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Positive Definite Kernels and Majorization

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1 Introduction

Definition 1.1 Let $f(t)$ be a real continuous function defined on I, and consider the functional calculus $f(X)$ for a Hermitian matrix X with eigenvalues in I.

- f is called an operator monotone function on I if $f(A) \leqq f(B)$ whenever $A \leqq B$ (of any order n).
- f is said to be operator decreasing if $-f$ is operator monotone.
- f is called an operator convex function on I if $f(sA+(1-s)B) \leqq sf(A)+(1-s)f(B) \quad (0<s<1)$
  for every pair of bounded Hermitian operators A and B whose spectra are both in I.
- An operator concave function is likewise defined.

Definition 1.2 Let $K(t, s)$ be a real, continuous and **symmetric** function defined on $I \times I$.

- $K(t,s)$ is called a positive semi-definite kernel on I if

$$\iint_{I\times I} K(t,s)\phi(t)\phi(s)dt\,ds \geqq 0 \tag{1}$$

for all real continuous functions ϕ with compact support in I.

Remark It is evident that $K(t,s)$ is positive semi-definite on I if and only if for each n and for all n points $t_i \in I$ the $n \times n$ matrices

$$\left(K(t_i,t_j)\right)_{i,j=1}^{n}$$

are positive semi-definite.

- Suppose $K(t,s) \geqq 0$ for every t,s in I. Then the kernel $K(t,s)$ is said to be infinitely divisible on I if $K(t,s)^r$ is a positive semi-definite kernel for every $r>0$, i.e.,

  $\iint_{I\times I} K(t,s)^r\phi(t)\phi(s)dt\,ds \geqq 0$

- A kernel $K(t,s)$ is said to be conditionally positive semi-definite on I if $\iint_{I\times I} K(t,s)\phi(t)\phi(s)dt\,ds \geqq 0$ for ϕ such that the support of ϕ is compact and $\int_I \phi(t)dt = 0$.

- A kernel $K(t,s)$ is said to be conditionally negative semi-definite on I if $-K(t,s)$ is conditionally positive semi-definite on I.

(Löwner) C^1 function f is operator monotone on I if and only if the Löwner kernel $K_f(t,s)$ defined by

$$K_f(t,s) = \frac{f(t)-f(s)}{t-s} \quad (t \neq s), \quad K_f(t,t) = f'(t),$$

is positive semi-definite on I. (**F. Krauss, J. Bendat- S. Shermann**) $g(t)$ is an operator convex function on I if and only if $g(t)$ is of class $C^2(I)$ and for each $t_0 \in I$, the function $f(t)$ defined by

$$f(t) = \frac{g(t) - g(t_0)}{t - t_0} \quad (t \neq t_0), \quad f(t_0) = g'(t_0)$$

is operator monotone on I .

2 Operator convex functions

Proposition 2.1 Let $f(t)$ be an operator monotone (or decreasing) function on I. Then the indefinite integral $\displaystyle\int f(t)dt$ is an operator convex (or concave) function on I.

Example 2.1 $\int \log t dt = t \log t - t$, *hence* $t \log t$ *and* $\log \Gamma(t) = \int \frac{\Gamma'(t)}{\Gamma(t)} dt$ *are both operator convex on* $(0, \infty)$

But the converse is not true; $\frac{1}{t}$ on $(0, \infty)$ is a counter example.

Proposition 2.2 *Let* $g(t)$ *be an operator convex function on* $(0, \infty)$. *Then* $g'(\sqrt{t})$ *is operator monotone there.*

(**Well-known**) Let $f(t) \geqq 0$ be defined on $[0, \infty)$. Then f is operator monotone $\Leftrightarrow$ $f(t)$ is operator concave.

Theorem 2.3 Let $f(t)$ be defined on (a, ∞) with $a \geqq -\infty$. Then

(i) $f(t)$ is operator decreasing $\Leftrightarrow$ $f(t)$ is operator convex and $f(\infty) = \lim\limits_{t \to \infty} f(t) < \infty$;

(ii) $f(t)$ is operator monotone $\Leftrightarrow$ $f(t)$ is operator concave and $f(\infty) > -\infty$.

In (ii) the condition "$f(\infty) > -\infty$" is indispensable; for instance, $f(t) = -t^2$ is operator concave on $(0, \infty)$ but not operator monotone there.

Corollary 2.4 Let $f(t)$ be defined on $(-\infty, b)$, where $b \leqq \infty$. Then

(i) $f(t)$ is operator monotone on $(-\infty, b)$ $\Leftrightarrow$ $f(t)$ is operator convex on $(-\infty, b)$ and $f(-\infty) < \infty$

(ii) $f(t)$ is operator decreasing on $(-\infty, b)$ $\Leftrightarrow$ $f(t)$ is operator concave on $(-\infty, b)$ and $f(-\infty) > -\infty$.

Corollary 2.5 **(Well-known)** Let $f(t)$ be defined on $(-\infty, \infty)$. Then $f(t)$ is operator monotone on $(-\infty, \infty)$
$\Leftrightarrow$ $f(t) = at + b$ $(a \geqq 0)$.

How about the case of finite intervals? $\tan t$ is operator monotone on $(-\pi/2, \pi/2)$.

Proposition 2.6 Let $f(t)$ be an operator monotone function on a finite interval (a, b). Then there is a decomposition of $f(t)$ such that

$$f(t) = f_+(t) + f_-(t) \quad (a < t < b)$$

where $f_+(t)$ and $f_-(t)$ are operator monotone on (a, ∞) and $(-\infty, b)$ respectively.

3 Löwner kernels

(**Bhatia and Sano**) Let $f(t)$ be a C^2 function on $[0,\infty)$ such that $f(t) \geqq 0$ and $f(0) = f'(0) = 0$. Then f is operator convex on $[0,\infty)$ $\Leftrightarrow$ the Löwner kernel $K_f(t,s)$ is conditionally negative semi-definite on $[0,\infty)$, where

$$K_f(t,s) = \frac{f(t) - f(s)}{t - s} \quad (t \neq s), \quad K_f(t,t) = f'(t),$$

Proposition 3.1 Let $f(t)$ be a C^1 function on (a,∞). Then

(i) $f(t)$ is operator convex on (a,∞) $\Leftrightarrow$

the Löwner kernel $K_f(t,s)$ is conditionally negative semi-definite and $\lim_{t\to\infty} \frac{f(t)}{t} > -\infty$;

(ii) $f(t)$ is operator concave on (a,∞) $\Leftrightarrow$ the Löwner kernel $K_f(t,s)$ is conditionally positive semi-definite and $\lim_{t\to\infty} \frac{f(t)}{t} < \infty$.

In (i) the condition "$\lim_{t\to\infty} \frac{f(t)}{t} > -\infty$" is indispensible: in fact, the Löwner kernel $K_f(t,s) = -(t^2 + st + s^2)$ of $f(t) = -t^3$ is conditionally negative on $(0,\infty)$, but $f(t)$ is not operator convex there.

Theorem 3.2 Let $f(t)$ be C^1 function on (a,∞). Then the following hold:

(i) the Löwner kernel $K_f(t,s)$ is positive semi-definite on (a,∞) if and only if $K_f(t,s)$ is conditionally positive semi-definite on (a,∞), $\lim_{t\to\infty} \frac{f(t)}{t} < \infty$, and $f(\infty) > -\infty$;

(ii) $K_f(t,s)$ is negative semi-definite on (a,∞) if and only if $K_f(t,s)$ is conditionally negative semi-definite on (a,∞), $\lim_{t\to\infty}\frac{f(t)}{t} > -\infty$, and $f(\infty) < \infty$.

Corollary 3.3 Let $f(t)$ be a C^1 function on $(-\infty,b)$. Then

(i) $f(t)$ is operator convex on $(-\infty,b)$ if and only if the Löwner kernel $K_f(t,s)$ is conditionally positive semi-definite; $\lim_{t\to-\infty}\frac{f(t)}{t} < \infty$.

(ii) $f(t)$ is operator concave on $(-\infty,b)$ if and only if the Löwner kernel $K_f(t,s)$ is conditionally negative semi-definite, and $\lim_{t\to-\infty}\frac{f(t)}{t} > -\infty$.

Corollary 3.4 Let $f(t)$ be C^1 function on $(-\infty,b)$. Then the following hold:

(i) the Löwner kernel $K_f(t,s)$ is positive semi-definite on $(-\infty,b)$ if and only if $K_f(t,s)$ is conditionally positive semi-definite on $(-\infty,b)$, $\lim_{t\to-\infty}\frac{f(t)}{t} < \infty$, and $f(-\infty) < \infty$;

(ii) the Löwner kernel $K_f(t,s)$ is negative semi-definite on $(-\infty,b)$ if and only if $K_f(t,s)$ is conditionally negative semi-definite on $(-\infty,b)$,
$\lim_{t\to-\infty}\frac{f(t)}{t} > -\infty$, and $f(-\infty) > -\infty$.

4 Majorization and kernel functions

Definition 4.1 Let $h(t)$ and $g(t)$ be C^1 functions on I, and suppose that $g(t)$ is increasing. Then h is said to be majorized by g and denoted by

$h \preceq g$ on I if

$h(A) \leqq h(B)$ whenever $g(A) \leqq g(B)$ for A, B whose spectra are both in I.

- $f(t) \preceq t$ on $I \Longleftrightarrow f(t)$ is operator monotone on I.

Definition 4.2 Let $h(t)$ and $g(t)$ be C^1 functions on I, and suppose that $g(t)$ is increasing. Then the kernel $K_{h,g}(t,s)$ defined by

$$K_{h,g}(t,s) = \frac{h(t)-h(s)}{g(t)-g(s)} \quad (s \neq t), \quad K_{h,g}(t,t) = \frac{h'(t)}{g'(t)}.$$

is continuous and symmetric.

- A Löwner kernel $K_f(t,s)$ can be written as $K_{f,t}(t,s)$.

Proposition 4.1 The following statements are equivalent:

(i) The kernel $K_{h,\,g}(t,s)$ is positive semi-definite on I.

(ii) There is an operator monotone function φ defined on $g(I)$ such that

$$h(t) = (\varphi \circ g)(t) \quad (t \in I).$$

(iii) $h \preceq g$ on I.

Lemma 4.2 Let $h(t)$ and $g(t)$ be positive C^1 functions on an open interval I. Suppose $h(t)g(t)$ is increasing and its range is $(0,\infty)$. Then the kernel $K_{h,hg}$ is positive semi-definite on I

if and only if so is the kernel $K_{g,hg}$.

Theorem 4.3 Let $h(t)$ and $g(t)$ be positive C^1 functions defined on I. Suppose g is increasing and its range is $(0, \infty)$. If the kernel $K_{h,g}$ is positive semi-definite on I, then
for $0 \leq i \leqq n,\ 0 \leqq j \leqq m,\ 1 \leqq m,\ i+j+1 \leqq n+m$

$$K_{h^i g^j,\, h^n g^m}(t, s) = \frac{h^i(t)g^j(t) - h^i(s)g^j(s)}{h^n(t)g^m(t) - h^n(s)g^m(s)}$$

is infinitely divisible.

Moreover, if f is a (not necessarily positive) C^1 function such that the kernel $K_{f,g}(t, s)$ is positive semi-definite, then the kernel

$$K_{g, e^f g}(t, s)$$

is infinitely divisible.

Example 4.1 (1). For $f(t) \preceq t$ on $(0, \infty)$

$$\frac{f(t)^i t^j - f(s)^i s^j}{f(t)^n t^m - f(s)^n s^m},$$

where $0 \leq i \leqq n,\ 0 \leqq j \leqq m,\ 1 \leqq m,\ i+j+1 \leqq n+m,$
$1 \leqq n+1 \leqq m,$

$$\frac{1}{t+s}(\text{Cauchy kernel}), \quad \frac{t-s}{te^{-1/t} - se^{-1/s}}$$

are all infinitely divisible kernels on $(0, \infty)$.

(2). Consider a polynomial
$p(t) := \Pi_{i=1}^n (t - a_i)$ with $a_1 \geqq a_2 \geqq \cdots \geqq a_n$. Then the kernel

$$K_{t, p(t)}(t, s) = \frac{t-s}{p(t) - p(s)}$$

is infinitely divisible on (a_1, ∞)

Theorem 4.4 Let $h(t)$ and $g(t)$ be positive C^1 functions defined on an open interval (a, b), where $-\infty \leqq a < b \leqq \infty$. Suppose the range of g is $(0, \infty)$. Then the following are equivalent:

(i) the kernel $K_{h,g}$ is conditionally negative;

(ii) there is an operator convex function φ defined on $(0, \infty)$ such that $\varphi(g(t)) = h(t)$ for $t \in (a, b)$.

(iii) $\dfrac{h(t) - h(a+0)}{g(t)} \preceq g(t) \quad (a < t < b)$