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Recent Advances on Brauer's Height Zero Conjecture

Gabriel Navarro
Universitat de València. SPAIN.

Let G be a finite group, let $\mathrm{Irr}(G)$ be the set of its irreducible complex characters, and let p be a prime. One of the main idea of the representation theory of finite groups is to find relations between global invariants (i.e., the properties of G) and the local invariants (i.e., the properties of the p-subgroups $Q > 1$ of G, and their normalizers).

In pure group theory, the perfect example of this interaction between global and local is provided by Frobenius theorem: A group G has a normal p-complement if and only if $\mathbf{N}_G(Q)$ has a normal p-complement for all p-subgroup $Q > 1$ of G.

In character theory, this paradigmatic example might be provided by the so called McKay conjecture: the number $|\mathrm{Irr}_{p'}(G)|$ of irreducible characters $\chi \in \mathrm{Irr}(G)$ of degree not divisible by p is the same as $|\mathrm{Irr}_{p'}(\mathbf{N}_G(P))|$, where $P \in \mathrm{Syl}_p(G)$. Another such an example is the Alperin Weight Conjecture which locally counts the number of **defect zero characters** of G (i.e., those $\chi \in \mathrm{Irr}(G)$ such that $\chi(1)_p = |G|_p$, the largest possible p-power for the irreducible characters of G). Finally, Dade's conjecture locally counts the number of all irreducible characters of G with any given p-part.

Long before these conjectures, R. Brauer proposed (in fact he asked) his celebrated Height Zero Conjecture. In order to state it, we need to introduce Brauer p-blocks. There are many ways to do this (blocks are indecomposable algebras over fields of characteristic p, overall), but for the purpose of this talk, let us say that two irreducible characters $\alpha, \beta \in \mathrm{Irr}(G)$ are **linked** if

$$\sum_{x \in G^0} \alpha(x)\overline{\beta(x)} \neq 0\,,$$

where G^0 is the set of elements $x \in G$ with order not divisible by p. The connected components of this linking are the **Brauer p-blocks** of G. Hence here a p-block, $\mathrm{Irr}(B)$, is a subset of $\mathrm{Irr}(G)$, and the set of p-blocks partition $\mathrm{Irr}(G)$.

R. Brauer proved that a p-block $\mathrm{Irr}(B)$ has canonically associated, up to G-conjugacy, a p-subgroup D of G which has deep influence on it, the so called **defect group** of B. Again, there are several ways of defining the defect groups, but we will not go into that here. The defect groups of the **principal block** (the block containing the trivial character, and considered the most important block) are the Sylow p-subgroups of G. (As a matter of fact, there is a whole theory that unifies defect groups and Sylow theory, developed by J. Alperin and M. Broué.) The following are some examples of the influence of the defect group D on B. For instance, if $\chi \in \mathrm{Irr}(B)$ and $g \in G$ is such that no G-conjugate of g_p lies in D, then $\chi(g) = 0$. Or

$$\min(\chi(1)_p \mid \chi \in \mathrm{Irr}(G)) = |G|_p/|D| .$$

Another connection among Brauer blocks and their defect group is stated in the famous $k(B)$-conjecture by Brauer, namely:

$$|\mathrm{Irr}(B)| \leq |D| .$$

Yet another connection between $\mathrm{Irr}(B)$ and D is that there is a bijection between the blocks of G with defect group D and the blocks of $\mathbf{N}_G(D)$ with defect group D, the so called Brauer's First Main Theorem. (This constitutes another perfect example of the global/local connection and it is the starting point of the so called Broué's conjecture. This conjecture explains some cases of the conjectures above from an structural point of view, by finding the exact relationship between the algebras correspondent by Brauer's First Main Theorem.)

Now, we can define heights. If $\mathrm{Irr}(B)$ has defect group D, $|G| = p^a$, $|D| = p^d$, and $\chi \in \mathrm{Irr}(B)$ then we already know that

$$\chi(1)_p = p^{a-d+h} ,$$

for a unique non-negative integer h, called the **height** of χ. For instance, in the key case of χ having p'-degree, we see that χ has height zero and lives in a block with defect group a Sylow p-subgroup of G (and conversely, of course).

Brauer's Height Zero Conjecture. *Let B be a p-block of a finite group G with defect group D. Then all $\chi \in \mathrm{Irr}(B)$ have height zero if and only if D is abelian.*

Our main contribution is to prove the conjecture when $p = 2$ and $D \in \mathrm{Syl}_2(G)$. (This is joint work with Pham Huu Tiep.)

Essentially, the preceding work on the Height Zero Conjecture is the following. In 1984 it was proven for p-solvable groups by D. Gluck and T. Wolf

[GW1,GW2] being the "only if" part extraordinary complicated. In 1988 the "if" implication was reduced to a question on quasisimple groups by T. Berger and R. Knörr ([BK]). P. Fong and M. Harris proved the "if" direction of the conjecture for the principal 2-block in [FH]. (In fact, they proved the Broué Conjecture for those blocks.) Several specific groups were treated in [BE].

Now, the recent advances on the McKay conjecture in [IMN], together with the recent and powerful results of M. Broué and J. Michel ([BM]) on unions of ℓ-blocks, of C. Bonnafé and R. Rouquier ([BR]) on Morita equivalences, and of course the Deligne-Lusztig theory [L], allow us to handle the full Brauer's Height Zero Conjecture for the 2-blocks of maximal defect.

THEOREM A (Navarro-Tiep). *Let B be a 2-block of G with defect group $P \in \mathrm{Syl}_2(G)$. Then $\chi(1)$ is odd for all $\chi \in \mathrm{Irr}(B)$ if and only if P is abelian.*

The "only if" part of Theorem A without a doubt constitutes the most complicated part of the proof of this theorem.

It is important to remark that the methods and the ideas used to prove Theorem A will definitely help to handle the general Height Zero Conjecture once the results in [IMN] are improved to general blocks to prove the Alperin-McKay conjecture, and the blocks of the quasisimple groups have been classified.

A major obstacle towards proving the Height Zero Conjecture for odd primes is the following purely character theoretical problem (which we consider one of the main problems in character theory right now).

CONJECTURE. *Suppose that N is a normal subgroup of G and let $\theta \in \mathrm{Irr}(N)$. Let p be a prime. If p does not divide $\chi(1)/\theta(1)$ for all $\chi \in \mathrm{Irr}(G)$ lying over θ, then P/N is abelian, where $P/N \in \mathrm{Syl}_p(G/N)$.*

This is a not so well-known consequence of the Brauer's Height Zero Conjecture, and as we say, is pure character theory (no blocks are involved, and probably no block theory will help in its solution). In the case, $p = 2$, we were fortunate to have a theorem of Alexander Moretó [M] proving that in this case G/N is solvable. However, this will not be the case if p is odd: for instance, $p = 5$ and $G = 2.A_5$ is a counterexample. Currently, we are working, with P. H. Tiep, in a proof of this conjecture.

Let us finally mention that a relationship between the Alperin-McKay Conjecture and the Height Zero Conjecture for principal blocks was observed in [Mu].

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