective spaces

A Banach space E is *injective* if for every Banach space X and every subspace Y of X, each operator $T\colon Y \to E$ admits an extension $\widehat{T}\colon X \to E$.

The space E is 1-*injective* if we can always get T with $\|\widehat{T}\| = \|T\|$.

Remark 1. *It is not difficult to show that E is injective if and only if every subspace of a Banach space isomorphic to E is complemented.*

The first examples of injective Banach spaces are obtained as a direct consequence of the Hahn-Banach Theorem:

(1) $\mathbb{K}$ is 1-injective.
(2) $\ell_\infty(I)$ is 1-injective. Indeed, for each $T\colon Y \to \ell_\infty(I)$ there exists a family $(y_i^*)_{i\in I} \subset Y^*$ so that $Ty = \big(y_i^*(y)\big)$ and $\|T\| = sup_{i\in I}\|x_i^*\|$.

Remark 2. *Every Banach space X can be embedded as a subspace of a $\ell_\infty(\Gamma)$ space with $|\Gamma| = dens(X)$. Indeed, let $\{x_i : i \in \Gamma\}$ be a dense subset of X and choose, for each $i \in \Gamma$, a norm-one $f_i \in X^*$ such that $f_i(x_i) = \|x_i\|$. Then the operator $T : X \longrightarrow \ell_\infty(\Gamma)$ defined by $T(x) := (f_i(x))$ is an isometric embedding.*

As a consequence, a Banach space E is injective if and only if it is isomorphic to a complemented subspace of $\ell_\infty(\Gamma)$ for some set Γ.

In the period 1950–58, a characterization of the 1-injective Banach spaces was obtained in several steps by Nachbin, Goodner, Kelley and Hasumi. Recall that a topological space is said to be *Stonian* if the closure of each open subset is open.

Theorem 3. [21, 10, 15, 12]
Every 1-injective space is isometrically isomorphic to some $C(K)$ space, where K is a Stonian compact.

However, the following problems have remained open:

(1) Is every injective space isomorphic to a 1-injective space?
(2) Is every injective space isomorphic to a $C(K)$ space?
(3) Which is the structure of an injective space?

Other examples of 1-injective spaces.

(1) Let I be a non-empty set endowed with the discrete topology. Denoting by βI the Stone-Čech compactification of I, $C(\beta I) \equiv \ell_\infty(I) \equiv c_0(I)^{**}$, a second dual space.
It was proved by Haydon [13] that every injective space isomorphic to a second dual is isomorphic to $\ell_\infty(I)$ for some set I.
(2) Let μ be a finite measure for which $L_1(\mu)$ non-separable. Then $L_\infty(\mu) \equiv L_1(\mu)^*$ is 1-injective, but it is not isomorphic to a second dual space.
(3) Rosenthal [23] proved that there exists a Stonian compact K_G such that $C(K_G)$ is not isomorphic to any dual space.

The following result of Rosenthal is helpful to show that some Banach spaces are not injective.

Proposition 4. [24]
(a) Every infinite dimensional injective Banach space contains a subspace isomorphic to ℓ_∞.
(b) If an injective space contains a subspace isomorphic to $c_0(I)$, then it also contains a subspace isomorphic to $\ell_\infty(I)$.

Corollary 5. *The quotient space ℓ_∞/c_0 is not injective.*

Proof. Let $\{A_i : i \in I\}$ be an uncountable family of infinite subsets of $\mathbb{N}$ such that $A_i \cap A_j$ is finite for $i \neq j$. The characteristic function of each A_i corresponds to an element $x_i \in \ell_\infty$. Let z_i denote the image of x_i in ℓ_∞/c_0.
The subspace generated by $\{z_i : i \in I\}$ in ℓ_∞/c_0 is isomorphic to $c_0(I)$. However, ℓ_∞/c_0 does not contain subspaces isomorphic to $\ell_\infty(I)$. □

3. Separably injective spaces

Let X be a Banach space and let Y be a subspace of X. We say that a Banach space E satisfies the *λ-extension property for* (X, Y) if each operator $T\colon Y \to E$ has an extension $\widehat{T}\colon X \to E$ with $\|\widehat{T}\| \leq \lambda\|T\|$.

Definition 6. *Let* $1 \leq \lambda < \infty$.

E is λ-separably injective *($E \in \Upsilon_\lambda$) if it satisfies the λ-extension property for (X, Y) when X is separable.*

E is λ-universally separably injective *($E \in \Upsilon_\lambda^{univ}$) if it satisfies the λ-extension property for (X, Y) when Y is separable.*

Notations: $\Upsilon := \bigcup_{\lambda\geq 1} \Upsilon_\lambda$, $\Upsilon^{univ} := \bigcup_{\lambda\geq 1} \Upsilon_\lambda^{univ}$.

Proposition 7. *The following implications hold:*

E injective $\Rightarrow E \in \Upsilon^{univ} \Rightarrow E \in \Upsilon \Rightarrow E \in \mathcal{L}_\infty$.

All the converse implications fail, in general.

If E is isomorphic to a dual space and $E \in \mathcal{L}_\infty$ then E is injective.

3.1. **Earlier results.** Several people have studied separably injective Banach spaces. Here we describe some of their results.

Proposition 8. *The following assertions hold:*

(1) *Let I be an infinite set. Then $c_0(I) \in \Upsilon_2$* (Sobczyk, [25]). *However, $c_0(I)$ is not universally separably injective.*
(2) *If $E \in \Upsilon$ is infinite dimensional and separable then E is isomorphic to c_0* (Zippin, [26]).
(3) *If E is infinite dimensional and $E \in \Upsilon_\lambda$ with $\lambda < 2$ then E is non-separable* (Ostrovskii, [22]).

Next we give a good description of the $C(K)$ spaces which are 1-separably injective due to several authors (see [5]). Recall that a compact space is a *F-space* if disjoint open F_σ subsets have disjoint closures.

Theorem 9. *For a compact space K, the following assertions are equivalent:*

(a) *$C(K)$ is 1-separably injective;*
(b) *given (f_i) and (g_j) in $C(K)$ with $f_i \leq g_j$ for each i, j there exists $h \in C(K)$ such that $f_i \leq h \leq g_j$ for each i, j;*
(c) *Every sequence of mutually intersecting balls in $C(K)$ has nonempty intersection;*
(d) *K is a F-space;*
(e) *Given $f \in C(K)$ there is $u \in C(K)$ such that $f = u|f|$.*

Observe that part (e) in the previous Theorem has several applications:

(1) A closed subset of a compact F-space is a F-space; in particular, $\ell_\infty/c_0 \equiv C(\beta\mathbb{N} \setminus \mathbb{N}) \in \Upsilon_1$.
(2) The space $B[0, 1]$ of bounded Borel functions on $[0, 1]$ belongs to Υ_1.

3.2. **Characterizations and properties.** We present several characterizations of the separably injective Banach spaces and describe some stability properties of the class Υ that allow us to obtain new examples from the previously known ones.

Proposition 10. *For a Banach space E, the following assertions are equivalent:*
(1) $E \in \Upsilon$;
(2) *if X/Y is separable, every $T : Y \to E$ extends to X;*
(3) $X \supset M \simeq E$, *X/M separable* $\Rightarrow$ *M complemented in X;*
(4) *if $Y \subset \ell_1$, every $T : Y \to E$ extends to ℓ_1.*

Proposition 11. *Let $E \in \Upsilon_\lambda$ infinite dimensional. Then:*
(i) *E is a $\mathcal{L}_{\infty,\lambda}$-space;*
(ii) *E contains a copy of c_0;*
(iii) *E has Pełczyński's* property (V): *every non-weakly compact $T : E \to Y$ is an isomorphism on a subspace of E isomorphic to c_0.*

We say that a class $\mathcal{C}$ of Banach spaces has the *three-space property* if the following condition is satisfied:

$$Y \subset X; \quad Y, X/Y \in \mathcal{C} \Rightarrow X \in \mathcal{C}.$$

We refer to [7] for information on classes of Banach spaces with the three-space property. The following properties allow us to construct examples of separably injective Banach spaces.

Proposition 12.
(i) *The class Υ has the three-space property;*
(ii) $X \supset M$, $X, M \in \Upsilon \Rightarrow X/M \in \Upsilon$;
(iii) $(E_n) \subset \Upsilon_\lambda \Rightarrow c_0(E_n) \in \Upsilon_{\lambda(1+\lambda)}$.

3.3. **On universally separably injective spaces.** First we describe a natural example of universally separably injective Banach space.

Let Γ be an uncountable set. We denote

$$\ell_\infty^c(\Gamma) := \{(a_i) \in \ell_\infty(\Gamma) : \text{supp } \big((a_i)\big) \text{ countable}\}.$$

Proposition 13. $\ell_\infty^c(\Gamma) \in \Upsilon_1^{univ}$, *but it is not injective.*

Proof. Given an infinite countable subset J of Γ, $\{(a_i) \in \ell_\infty^c(\Gamma) : \text{supp } \big((a_i)\big) \subset J\}$ is a subspace of $\ell_\infty^c(\Gamma)$ isometric to ℓ_∞. Therefore, every separable subspace of $\ell_\infty^c(\Gamma)$ is contained in a subspace isometric to ℓ_∞. From this fact, it follows that $\ell_\infty^c(\Gamma) \in \Upsilon_1^{univ}$.

The space $\ell_\infty^c(\Gamma)$ is not injective because it contains a subspace isomorphic to $c_0(\Gamma)$, but it does not contain subspaces isomorphic to $\ell_\infty(\Gamma)$. □

Surprisingly, the property that allowed us to show that $\ell_\infty^c(\Gamma) \in \Upsilon^{univ}$ characterizes the universally separably injective spaces.

Theorem 14 (Structure). *$E \in \Upsilon^{univ}$ if and only if each separable subspace of E is contained in another subspace isomorphic to ℓ_∞.*

The following result shows that infinite dimensional spaces in Υ^{univ} are big.

Proposition 15. *If $E \in \Upsilon^{univ}$, every non-weakly compact operator $T : E \to Y$ is an isomorphism on a subspace of E isomorphic to ℓ_∞.*

Recall that two Banach spaces E and F are *essentially incomparable* if given operators $T : E \longrightarrow F$ and $S : F \longrightarrow E$, $I_E - ST$ (equivalently, $I_F - TS$) is a bijective isomorphism up to some finite dimensional subspaces [8].

Corollary 16. *Every infinite dimensional space in* Υ^{univ} *contains a subspace isomorphic to* ℓ_∞.

If $E \in \Upsilon^{univ}$ *and* F *contains no subspaces isomorphic to* ℓ_∞ *then* E *and* F *are essentially incomparable.*

The following result describe some stability properties of the class Υ^{univ}

Proposition 17 (Construction of examples).

(i) *The class* Υ^{univ} *has the three-space property:*

$$Y \subset X; \quad Y, X/Y \in \Upsilon^{univ} \Rightarrow X \in \Upsilon^{univ};$$

(ii) $X \supset M$, $X \in \Upsilon^{univ}$, $M \in \Upsilon \Rightarrow X/M \in \Upsilon^{univ}$.

3.4. **Special properties of spaces in Υ_1.** Here we present some properties of the 1-separably injective Banach spaces in which special axioms of set theory are involved.

We denote by C.H. the *continuum hypothesis:* $\mathfrak{c} = \aleph_1$, and Z.F.C. represents the *Zermelo-Fraenkel axioms,* including Choice.

Proposition 18. *Let* E *be a* 1*-separably injective space. Then*

(1) E *is Grothendieck; i.e., every operator from* E *into* c_0 *is weakly compact;*
(2) E *is a Lindenstrauss space; i.e.,* E^* *is linearly isometric to some* $L_1(\mu)$ *space;*
(3) *if* E *is infinite dimensional, then dens*$(E) \geq \mathfrak{c}$.

The following result is a direct application of an argument of Lindenstrauss [17].

Proposition 19. *Under C.H., the classes* Υ_1 *and* Υ_1^{univ} *coincide.*

Corollary 20. *Under C.H., every* $E \in \Upsilon_1$ *contains a subspace isomorphic to* ℓ_∞.

In the following result we show that C.H. is necessary for the coincidence of Υ_1 and Υ_1^{univ}.

Theorem 21. *Under Z.F.C.* $+\, \mathfrak{c} = \aleph_2$, *there exists a compact space* K_0 *such that* $C(K_0) \in \Upsilon_1$ *but* $C(K_0) \notin \Upsilon_1^{univ}$.

We observe that we do not know if the space $C(K_0)$ in the previous Theorem belongs to Υ^{univ}.

Let $\mathcal{K}u$ denote the Banach space of universal disposition for separable spaces constructed by Kubis [16]. Observe that $\mathcal{K}u$ is not isomorphic to any $C(K)$ space.

Proposition 22. *Under C.H.,* $\mathcal{K}u \in \Upsilon_1^{univ}$.

3.5. **Ideals and M-ideals.** Here we give some results for closed ideals of $C(K)$ spaces and the corresponding quotients. We also give some related abstract results in terms of M-ideals in Banach spaces.

Let M be a closed subset of a compact K. Then $L := K \setminus M$ is locally compact. Moreover, $C_0(L)$ is a closed ideal in $C(K)$ and the quotient space $C(K)/C_0(L)$ is isometric to $C(M)$. Consequently, we have an exact sequence

$$0 \longrightarrow C_0(L) \longrightarrow C(K) \longrightarrow C(M) \longrightarrow 0.$$

Theorem 23. *Let M be a closed subset of a compact K.*
(1) $C(K) \in \Upsilon_\lambda^{univ} \Rightarrow C(M) \in \Upsilon_\lambda^{univ}$;
(2) $C(K) \in \Upsilon_\lambda \Rightarrow C_0(L) \in \Upsilon_{2\lambda}$.

Recall that a closed subspace J of a Banach space E is a *M-ideal* if $E^* = J^\perp \oplus_1 N$ for some closed subspace N.

Theorem 24. *Let J be a M-ideal in E.*
(1) $E \in \Upsilon_\lambda^{univ} \Rightarrow E/J \in \Upsilon_{\lambda^2}^{univ}$;
(2) $E \in \Upsilon_\lambda \Rightarrow J \in \Upsilon_{2\lambda^2}$.

3.6. **Ultraproducts of Banach spaces.** Here we give some results involving ultraproducts of Banach spaces. First, we recall the concept of ultraproduct. For additional information, we refer to [14] or [9, A4].

Let I be an infinite set and let $\mathcal{U}$ be a countably incomplete ultrafilter on I. Recall that $\mathcal{U}$ is countably incomplete if and only if there exists a sequence (I_n) of subsets of I in $\mathcal{U}$ such that $\cap_{n=1}^\infty I_n = \emptyset$.

Let $(X_i)_{i\in I}$ be a family of Banach spaces. Then

$$\ell_\infty(X_i) := \{(x_i) : x_i \in X_i,\ \sup_i \|x_i\| < \infty\},$$

endowed with the supremum norm, is a Banach space, and

$$c_0^{\mathcal{U}}(X_i) := \{(x_i) \in \ell_\infty(X_i) : \lim_{i\to\mathcal{U}} \|x_i\| = 0\}$$

is a closed subspace of $\ell_\infty(X_i)$.

The *ultraproduct of $(X_i)_{i\in I}$ following $\mathcal{U}$* is defined as the quotient

$$(X_i)_{\mathcal{U}} := \frac{\ell_\infty(X_i)}{c_0^{\mathcal{U}}(X_i)}.$$

If $[x_i]$ is the element of $(X_i)_{\mathcal{U}}$ which has (x_i) as a representative then

$$\|[x_i]\| = \lim_{i\to\mathcal{U}} \|x_i\|.$$

In the case $X_i = X$ for all i, we denote the ultraproduct by $X_{\mathcal{U}}$, and call it the *ultrapower of X following $\mathcal{U}$*.

3.7. **Ultraproducts which are $\mathcal{L}_\infty$-spaces.** Recall that X is a $\mathcal{L}_\infty$ space if there exists λ $(1 \leq \lambda < \infty)$ such that every finite dimensional subspace of X is contained in another finite dimensional subspace F so that $\mathrm{dist}_{BM}(F, \ell_\infty^{\dim F}) \leq \lambda$. We refer to [6, 19] for information on $\mathcal{L}_\infty$-spaces.

Our first result says that non-trivial ultraproducts are never injective.

Theorem 25. *$(X_i)_{\mathcal{U}}$ is injective if and only if it is finite dimensional.*

Our second results says that ultraproducts which are $\mathcal{L}_\infty$-spaces belong to Υ^{univ}.

Theorem 26. *$(X_i)_{\mathcal{U}}$ $\mathcal{L}_\infty$-space $\Rightarrow (X_i)_{\mathcal{U}} \in \Upsilon^{univ}$.*

Let us state some consequences:

(1) $(X_i) \subset \mathcal{L}_{\infty,\lambda+} \Rightarrow (X_i)_{\mathcal{U}} \in \Upsilon_\lambda^{univ}$.

(2) (X_i) Lindenstrauss spaces (e.g., $C(K)$ spaces) $\Rightarrow (X_i)_{\mathcal{U}} \in \Upsilon_1^{univ}$.

Let $\mathcal{G}u$ denote the Banach space of universal disposition for finite dimensional spaces constructed by Gurarii [11].

Theorem 27. *The ultrapower $(\mathcal{G}u)_{\mathcal{U}}$ belongs to Υ_1^{univ}, but it is not isomorphic to a complemented subspace of any $C(K)$ space.*

It was proved by Kubis [16] that, under C.H., there is only one Banach space of universal disposition for separable spaces with density character $\aleph_1$. As a consequence, we derive the following result.

Proposition 28. *Under C.H., for each non-trivial ultrafilter $\mathcal{U}$ on $\mathbb{N}$, $\mathcal{K}u = (\mathcal{G}u)_{\mathcal{U}}$.*

3.8. **Automorphic character.** Let $\mathcal{C}$ be a class of Banach spaces.

We say that a Banach space E is *automorphic for $\mathcal{C}$* if, given subspaces M_1 and M_2 of E with $M_1 \simeq M_2 \in \mathcal{C}$ and $\mathrm{dens}(E/M_1) = \mathrm{dens}(E/M_2) \geq \aleph_0$, each bijective isomorphism $j : M_1 \longrightarrow M_2$ extends to an automorphism of E.

The following list contains all known examples of Banach spaces which are automorphic for all their subspaces:

(1) $\ell_2(I)$ (trivial);
(2) c_0 (Lindenstrauss-Rosenthal [18]);
(3) $c_0(I)$ (Moreno-Plichko [20]).

Remark 29. *It would be interesting to know if c_0 and ℓ_2 are the only infinite dimensional separable spaces which are automorphic for all their subspaces.*

Let us see the relations between the automorphic character of a space and its extension properties.

Proposition 30. *Let E be a Banach space automorphic for separable spaces.*

(1) *If E contains a subspace isomorphic to ℓ_1 then $E \in \Upsilon$.*
(2) *If E contains a subspace isomorphic to ℓ_∞ then $E \in \Upsilon^{univ}$.*

3.9. **Automorphic character of Banach spaces in Υ^{univ}.** Recall that an operator $U : X \longrightarrow Y$ is *Fredholm* if the kernel $\ker(U)$ and the cokernel $Y/U(X)$ are finite dimensional (hence U(X) is closed). In this case, we define the *index* of U by

$$\mathrm{ind}(U) := \dim \ker(U) - \dim Y/U(X).$$

The following two Propositions were proved by Lindenstrauss and Rosenthal [18] for $E = \ell_\infty$.

Proposition 31. *Let M be a subspace of $E \in \Upsilon^{univ}$.*

If $j : M \to E$ is an isomorphism and E/M and $E/j(M)$ are reflexive, then there are extensions $U : E \longrightarrow E$ of j.

All the extensions are Fredholm operators with the same index.

Proposition 32. *Each $E \in \Upsilon^{univ}$ is automorphic for separable spaces.*

We say that a subspace M of a Banach space E is *$c_0(I)$-supplemented* if there exists another subspace N of E isomorphic to $c_0(I)$ such that $M \cap N = \{0\}$ and $M + N$ closed.

Proposition 33. *Every $E \in \Upsilon^{univ}$ is automorphic for subspaces of $\ell_\infty(I)$ which are $c_0(I)$-supplemented; i.e., if M_1 and M_2 are $c_0(I)$-supplemented subspaces of E and $M_1 \simeq M_2 \simeq N \subset \ell_\infty(I)$, then each isomorphism from M_1 onto M_2 extends to an automorphism of E.*

4. Open problems

Here we describe some questions which remain unsolved.

(1) $X, Y \in \Upsilon \Rightarrow X\hat{\otimes}_\varepsilon Y \in \Upsilon$? ($\hat{\otimes}_\varepsilon$: injective tensor product)

We have a positive answer to (1) in a special case: $Y \in \Upsilon \Rightarrow c_0(Y) \equiv c_0\hat{\otimes}_\varepsilon Y \in \Upsilon$. The corresponding implication for Υ^{univ} fails because $X \supset c_0$ and $\dim Y = \infty$ imply $X\hat{\otimes}_\varepsilon Y \supset c_0$ complemented.

We do not know the answer in the case $X = Y = \ell_\infty$.

(2) Characterize the compact spaces K for which $C(K) \in \Upsilon^{univ}$.

We conjecture that K σ-Stonian $\Rightarrow C(K) \in \Upsilon^{univ}$, where K is *σ-Stonian* if the closure of each open F_σ-set is open.

(3) Is $\ell_\infty/C[0,1]$ separably injective?

Note that, since ℓ_∞ is automorphic for separable spaces, the quotient $\ell_\infty/C[0,1]$ does not depend on the way we embed $C[0,1]$ into ℓ_∞.

References

[1] S.A. Argyros. *Weak compactness in $L^1(\lambda)$ and injective Banach spaces.* Israel J. Math. 37 (1980), 21–33.

[2] S.A. Argyros. *On the dimension of injective Banach spaces.* Proc. Amer. Math. Soc. 78 (1980), 267–268.

[3] S.A. Argyros. *On nonseparable Banach spaces.* Trans. Amer. Math. Soc. 270 (1982), 193–216.

[4] S.A. Argyros. *On the space of bounded measurable functions.* Quart. J. Math. Oxford Ser. (2) 34 (1983), 129-132.

[5] A. Avilés, F. Cabello, J.M.F. Castillo, M. González and Y. Moreno. *On separably injective Banach spaces.* Preprint 2009, 103 pages.

[6] J. Bourgain. *New classes of $\mathcal{L}^p$-spaces.* Lecture Notes in Math. 889. Springer-Verlag, 1981.

[7] J.M.F. Castillo and M. González. *Three-space problems in Banach space theory.* Lecture Notes in Math. 1667. Springer-Verlag, 1997.

[8] M. González. *On essentially incomparable Banach spaces.* Math. Zeits. 215 (1994), 621–629.

[9] M. González and A. Martínez-Abejón. *Tauberian operators.* Operator Theory: Advances and applications, vol. 194. Birkhäuser Verlag, Basel. 2010.

[10] D.B. Goodner. *Projections in normed linear spaces.* Trans. Amer. Math. Soc. 69 (1950), 89–108.

[11] V.I. Gurariĭ. *Spaces of universal placement, isotropic spaces and a problem of Mazur on rotations of Banach spaces (Russian).* Sibirsk. Mat. Ž. 7 (1966), 1002–1013.

[12] M. Hasumi. *The extension property of complex Banach spaces.* Tôhoku Math. J. 10 (1958), 135–142.

[13] R.G. Haydon. *On dual L_1-spaces and injective bidual Banach spaces.* Israel J. Math. 31 (1978), 142–152.

[14] S. Heinrich. *Ultraproducts in Banach space theory.* J. Reine Angew. Math. 313 (1980), 72–104.

[15] J.L. Kelley. *Banach spaces with the extension property.* Trans. Amer. Math. Soc. 72 (1952), 323–326.

[16] W. Kubis, *Fraïssé sequences - a category-theoretic approach to universal homogeneous structures*, arXiv:0711.1683v1, Nov 2007.

[17] J. Lindenstrauss. *On the extension of operators with range in a $C(K)$ space.* Proc. Amer. Math. Soc. 15 (1964), 218–225.

[18] J. Lindenstrauss and H.P. Rosenthal. *Automorphisms in c_0, ℓ_1 and m.* Israel J. Math. 9 (1969), 227–239.

[19] J. Lindenstraus and L. Tzafriri, *Classical Banach spaces.* Lecture Notes in Math. 338. Springer-Verlag, 1973.

[20] Y. Moreno and A. Plichko. *On automorphic Banach spaces.* Israel J. Math. 169 (2009) 29-45.

[21] L. Nachbin. *A theorem of the Hahn-Banach type for linear transformations.* Trans. Amer. Math. Soc. 68 (1950), 28–46

[22] M. I. Ostrovskii. *Separably injective Banach spaces.* Functional Anal. i. Prilozhen 20 (1986), 80-81. English transl.: Functional Anal. Appl. 20 (1986), 154-155.

[23] H.P. Rosenthal. *On injective Banach spaces and the spaces $L^\infty(\mu)$ for finite measures μ.* Acta Math. 124 (1970), 205-248.

[24] H.P., Rosenthal. *On relatively disjoint families of measures, with some applications to Banach space theory.* Studia Math. 37 (1970), 13–36.

[25] A. Sobczyk. *On the extension of linear transformations.* Trans. Amer. Math. Soc. 55 (1944), 153–169.

[26] M. Zippin. *The separable extension problem.* Israel J. Math. 26 (1977), 372–387.

Departamento de Matemáticas, Facultad de Ciencias, Universidad de Cantabria, E-39071 Santander, España

E-mail address: gonzalem@unican.es