数理解析研究所講究録
第 1678 巻 2010 年 92-96

Operator equations via an order preserving operator inequality

Takayuki Furuta

Department of Information Science

Tokyo University of Science

§1 Introduction

An operator T is said to be *positive semidefinite* (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$. Löwner-Heinz inequality (denoted by (LH) briefly) states *if* $A \geq B \geq 0$ *holds, then* $A^{\alpha} \geq B^{\alpha}$ *for any* $\alpha \in [0,1]$. Unfortunately $A^p \geq B^p$ *does not always hold for* $p > 1$. The following result has been obtained from this point of view.

Theorem A (1987).
If $A \geq B \geq 0$, *then for each* $r \geq 0$,

(i) $$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ *and* $q \geq 1$ *with* $(1+r)q \geq p+r$.

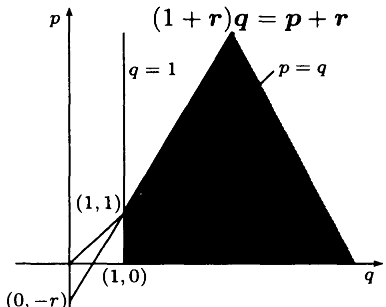

FIGURE 1

The original proof of Theorem A is shown in [4], an elementary one-page proof is in [5] and alternative ones are in [3],[8] and [6]. It is shown in [11] that the conditions p, q and r in FIGURE 1 are best possible. On the other hand we have the following result.

Theorem B [2]. *Let A be a positive definite matrix and B a positive semidefinite matrix. The solution X of the following matrix equation is always positive semidefinite*:

$$A^2X + XA^2 = AB + BA. \tag{1.1}$$

In [2] the following question was posed associated with Theorem B: *How can one characterize all the functions f such that the solution of the matrix equation*

$$f(A)X + Xf(A) = AB + BA \tag{1.2}$$

is positive semidefinite?

We shall discuss the solutions of the following operator equation related to (1.1) and (1.2):

$$\sum_{j=1}^{n} A^{n-j} X A^{j-1} = B$$

where B is of special type.

The proofs and related results in this paper are found in [7].

§2 Operator equations $\sum_{j=1}^{n} A^{n-j} X A^{j-1} = B$ via Theorem A

As an application of Theorem A we shall obtain the following operator equation.

Theorem 2.1 [7]. *Let A be positive definite operator and B be positive semidefinite operator. Let m and n be natural numbers. There exists positive semidefinite operator solution X of the following operator equation:*

$$\sum_{j=1}^{n} A^{n-j} X A^{j-1} = A^{\frac{nr}{2(m+r)}} \Big(\sum_{j=1}^{m} A^{\frac{n(m-j)}{m+r}} B A^{\frac{n(j-1)}{m+r}} \Big) A^{\frac{nr}{2(m+r)}} \tag{2.1}$$

$$\text{for } r \text{ such that } \begin{cases} r \geq 0 & \text{if } n \geq m \quad \text{(i)} \\ r \geq \dfrac{m-n}{n-1} & \text{if } m \geq n \geq 2 \quad \text{(ii).} \end{cases}$$

Theorem 2.1 easily implies the following result.

Corollary 2.2 [7]. *Let A be positive definite operator and B be positive semidefinite operator. There exists positive semidefinite operator solution X of the following operator equation* (i),(ii), (iii), (iv) *and* (v) *respectively:*

(i) $A^{\frac{2+r}{2}} X + X A^{\frac{2+r}{2}} = A^{\frac{r}{2}} (AB + BA) A^{\frac{r}{2}}$ *for* $r \geq 0$.

(ii) $A^{\frac{(2+r)2}{3}} X + A^{\frac{2+r}{3}} X A^{\frac{2+r}{3}} + X A^{\frac{(2+r)2}{3}} = A^{\frac{r}{2}} (AB + BA) A^{\frac{r}{2}}$ *for* $r \geq 0$.

(iii) $A^{\frac{(3+r)2}{3}} X + A^{\frac{3+r}{3}} X A^{\frac{3+r}{3}} + X A^{\frac{(3+r)2}{3}} = A^{\frac{r}{2}} (A^2B + ABA + BA^2) A^{\frac{r}{2}}$ *for* $r \geq 0$.

(iv) $A^{\frac{3+r}{2}} X + X A^{\frac{3+r}{2}} = A^{\frac{r}{2}} (A^2B + ABA + BA^2) A^{\frac{r}{2}}$ *for* $r \geq 1$.

(v) $A^{\frac{5+r}{2}} X + X A^{\frac{5+r}{2}} = A^{\frac{r}{2}} (A^4B + A^3BA + A^2BA^2 + ABA^3 + BA^4) A^{\frac{r}{2}}$ *for* $r \geq 3$.

§3 Concrete examples of positive semidefinite matrices

Proposition 3.1 [7]. *Let the diagonal matrix* $A = \mathrm{diag}(a_1, a_2, \cdots, a_l)$ *with each* $a_j > 0$ *and* B *be the* $l \times l$ *matrix all of whose entries are* 1. *Let* m *and* n *be natural numbers. There exists positive semidefinite matrix solution* X *of the following matrix equation:*

$$\sum_{j=1}^{n} A^{\frac{(m+r)(n-j)}{n}} X A^{\frac{(m+r)(j-1)}{n}} = A^{\frac{r}{2}}\Big(\sum_{j=1}^{m} A^{m-j} B A^{j-1}\Big) A^{\frac{r}{2}} \tag{2.1}$$

$$\text{for } r \text{ such that } \begin{cases} r \geq 0 & \text{if } n \geq m \qquad \text{(i)} \\ r \geq \dfrac{m-n}{n-1} & \text{if } m \geq n \geq 2 \qquad \text{(ii).} \end{cases}$$

The poisitive semidefinite matrix solution X *of* (2.1) *can be expressed as:*

$$X = \left(\frac{a_i^{\frac{r}{2}} a_j^{\frac{r}{2}} \left(\sum_{k=1}^{m} a_i^{m-k} a_j^{k-1}\right)}{\sum_{k=1}^{n} a_i^{\frac{(m+r)(n-k)}{n}} a_j^{\frac{(m+r)(k-1)}{n}}} \right)_{i,j=1,2,\ldots,l}. \tag{3.1}$$

Let the diagonal matrix $A = (a_1, a_2, \cdots, a_n)$ *with each* $a_j > 0$ *and* B *be* $n \times n$ *matrix all of whose entries are* 1. *Then the positive semidefinite solutions* X_i *of* (i),(ii),(iii),(iv) *and* (v) *of Corollary 2.2 are given by:*

$$X_1 = \left(\frac{a_i^{\frac{r}{2}} a_j^{\frac{r}{2}} (a_i + a_j)}{a_i^{\frac{2+r}{2}} + a_j^{\frac{2+r}{2}}} \right)_{i,j=1,2,\ldots,n} \qquad \text{for } r \geq 0.$$

$$X_2 = \left(\frac{a_i^{\frac{r}{2}} a_j^{\frac{r}{2}} (a_i + a_j)}{a_i^{\frac{2(2+r)}{3}} + a_i^{\frac{2+r}{3}} a_j^{\frac{2+r}{3}} + a_j^{\frac{2(2+r)}{3}}} \right)_{i,j=1,2,\ldots,n} \qquad \text{for } r \geq 0.$$

$$X_3 = \left(\frac{a_i^{\frac{r}{2}} a_j^{\frac{r}{2}} (a_i^2 + a_i a_j + a_j^2)}{a_i^{\frac{2(3+r)}{3}} + a_i^{\frac{3+r}{3}} a_j^{\frac{3+r}{3}} + a_j^{\frac{2(3+r)}{3}}} \right)_{i,j=1,2,\ldots,n} \qquad \text{for } r \geq 0.$$

$$X_4 = \left(\frac{a_i^{\frac{r}{2}} a_j^{\frac{r}{2}} (a_i^2 + a_i a_j + a_j^2)}{a_i^{\frac{3+r}{2}} + a_j^{\frac{3+r}{2}}} \right)_{i,j=1,2,\ldots,n} \qquad \text{for } r \geq 1.$$

$$X_5 = \left(\frac{a_i^{\frac{r}{2}} a_j^{\frac{r}{2}} (a_i^4 + a_i^3 a_j + a_i^2 a_j^2 + a_i a_j^3 + a_j^4)}{a_i^{\frac{5+r}{2}} + a_j^{\frac{5+r}{2}}} \right)_{i,j=1,2,\ldots,n} \qquad \text{for } r \geq 3.$$

We would like to state that we can obtain many concrete examples of positive semidefinite matrices as stated in §3 by applying Theorem 2.1.

We remark that many types of useful operator equations related to Lyapunov equation are discussed in [9] and [10].

Also we can find the following example quite similar to our Example X_2 in §3:

Let $a_1, a_2, \ldots., a_n$ be positive numbers, $-1 \leq r \leq 1$, and $-2 < t \leq 2$. Then $n \times n$ matrix

$$W = \left(\frac{a_i^r + a_j^r}{a_i^2 + t a_i a_j + a_j^2} \right)_{i,j=1,2,\cdots,n}$$

is positive semidefinite. [12, **Lemma 4.23**].

Other ueseful examples of positive semidefinite matrices are found in [13, page 197, Problem 21]. The following more general type operator equation is discussed in [1]:

$$\sum_{j=1}^{n} A^{n-j} X B^{j-1} = Y.$$

References

[1] R. Bhatia and M. Uchiyama, The operator equation $\sum_{i=0}^{n} A^{n-i} X B^{i} = Y$, Expo. Math., **27**(2009), 251-255.

[2] N.N. Chan and M.K. Kwong, Hermitian matrix inequalities and a conjecture, Amer. Math. Monthly, **92**(1985), 533-541.

[3] M. Fujii, Furuta's inequality and its mean theoretic approach, J. Operator Theory, **23**(1990), 67-72.

[4] T. Furuta, $A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0, p \geq 0, q \geq 1$ with $(1+2r)q \geq p+2r$, Proc. Amer. Math. Soc.,**101** (1987),85-88.

[5] T. Furuta, Elementary proof of an order preserving inequality, Proc. Japan Acad.,**65**(1989),126.

[6] T. Furuta, Invitation to Linear Operators, Taylor & Francis, London **2001**.

[7] T. Furuta, The positive semidefinite solution of the operator equation $\sum_{j=1}^{n} A^{n-j} X A^{j-1} = B$, to appear in Linear Alg and Its Appl.

[8] E. Kamei, A sattelite to Furuta's inequality, Math. Japon., **33**(1988), 883-886.

[9] M.K. Kwong, On the definiteness of the solutions of certain matrix equations, Linear Alg and Its Appl., **108**(1988), 177-197.

[10] M.K. Kwong, Some results on matrix monotone functions, Linear Alg and Its Appl., **118**(1989),129-153.

[11] K. Tanahashi, Best possibility of the Furuta inequality, Proc. Amer. Math. Soc., **124**(1996),141-146.

[12] X. Zhan, Matrix Inequalities, Springer-Verlag, Berlin, 2002.

[13] F. Zhang, Matrix Theory, Springer-Verlag, New York, 1999.