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On Coarse fibre structure

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1 Introduction

Gromov's programs for classifying finitely generated groups by quasi-isometries are major studying objects in geometric group theory : (i) Classify finitely generated groups by quasi-isometry. (ii) Classify finitely generated groups by group isomorphisms which are quasi-isometric to each finitely generated group. Because quasi-isometry invariant geometry is the *asymptotic* or *coarse* geometry of non-compact metric spaces, intuitively this classification surely ignores the "finite" difference of groups. For now many good partial answers for Gromov's programs exist, but they are far from complete. For example answers for (i) and (ii) about finite groups, finitely generated abelian groups and finitely generated free groups are known [1] [4]. For another example answers for (ii) about finitely generated nilpotent groups and irreducible lattice of semi-simple Lie groups are known [6] [3]. But above class of groups are small parts of all finitely generated groups, and unfortunately we have few good general theory for studying the Gromov's programs on general groups. So we must study the general theory for general groups. In this article we suggest the method for understanding the coarse geometry of finitely generated groups given by their normal subgroups. At first we define the *prototype fibre structures* which are obtained by picking out the quasi-isometry invariant properties of finitely generated groups and normal subgroups. These structures are too weak to characterize normal subgroups. So second we define the *semi-flat fibre structures* which are obtained by picking out the quasi-isometry invariant properties of semi-direct products. In case of general normal subgroups we can probably construct similar structures. After all we were not be able to characterize normal subgroups and even semi-direct products in the view of quasi-isometry, but we get the necessary condition to characterize normal subgroups. Then any hope are remain to characterize normal subgroups. So at last we remark about our current study by using algebraic method to solve this problem.

2 Notations

First we think finitely generated groups as discrete metric spaces. Usually we ignore the difference between "left" and "right" word metrics, but for our constructions its difference is very important. So in this article word metric means left word metric :

Notation 2.1. (left word metric) Let G be a finitely generated group, and S is its finite generating system. We define $d_S : G\times G \longrightarrow \mathbf{R}$ as flows, $d_S(g,h) =\| h^{-1}g \|_S= \inf\{n \mid h^{-1}g = s_1^{\epsilon_1}...s_n^{\epsilon_n}, \epsilon_i \in \{-1,1\}, s_i \in S\}$. d_S is distance function on G, then d_S is called left word metric on G by S.

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Metric spaces which are made by word metrics satisfy two regular conditions in the class of discrete metric spaces : they are *uniformly discrete metric spaces* and *coarse path metric spaces.*

Notation 2.2. (uniformly discrete metric space) A metric space (X, d_X) is called uniformly discrete if there is a real number $\Delta > 0$ such that for all $x, y \in X$, $x \neq y$ implies $d_X(x, y) \geq \Delta$.

Notation 2.3. (coarse path metric space) Given a metric space (X, d_X), we call X a coarse path metric space, if there are real numbers $\delta > 0$, $P \geq 1$, and $Q \geq 0$ such that for all $x, y \in X$ we can find a sequence $x = x_1, ..., x_{n+1} = y$ in X which satisfies $d_X(x_i, x_{i+1}) < \delta$ for each $1 \leq i \leq n$ and $d_X(x, y) \geq P \sum_{i=1}^{n} d_X(x_i, x_{i+1}) - Q$.

Second we recall the definition of *quasi-isometry.*

Notation 2.4. (quasi-isometry) Given metric spaces (X, d_X) and (Y, d_Y), a function $f : X \longrightarrow Y$ is called quasi-isometry if there are real numbers K, L, and C $(K \geq 1, L, C \geq 0)$ such that they satisfy following two conditions. (i) For all $x, x' \in X$, $\frac{1}{K} d_X(x, x') - L \leq d_Y(f(x), f(x')) \leq K d_X(x, x') + L$. (ii) $f(X)$ is C-caorsely dense in Y : for all $y \in Y$ there is $x \in X$ such that $d_Y(f(x), y) \leq L$.

If there is a quasi-isometry between (X, d_X) and (Y, d_Y), we say that they are quasi-isometric. This is a equivalence relation among metric spaces. And left word metrics on a finitely generated group which are made by the two finitely generating systems are quasi-isometric. At last we recall the definition of *Hausdorrf distance* on uniformly discrete metric spaces.

Notation 2.5. (Hausdorrf distance) Given an uniformly discrete metric space (X, d_X), We define the hausdorrf distance d_X^H for A, B which are subspaces of X as follows. $d_X^H(A, B) = \max\{\inf\{r | A \subseteq \mathcal{N}_r(B)\}, inf\{r | B \subseteq \mathcal{N}_r(A)\}\}$.

Because X is uniformly discrete, d_X^H is an extended distance function on all the subspaces of X. So subset of all the subspaces of X of which elements are mutually finite distance for d_X^H can be a metric space by the restriction of d_X^H.

3 Prototype fibre structure

Given a finitely generated group G, H which is a normal subgroup of G. G is a metric space by a left word metric. Then we want to think G a fibre space which base is right coset class $H \setminus G$ and fibres are right cosets $\{Hg | [g] \in H \setminus G\}$. So if we will carefully take these properties, we can get a quasi-isometrically invariant concept like fibre space. We call it a prototype fibre structure :

Definition 3.1. (prototype fibre structure) Given an uniformly discrete metric space (X, d_X), a set Y, metric spaces A, B, and X_α which is a subspace of X for each $\alpha \in Y$. We call $(Y, \{X_\alpha\}_{\alpha \in Y})$ a prototype fibre structure on X which has a base B and a fibre A if it satisfies following four conditions.
(i) $\cup_{\alpha \in Y} X_\alpha$ which is a subspace of X is coarsely dense in X .
(ii) $d_X^H(X_\alpha, X_\beta)$ is finite for each pair $\alpha, \beta \in Y$.
(iii) X_α is quasi-isometric to A for each $\alpha \in Y$.
(iv) If we define a pseudo metric on Y as $d_Y(\alpha, \beta) = d_X^H(X_\alpha, X_\beta)$ for each $\alpha, \beta \in Y$, then its quotient metric space $(Y, d_Y)/\sim$ is quasi-isometric to B.

These structures have the following invariance under quasi-isometry :

Proposition 3.2. (invariance) [8] Given uniformly discrete metric spaces (X, d_X), $(X', d_{X'})$, metric spaces A, B, a quasi-isometry $f : X \longrightarrow X'$. If $(Y, \{X_\alpha\}_{\alpha \in Y})$ is a prototype fibre structure on X which has a base B and a fibre A, then $(Y, \{f(X_\alpha)\}_{\alpha \in Y})$ is a prototype fibre structure on X' which has a base B and a fibre A.

Actually the existence of a normal subgroup implies the existence of this structure :

Proposition 3.3. (normal subgroup $\Rightarrow$ prototype fibre structure) [8] Given a finitely generated group G, d_G a left word metric on G, H a normal subgroup of G. Because $H\backslash G$ is finitely generated, we can take d_q a left word metric on $H\backslash G$. We take $H\backslash G$ as Y, and X_α as α for each $\alpha \in Y$, then $(Y, \{X_\alpha\}_{\alpha \in Y})$ is a prototype fibre structure on (G, d_G) which has a base $(H\backslash G, d_q)$ and a fibre (H, d_G).

The existence of a subgroup which is not necessary normal also implies the existence of a prototype fibre structure :

Corollary 3.4. (subgroup $\Rightarrow$ prototype fibre structure) [8] Given a finitely generated group G, d_G a left word metric on G, K a subgroup of G. We take $K\backslash G$ as Y, and X_α as α for each $\alpha \in Y$, then $(Y, \{X_\alpha\}_{\alpha \in Y})$ is a prototype fibre structure on (G, d_G) which has a base $(K\backslash G, d^H_{(G,d_G)})$ and a fibre (K, d_G).

Because in the definition of the prototype fibre structures we don't think the informations which reconstruct the total space by the base and the fibres, these structures are too weak to characterize normal subgroups. Actually in classical group extension theory the total group is reconstructed from the "base" and the "fibres"[7]. So the next section we consider the structures which have more informations than the prototype fibre structures to reconstruct the total spaces.

4 Semi-flat fibre structure

We consider in the special case : semi-direct products. Let G be a finitely generated group. Given $1 \to A \to G \to B \to 1$(split exact as group). Consider a left word metric on G and B, and A as a subspace of G. So for each $\beta \in B$ there exists $g_\beta \in G$ which satisfies $Ag_\beta = \beta$, such that for all $\alpha, \beta \in B$, $g_\alpha g_\beta = g_{\alpha.\beta}$. Then we notice the two properties : (i) for each $\alpha, \beta \in B$ a function $Ag_\alpha \to Ag_\beta$: $ag_\alpha \mapsto ag_\alpha(g_\alpha^{-1}g_\beta)$ is quasi-isometry which has certain common properties depending on only the distance of α and β. And (ii) G is generated by a generator of A and finite subset $\{g_\gamma \in G | d_B(\gamma, e) = 1\}$. So we can create quasi-isometry fibre structures which have more properties than prototype fibre structures. We call them semi-flat fibre structures.

Definition 4.1. (Semi-flat fibre structure) Given an uniformly discrete metric space (X, d_X), a set Y, a metric space A, a coarse path metric space B, X_α which is a subspace of X for each $\alpha \in Y$, and $\varphi_{\alpha,\beta} : X_\alpha \longrightarrow X_\beta$ a quasi-isometry for each pair $\alpha, \beta \in Y$. We call $(Y, \{X_\alpha\}_{\alpha \in Y}, \{\varphi_{\alpha,\beta}\}_{\alpha,\beta \in Y})$ a semi-flat fibre structure on X which has a base B and a fibre A if it satisfies following five conditions.
(o) $(Y, \{X_\alpha\}_{\alpha \in Y})$ is a prototype fibre structure on X which has a base B and a fibre A.
(i)(a) There are real numbers $K \geq 1$ and $t \geq 0$, for all real numbers $R \geq 0$, there is a real number $L_R \geq 0$

such that for all $\alpha, \beta \in Y$, if $d_{Y/\sim}(\alpha,\beta) \leq R$ then $\varphi_{\alpha,\beta}$ is a quasi-isometry with constants (K, L_R, t).
(b) There are real numbers $p \geq 1$ and $q \geq 0$ such that for all $\alpha, \beta \in Y$ and $x \in X$

$$d_X(\varphi_{\alpha,\beta}(x), x) \leq p d_X^H(X_\alpha, X_\beta) + q.$$

(c) For all real numbers $R \geq 0$, there is a real number $M_R \geq 0$ such that

$$d_{Y/\sim}(\alpha,\beta) \leq R \text{ and } d_{Y/\sim}(\beta,\gamma) \leq R$$
$$\Longrightarrow sup_{\alpha,\beta,\gamma \in Y} sup_{x \in X_\alpha} d_X(\varphi_{\beta,\gamma} \circ \varphi_{\alpha,\beta}(x), \varphi_{\alpha,\gamma}(x)) \leq M_R.$$

(preparing for (ii))
For each $R \geq 0$ we will make a path metric D_R on $T = \sqcup_{\alpha \in Y} X_\alpha$ from the following weighted graph structure on T. We take an edge set E_R as the following two types.
(type 1) For each α, $\beta \in Y$ such that $d_{Y/\sim}(\alpha, \beta) \leq R$, and $x \in X_\alpha$

$$\begin{cases} \text{edge} : x \longrightarrow \varphi_{\alpha,\beta}(x), \ \text{weight} : d_X^H(X_\alpha, X_\beta) & (\text{if } d_X^H(X_\alpha, X_\beta) \neq 0), \\ \text{edge} : x \longrightarrow \varphi_{\alpha,\beta}(x), \ \text{weight} : 1 & (\text{if } d_X^H(X_\alpha, X_\beta) = 0). \end{cases}$$

(type 2) For each $\alpha \in Y$, $x \neq x' \in X_\alpha$

$$\text{edge} : x \longrightarrow x', \ \text{weight} : d_X(x, x').$$

(ii) Let $\pi_R : (T, D_R) \longrightarrow (\cup_{\alpha \in Y} X_\alpha, d_X)$ be a natural map, then there is a real number $R_0 \geq 0$ such that for all real numbers $R \geq R_0$, π_R is a quasi-isometry.

Definition 4.2. (induced structure) Given uniformly discrete metric spaces (X, d_X) and $(X', d_{X'})$, a quasi-isometry $f : X \longrightarrow X'$, and a semi-flat fibre structure on X which has a base B and a fibre A : $(Y, \{X_\alpha\}_{\alpha \in Y}, \{\varphi_{\alpha,\beta}\}_{\alpha,\beta \in Y})$. Then we can define a quasi-isometry $\varphi'_{\alpha,\beta} : f(X_\alpha) \longrightarrow f(X_\beta)$ for each pair $\alpha, \beta \in Y$: $\varphi'_{\alpha,\beta} = f \circ \varphi_{\alpha,\beta} \circ (f_{|X_\alpha})^{-1}$. We call $(Y, \{f(X_\alpha)\}_{\alpha \in Y}, \{\varphi'_{\alpha,\beta}\}_{\alpha,\beta \in Y})$ a induced triple by f.

Proposition 4.3. (invariance) [8] The induced triple by f is a semi-flat fibre structure on X' which has a base B and a fibre A.

Proposition 4.4. (semi-direct product $\Rightarrow$ semi-flat fibre structure) [8] Given a finitely generated group G, d_G a left word metric on G, H a normal subgroup of G which is split. Because $H\backslash G$ is finitely generated, we can take d_q a left word metric on $H\backslash G$. We take $H\backslash G$ as Y, X_α as α for each $\alpha \in Y$, and $ag_\alpha \mapsto ag_\alpha(g_\alpha^{-1} g_\beta)$ as $\varphi_{\alpha,\beta}$ for a certain section $\{g_\gamma \in G | \gamma \in H\backslash G\}$. Then $(Y, \{X_\alpha\}_{\alpha \in Y}, \{\varphi_{\alpha,\beta}\}_{\alpha,\beta \in Y})$ is a semi-flat fibre structure on (G, d_G) which has a base $(H\backslash G, d_q)$ and a fibre (H, d_G).

In special case of semi-direct products we can think direct products, so we can construct fibre structures which correspond to direct products : "$R \longrightarrow \infty$" in the above definition of the semi-flat fibre structures [8]. These structures provide a general interpretation of Gersten's method [5].

5 Remark

In studying the quasi-isometric invariant properties for general groups, we want to consider more algebraic. So for example we notice the Morita theory of rings. Sauer thinks this method for the proof of quasi-isometry invariance of cohomological dimension [10]. In above thesis Sauer use the Morita theory for studying module category of the group ring. We can study more universal than Sauer's to use the groupoid characterization of quasi-isometry and the topos theory [9] [11] [2] : We think the skew group ring $\mathcal{R}(G) = G \ltimes l^\infty(G, \mathbf{Z})$. So it is proved that if finitely generated groups G and G' are quasi-isometric then $\mathcal{R}(G)$ and $\mathcal{R}(G')$ are Morita equivalent. These rings probably lose the few quasi-isometry invariant properties.

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