数理解析研究所講究録
第 1678 巻 2010 年 84-91

Mean theoretic approach to a further extension of grand Furuta inequality

前橋工科大学　伊藤 公智 (Masatoshi Ito)
Maebashi Institute of Technology
前橋工科大学　亀井 栄三郎 (Eizaburo Kamei)
Maebashi Institute of Technology

This report is based on "M. Ito and E. Kamei, *Mean theoretic approach to a further extension of grand Furuta inequality*, to appear in J. Math. Inequal.."

Abstract

Very recently, Furuta has shown a further extension of grand Furuta inequality. In this report, we obtain a more precise and clear expression of Furuta's extension by considering a mean theoretic proof of grand Furuta inequality.

1 Introduction

In what follows, A and B are positive operators on a complex Hilbert space, and we denote $A \geq 0$ (resp. $A > 0$) if A is a positive (resp. strictly positive) operator.

Löwner-Heinz theorem *"$A \geq B \geq 0$ ensures $A^{\alpha} \geq B^{\alpha}$ for any $\alpha \in [0,1]$"* is very famous as an order preserving operator inequality. As an extension of Löwner-Heinz theorem, Furuta [8] established the following result called Furuta inequality (see also [2, 3, 9, 12, 18, 20]).

Theorem 1.A (Furuta inequality [8]).

If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) $\qquad (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$

and

(ii) $\qquad (A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

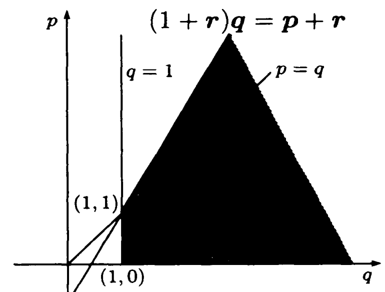

FIGURE 1

Theorem 1.B ([3]). *Let $A \geq B \geq 0$ with $A > 0$. Then*

$$f(p,r) = A^{\frac{-r}{2}}(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}}A^{\frac{-r}{2}} \tag{1.1}$$

is decreasing for $p \geq 1$ and $r \geq 0$.

In [10], Furuta has shown an extension of Furuta inequality, which is called grand Furuta inequality (see also [5, 7, 11, 12, 13, 16, 21, 22, 23]). We remark that grand Furuta inequality is also an extension of Ando-Hiai inequality [1] which is equivalent to the main result of log majorization, and we are also discussing Furuta inequality and Ando-Hiai inequality in [4, 6, 17].

Theorem 1.C (Grand Furuta inequality [10]). *If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$ and $p \geq 1$,*

$$F(r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing for $r \geq t$ and $s \geq 1$, and

$$A^{1-t+r} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$$

holds for $r \geq t$ and $s \geq 1$.

For $A > 0$ and $B \geq 0$, α-power mean $\sharp_\alpha$ for $\alpha \in [0,1]$ is defined by $A \sharp_\alpha B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^\alpha A^{\frac{1}{2}}$. In this report, we use this operator mean as our main tool. We remark that the operator mean theory was established by Kubo-Ando [19].

It is known that α-power mean is very usful for investigating Furuta inequality. As stated in [18], when $A > 0$ and $B \geq 0$, Theorem 1.A can be arranged in terms of α-power mean as follows: If $A \geq B \geq 0$ with $A > 0$, then

$$A \geq B \geq A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \quad \text{for } p \geq 1 \text{ and } r \geq 0.$$

We can also rewrite (1.1) in Theorem 1.B by

$$f(p,r) = A^{-r} \sharp_{\frac{1+r}{p+r}} B^p. \tag{1.1'}$$

Similarly, by putting $\beta = (p-t)s+t$ and $\gamma = r-t$, we can arrange Theorem 1.C in terms of α-power mean as follows [5]: If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$ and $p \geq 1$ with $p \neq t$,

$$\hat{F}(\beta,\gamma) = A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta+\gamma}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \quad \text{is decreasing for } \beta \geq p \text{ and } \gamma \geq 0,$$

and

$$A \geq B \geq A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta+\gamma}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \quad \text{for } \beta \geq p \text{ and } \gamma \geq 0, \tag{1.2}$$

where $A \natural_s B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^sA^{\frac{1}{2}}$ for a real number s. (If $s \in [0,1]$, then $\natural_s = \sharp_s$.)

Very recently, Furuta [14, 15] has dug for a further extension of grand Furuta inequality, which is the following Theorem 1.D. We call this "FGF inequality" here.

Theorem 1.D (FGF inequality [14, 15]). *Let $A \geq B \geq 0$ with $A > 0$, $t \in [0,1]$ and $p_1, p_2, \ldots, p_{2n-1} \geq 1$ for natural number n. Then*

$$\begin{aligned} G(r, p_{2n}) = A^{\frac{-r}{2}} \Big[A^{\frac{r}{2}} \big(A^{\frac{-t}{2}} \big\{ A^{\frac{t}{2}} \cdots (A^{\frac{-t}{2}} \{ A^{\frac{t}{2}} \\ \times (A^{\frac{-t}{2}} B^{p_1} A^{\frac{-t}{2}})^{p_2} A^{\frac{t}{2}} \}^{p_3} A^{\frac{-t}{2}})^{p_4} \cdots A^{\frac{t}{2}} \big\}^{p_{2n-1}} A^{\frac{-t}{2}} \big)^{p_{2n}} A^{\frac{r}{2}} \Big]^{\frac{1-t+r}{q[2n]-t+r}} A^{\frac{-r}{2}} \end{aligned} \tag{1.3}$$

is decreasing for $r \geq t$ and $p_{2n} \geq 1$, and

$$\begin{aligned} A^{1-t+r} \geq \Big[A^{\frac{r}{2}} \big(A^{\frac{-t}{2}} \big\{ A^{\frac{t}{2}} \cdots (A^{\frac{-t}{2}} \{ A^{\frac{t}{2}} \\ \times (A^{\frac{-t}{2}} B^{p_1} A^{\frac{-t}{2}})^{p_2} A^{\frac{t}{2}} \}^{p_3} A^{\frac{-t}{2}})^{p_4} \cdots A^{\frac{t}{2}} \big\}^{p_{2n-1}} A^{\frac{-t}{2}} \big)^{p_{2n}} A^{\frac{r}{2}} \Big]^{\frac{1-t+r}{q[2n]-t+r}} \end{aligned} \tag{1.4}$$

holds for $r \geq t$ and $p_{2n} \geq 1$, where

$$q[2n] = \big(\{ \cdots (\{(p_1 - t)p_2 + t\}p_3 - t)p_4 + \cdots + t\}p_{2n-1} - t \big) p_{2n} + t.$$

In this report, we obtain a more precise and clear expression of FGF inequality by considering a mean theoretic proof of grand Furuta inequality. Moreover, we get a variant of FGF inequality by scrutinizing the former argument.

2 FGF inequality

Firstly, we show that a sequence $\{B_i\}$ such that $B_i = (A^t \natural_{\frac{\beta_i - t}{\alpha_i - t}} B_{i-1}^{\alpha_i})^{\frac{1}{\beta_i}}$ is decreasing. Theorem 2.1 is a key result in the proof of FGF inequality.

Theorem 2.1. *Let $A \geq B \geq 0$ with $A > 0$ and n be a natural number. Then for $t \in [0,1]$, $\beta_i \geq \alpha_i \geq 1$ and $\alpha_i \neq t$ for $i = 1, 2, \ldots, n$,*

$$A \geq B \geq B_1 \geq \cdots \geq B_{n-1} \geq B_n,$$

where $B_0 = B$ and $B_i = (A^t \natural_{\frac{\beta_i - t}{\alpha_i - t}} B_{i-1}^{\alpha_i})^{\frac{1}{\beta_i}}$.

Lemma 2.A ([5]). *Let $A \geq B \geq 0$ with $A > 0$. Then*

$$A \geq B \geq (A^t \natural_{\frac{\beta - t}{p - t}} B^p)^{\frac{1}{\beta}}$$

holds for $t \in [0,1]$, $\beta \geq p \geq 1$ and $p \neq t$.

We remark that Lemma 2.A plays an important role in the proof of grand Furuta inequality (1.2).

Proof of Theorem 2.1. By applying Lemma 2.A to that $A \geq B \geq 0$ with $A > 0$, we have

$$A \geq B \geq (A^t \natural_{\frac{\beta_1 - t}{\alpha_1 - t}} B^{\alpha_1})^{\frac{1}{\beta_1}} = B_1$$

for $t \in [0,1]$, $\beta_1 \geq \alpha_1 \geq 1$ and $\alpha_1 \neq t$, and also by applying Lemma 2.A repeatedly to that $A \geq B_{i-1} \geq 0$ with $A > 0$ for $i = 1, 2, \ldots, n$, we have

$$B_{i-1} \geq (A^{t_i} \natural_{\frac{\beta_i - t}{\alpha_i - t}} B_{i-1}^{\alpha_i})^{\frac{1}{\beta_i}} = B_i$$

for $t \in [0,1]$, $\beta_i \geq \alpha_i \geq 1$ and $\alpha_i \neq t$, so that

$$A \geq B \geq B_1 \geq \cdots \geq B_{n-1} \geq B_n.$$

Hence the proof is complete. □

Furuta [15] has given an extension of Lemma 2.A as an application of Theorem 1.D.

Theorem 2.B ([15]). *Let $A \geq B \geq 0$ with $A > 0$, $t \in [0,1]$ and $p_1, p_2, \ldots, p_{2n-1}, p_{2n} \geq 1$ for natural number n. Then*

$$A \geq B \geq \{A^{\frac{t}{2}}(A^{\frac{-t}{2}} B^{p_1} A^{\frac{-t}{2}})^{p_2} A^{\frac{t}{2}}\}^{\frac{1}{q[2]}} \geq \cdots \geq$$
$$[A^{\frac{t}{2}}(A^{\frac{-t}{2}}\{A^{\frac{t}{2}} \cdots (A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}} B^{p_1} A^{\frac{-t}{2}})^{p_2} A^{\frac{t}{2}}\}^{p_3} A^{\frac{-t}{2}})^{p_4} \cdots A^{\frac{t}{2}}\}^{p_{2n-1}} A^{\frac{-t}{2}})^{p_{2n}} A^{\frac{t}{2}}]^{\frac{1}{q[2n]}},$$

where

$$q[2n] = (\{\cdots(\{(p_1 - t)p_2 + t\}p_3 - t)p_4 + \cdots + t\}p_{2n-1} - t)p_{2n} + t.$$

We can rewrite Theorem 2.B by putting

$$\beta_0 = 1,\ \alpha_i = \beta_{i-1}p_{2i-1},\ \beta_i = (\alpha_i - t)p_{2i} + t \text{ and } \gamma = r - t \tag{2.1}$$

as follows:

Theorem 2.B'. *Let $A \geq B \geq 0$ with $A > 0$ and n be a natural number. Then for $t \in [0,1]$, $\beta_n \geq \alpha_n \geq \beta_{n-1} \geq \alpha_{n-1} \geq \cdots \geq \beta_1 \geq \alpha_1 \geq 1$ and $\alpha_i \neq t$ for $i = 1, 2, \ldots, n$,*

$$A \geq B \geq B_1 \geq \cdots \geq B_{n-1} \geq B_n,$$

where $B_0 = B$ and $B_i = (A^t \natural_{\frac{\beta_i - t}{\alpha_i - t}} B_{i-1}^{\alpha_i})^{\frac{1}{\beta_i}}$.

Therefore we recognize that Theorem 2.1 is a fine extension of Theorem 2.B. More precisely, $\beta_i \geq \alpha_i \geq 1$ in Theorem 2.1 is looser than $\beta_n \geq \alpha_n \geq \beta_{n-1} \geq \alpha_{n-1} \geq \cdots \geq \beta_1 \geq \alpha_1 \geq 1$ in Theorem 2.B.

By using Theorem 2.1, we obtain an improvement of (1.4) in Theorem 1.D and Theorem 2.B. Theorem 2.2 is a satellite form of Theorem 1.D in our sense. Theorem 2.2 leads (1.4) in Theorem 1.D by the same replacement to (2.1).

Theorem 2.2. *Let $A \geq B \geq 0$ with $A > 0$ and n be a natural number. Then for $t \in [0,1]$, $\beta_n \geq \alpha_n \geq \beta_{n-1} \geq \alpha_{n-1} \geq \cdots \geq \beta_1 \geq \alpha_1 \geq 1$, $\gamma \geq 0$ and $\alpha_1 \neq t$,*

$$\begin{aligned} A \geq B &\geq A^{-\gamma} \sharp_{\frac{1+\gamma}{\alpha_1+\gamma}} B^{\alpha_1} \geq A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_1+\gamma}} B_1^{\beta_1} \geq A^{-\gamma} \sharp_{\frac{1+\gamma}{\alpha_2+\gamma}} B_1^{\alpha_2} \geq A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_2+\gamma}} B_2^{\beta_2} \\ &\geq \cdots \geq A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_{n-1}+\gamma}} B_{n-1}^{\beta_{n-1}} \geq A^{-\gamma} \sharp_{\frac{1+\gamma}{\alpha_n+\gamma}} B_{n-1}^{\alpha_n} \geq A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_n+\gamma}} B_n^{\beta_n}, \end{aligned}$$

where $B_0 = B$ and $B_i = (A^t \natural_{\frac{\beta_i - t}{\alpha_i - t}} B_{i-1}^{\alpha_i})^{\frac{1}{\beta_i}}$.

Proof. Let $\beta_0 = 1$. By Theorem 2.1, $A \geq B_{i-1}$ holds for $i = 1, 2, \ldots, n$, so that we have

$$\begin{aligned} & A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_{i-1}+\gamma}} B_{i-1}^{\beta_{i-1}} \\ \geq\ & A^{-\gamma} \sharp_{\frac{1+\gamma}{\alpha_i+\gamma}} B_{i-1}^{\alpha_i} && \text{by Theorem 1.B} \\ \geq\ & A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_i+\gamma}} (A^t \natural_{\frac{\beta_i-t}{\alpha_i-t}} B_{i-1}^{\alpha_i}) = A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_i+\gamma}} B_i^{\beta_i} && \text{by Theorem 1.C} \end{aligned}$$

since $\beta_i \geq \alpha_i \geq \beta_{i-1} \geq 1$. Hence the proof is complete. □

3 Variant of FGF inequality

In this section, we obtain a variant of FGF inequality by scrutinizing the argument in Section 2, and also we have a result on a FGF-type operator function. We omit their proofs here.

Theorem 3.1. *Let $A \geq B \geq 0$ with $A > 0$ and n be a natural number. Then for $t \in [0,1]$, $\alpha_i \geq 1$, $1 \leq \frac{\beta_i - t}{\alpha_i - t} \leq 2$ and $\alpha_i \neq t$ for $i = 1, 2, \ldots, n$,*

$$B_{i-1}^{\beta_i} \geq B_i^{\beta_i},$$

where $B_0 = B$ and $B_i = (A^t \natural_{\frac{\beta_i - t}{\alpha_i - t}} B_{i-1}^{\alpha_i})^{\frac{1}{\beta_i}}$.

Theorem 3.2. *Let $A \geq B \geq 0$ with $A > 0$ and n be a natural number. Then for $t \in [0,1]$, $\alpha_i \geq 1$, $\beta_n \geq \cdots \geq \beta_2 \geq \beta_1 \geq 1$, $1 \leq \frac{\beta_i - t}{\alpha_i - t} \leq 2$, $\gamma \geq 0$ and $\alpha_i \neq t$ for $i = 1, 2, \ldots, n$,*

$$\begin{aligned} A \geq B \geq A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_1+\gamma}} B^{\beta_1} &\geq A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_1+\gamma}} B_1^{\beta_1} \geq A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_2+\gamma}} B_1^{\beta_2} \geq A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_2+\gamma}} B_2^{\beta_2} \\ &\geq \cdots \geq A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_{n-1}+\gamma}} B_{n-1}^{\beta_{n-1}} \geq A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_n+\gamma}} B_{n-1}^{\beta_n} \geq A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_n+\gamma}} B_n^{\beta_n}, \end{aligned}$$

where $B_0 = B$ and $B_i = (A^t \natural_{\frac{\beta_i - t}{\alpha_i - t}} B_{i-1}^{\alpha_i})^{\frac{1}{\beta_i}}$.

Theorem 3.3. *Let $A \geq B \geq 0$ with $A > 0$ and n be a natural number. Then for $t \in [0,1]$, $\beta_i \geq \alpha_i \geq 1$ for $i = 1, 2, \ldots, n-1$, $\alpha_n \geq 1$, $\gamma \geq 0$ and $\alpha_i \neq t$ for $i = 1, 2, \ldots, n$,*

$$\hat{G}(\beta_n) = A^{-\gamma} \sharp_{\frac{1+\gamma}{\beta_n+\gamma}} (A^t \natural_{\frac{\beta_n - t}{\alpha_n - t}} B_{n-1}^{\alpha_n}) \tag{3.1}$$

is decreasing for $\beta_n \geq \alpha_n$, where $B_0 = B$ and $B_i = (A^t \natural_{\frac{\beta_i - t}{\alpha_i - t}} B_{i-1}^{\alpha_i})^{\frac{1}{\beta_i}}$.

Remark. (3.1) is also decreasing for $\gamma \geq 0$ by Theorem 1.B since $A \geq B \geq 0$ with $A > 0$ ensures $A \geq B_n = (A^t \natural_{\frac{\beta_n - t}{\alpha_n - t}} B_{n-1}^{\alpha_n})^{\frac{1}{\beta_n}}$ by Theorem 2.1. Therefore, similarly to Theorem 2.1, we recognize that Theorem 3.3 is a slight extension of (1.3) in Theorem 1.D.

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(Masatoshi Ito) Maebashi Institute of Technology, 460-1 Kamisadorimachi, Maebashi, Gunma 371-0816, JAPAN
E-mail address: `m-ito@maebashi-it.ac.jp`

(Eizaburo Kamei) Maebashi Institute of Technology, 460-1 Kamisadorimachi, Maebashi, Gunma 371-0816, JAPAN
E-mail address: `kamei@maebashi-it.ac.jp`