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Independent Subbases of the Sierpinski Gasket

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1 Introduction

Let $\mathbb{T}$ denote the set $\{0, 1, \perp\}$, where $\perp$ is a special value which means undefinedness between 0 and 1. We consider the space $\mathbb{T}^\omega$ of infinite sequences of $\mathbb{T}$ and call each element of $\mathbb{T}^\omega$ a bottomed sequence. We call each appearance of 0 and 1 in a bottomed sequence a digit, and call a bottomed sequence with finite number of digits a finite bottomed sequence. We denote by $\mathbb{T}^*$ the set of finite bottomed sequences.

In [2], in order to realize computation on the unit interval $\mathbb{I}$, the author introduced an embedding of $\mathbb{I}$ into $\mathbb{T}^\omega$, and then generalized this idea to metrizable spaces in general[3]. When φ is an embedding of a metric space X in $\mathbb{T}^\omega$, those sets $S_{n,i} = \{x \in X : \varphi(x)(n) = i\}$ for $n < \omega$ and $i < 2$ form a subbase of X, since those sets $\{p \in \mathbb{T}^\omega : p(n) = i\}$ for $n < \omega$ and $i < 2$ form a subbase of $\mathbb{T}^\omega$. On the other hand, if $S = \{S_{n,i} : n < \omega, i < 2\}$ is a subbase such that $S_{n,0} \cap S_{n,1}$ is empty for $n < \omega$, then we have a corresponding embedding φ_S of X in $\mathbb{T}^\omega$ defined as

$$\varphi_S(x)(n) = \begin{cases} 0 & (x \in S_{n,0}) \\ 1 & (x \in S_{n,1}) \\ \perp & (otherwise) \end{cases}.$$

Therefore, we identify an embedding of X in $\mathbb{T}^\omega$ with such a subbase. Among such subbases, we are interested in those with the following property.

Definition 1 ([4], [5]) *An independent subbase of a space* X *is a subbase* $S = \{S_{n,i} : n < \omega, i < 2\}$ *of* X*, such that*

$$(\forall n < \omega)(S_{n,0} \cap S_{n,1} = \emptyset), \textit{ and} \tag{1}$$

$$(\forall n < \omega)(\forall p \in 2^n)\left(\bigcap_{k<n} S_{k,p(k)} \neq \emptyset\right). \tag{2}$$

We can show that $S_{n,1}$ is the exterior of $S_{n,0}$ for every $n < \omega$ for an independent subbase S. For an independent subbase S and a bottomed sequence p, we denote by $S(p)$ and $\bar{S}(p)$ the sets

$$S(p) = \bigcap_{m \in \mathrm{dom}(p)} S_{m,p(m)},$$
$$\bar{S}(p) = \bigcap_{m \in \mathrm{dom}(p)} \mathrm{cl}\ S_{m,p(m)}.$$

Here, $\mathrm{dom}(p) = \{n \in \omega : p(n) \neq \perp\}$. The family of open sets $\{S(p) : p \in \mathbb{T}^\omega\}$ forms the open base corresponding to S. One can prove that an independent subbase satisfies the following condition which is stronger than (2).

$$(\forall p \in \mathbb{T}^*)(S(p) \neq \emptyset).$$

This property is related to non-redundancy of each coding sequence $\varphi_S(x), x \in X$[4].

An independent subbase also satisfies the condition cl $S(p) = \bar{S}(p)$ for $p \in \mathbb{T}^*$. That is, it is a proper dyadic subbase in [4], and thus it coincides with the definition of an independent subbase in [4].

For a non-negative integer m, an independent subbase $S = \{S_{n,i} : n < \omega, i < 2\}$ is *of dimension* m if $\mathrm{ord}\{\mathrm{cl}\ S_{n,0} \setminus S_{n,0} : n < \omega\} \leq m - 1$, where $\mathrm{ord}\,\mathcal{A}$ means the largest integer m such that the collection $\mathcal{A}$ contains $m + 1$ sets with a non-empty intersection. It is obvious that the degree of an independent subbase cannot be smaller than the small inductive dimension of the space, and we are interested in independent subbases with the same dimension as the small inductive dimension of the space.

In [4], starting with the Gray subbase of $\mathbb{I}$, he constructed a lot of examples of independent subbases of topological spaces. He constructed a dyadic subbase for the Cantor set, the unit interval $\mathbb{I}$, the products $\mathbb{I}^n$, the Hilbert cube $\mathbb{I}^\omega$, the circle S^1, and several surfaces such as S^2, the torus T^2 and the n-torus nT^2.

Then, in [5], they proved that a n-dimensional separable metrizable space X has an independent subbase of dimension n if and only if X is dense in itself. From this theorem, for example, the Sierpinski Gasket has an independent subbase of dimension 1. However,

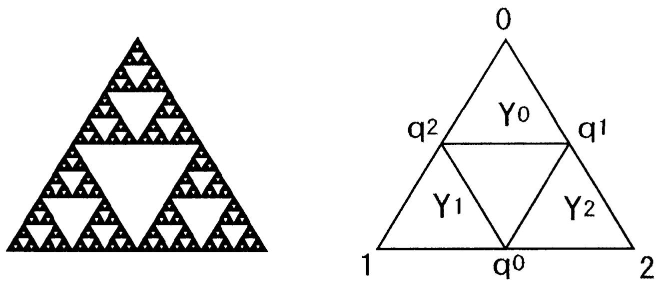

図 1: Sierpinski Gasket Y, the three vertices 0,1, and 2, and the three components Y_0, Y_1, and Y_2.

this theorem does not provide an independent subbase of the Sierpinski Gasket which is defined following the recursive structure the Gasket has.

The Sierpinski Gasket Y is a well-known fractal defined by the iteration function system (IFS for short) (f_0, f_1, f_2) where f_i $(i = 0, 1, 2)$ are dilations (i.e., similarity functions which do not rotate the object) with the ratio $1/2$ and with the centers the three vertices of a regular triangle (Figure 1). By f_i $(i = 0, 1, 2)$, Y is mapped to half-sized copies Y_i of Y. Note that dilations are not the only similarity maps which form this fractal, and there are six candidates for each of them because the symmetry group of Y has order 6.

If a similarity function (or more generally a homeomorphism) $f : Y \to Y_i$ is given, one can transform an independent subbase S of Y to an independent subbase $f(S)$ of Y_i $(i = 0, 1, 2)$. On the other hand, as we will show, there is a method to construct an independent subbase of Y from independent subbases of Y_i $(i = 0, 1, 2)$ if the three subbases satisfy some conditions. Our question is whether there is an independent subbase S of Y and an IFS (f_0, f_1, f_2) of the Gasket such that the independent subbases $f_0(S)$, $f_1(S)$, $f_2(S)$ satisfy the above mentioned condition and the independent subbase constructed from $f_0(S)$, $f_1(S)$, and $f_2(S)$ on Y coincides with S. Computationally, among dyadic subbases, only such a recursively-defined one is meaningful. With other subbases, it would be difficult to express a function recursively defined according to the structure of the Gasket as a recursively-defined program to input/output the $\mathbb{T}^\omega$-code.

In this article, we show that there are three such subbases, modulo rotational and reflective transformations of the Gasket and switching the role of $S_{n,0}$ and $S_{n,1}$ for $n < \omega$.

2 Gray subbase of $[0, 1]$

In order to clarify the problem, let us first explain the Gray subbase G of the unit interval $\mathbb{I}$. From the definition of an independent subbase, $\mathbb{I}$ is divided into $G(0) = G_{0,0}$, $G(1) = G_{0,1}$, and their boundary (i.e., $\{1/2\}$). The set $G(0)$ is again divided into two parts $G(00)$, $G(01)$, and their boundary ($\{1/4\}$), and $G(1)$ is divided into two parts

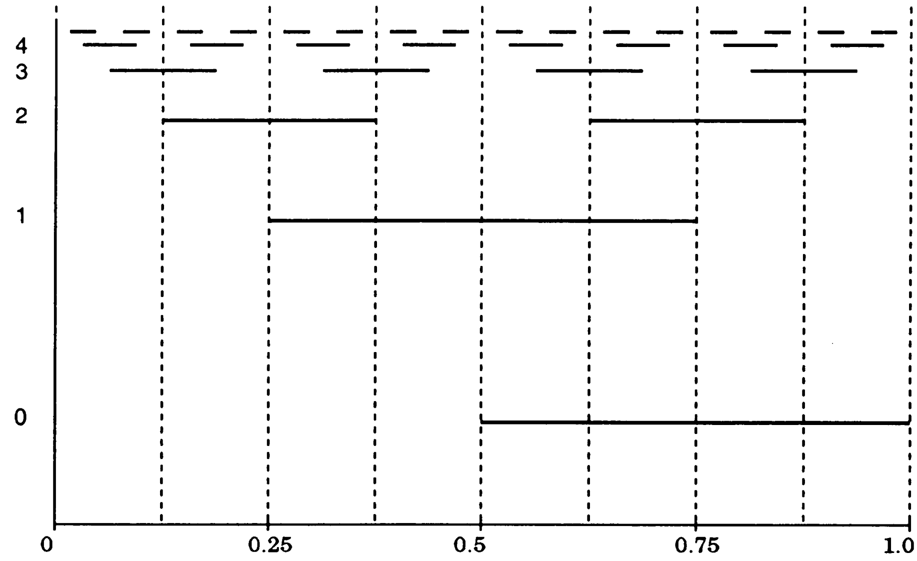

図 2: Gray subbase of the unit interval [0,1]. Here, the union of interiors of line segments at level n is $G_{n,1}$ and the interior of the rest is $G_{n,0}$.

$G(10)$, $G(11)$ and their boundary ($\{3/4\}$). Here we express by a finite sequence d the finite bottomed sequence $d\bot^\omega$. Afterwards, each of them is divided in the same way. Note that for $x = 1/2$, $\varphi_G(x) = \bot 10^\omega$ and those open sets $G(\bot 10^n)$ ($n < \omega$) form a basic neighbourhood system of $1/2$, in order that S is a subbase. Therefore, the assignments of the codes 00, 01, 10, 11 to the four regions is not arbitrary, and, furthermore, in order to use at most one $\bot$ to each sequence, connected regions have sequences with one digit difference.

This Gray subbase is defined through the fractal structure of $\mathbb{I}$; $\mathbb{I}$ is composed of two parts $[0, 1/2]$ and $[1/2, 1]$ which are half-sized copies of $\mathbb{I}$. We give two explanations of the recursive structure of this subbase.

The first one is to use a similarity map different from the ordinary one. Normally, the unit interval $\mathbb{I}$ is considered as the fractal generated from the IFS (f_0, f_1) where $f_0 : \mathbb{I} \to [0, 1/2]$ and $f_1 : \mathbb{I} \to [1/2, 1]$ are defined as $f_0(x) = x/2$ and $f_1(x) = 1/2 + x/2$. Binary expansion is defined according to this fractal structure, and each point x of $\mathbb{I}$ is given (multiple) codes $c(x)$; each point in $[0,\ 1/2)$ is given code(s) $0 : c(f_0(x))$, each point in $(1/2,1]$ is given code(s) $1 : c(f_1(x))$, and $1/2$ is given both codes $0 : c(f_0(1/2)) = 01^\omega$ and $1 : c(f_1(1/2)) = 10^\omega$. Here, we denote by $a : p$ an infinite sequence with the head a and the tail p, and sometimes we omit $:$ and simply denote it by ap. According to this code, $\mathbb{I}$ is homeomorphic to the quotient of the Cantor space $\{0, 1\}^\omega$ with the equivalence relation $p01^\omega \sim p10^\omega$ for $p \in 2^*$. With this binary expansion, the two codes of $1/2$ are completely different, and we cannot express them as one bottomed sequence.

For Gray subbase, we consider that $\mathbb{I}$ is defined as a fractal generated by another IFS with a different similarity map from $\mathbb{I}$ to $[1/2, 1]$. We consider that $\mathbb{I}$ is the fractal generated from two similarity maps $g_0 : \mathbb{I} \to [0, 1/2]$ and $g_1 : \mathbb{I} \to [1/2, 1]$ defined as $g_0(x) = x/2$ and $g_1(x) = 1 - x/2$. Then, the two codes given to $1/2$ become 010^ω and 110^ω. That is, they are the same except for the first digit. Since the first digit does not contribute in determining the point and all the rest determines that the point is $1/2$, we define the code of $1/2$ as $\perp 10^\omega$ and define Gray embedding φ_G from $\mathbb{I}$ to $\mathbb{T}^\omega$ as follows.

$$\varphi_G(x) = \begin{cases} 0 : \varphi_G(g_0^{-1}(x)) = 0 : \varphi_G(2x) & (x < 1/2) \\ 1 : \varphi_G(g_1^{-1}(x)) = 1 : \varphi_G(2(1-x)) & (x > 1/2) \\ \perp : 1 : 0^\omega (= \perp : \varphi_G(g_0^{-1}(x)) = \perp : \varphi_G(g_1^{-1}(x))) & (x = 1/2) \end{cases}$$

The corresponding subbase $G = \{G_{n,i} : n < \omega, i < 2\}$ defined as $G_{n,i} = \{x \in \mathbb{I} : \varphi_G(x)(n) = i\}$ is the Gray subbase. Note that $g_0(G)$ and $g_1(G)$, as defined in the last part of Section 1, are independent subbases of $[0, 1/2]$ and $[1/2, 1]$, respectively, and $\varphi_{g_0(G)}(1/2)$ and $\varphi_{g_1(G)}(1/2)$ agree (to 10^ω). This suggests a condition we need for a recursively-defined independent subbase of the Sierpinski Gasket.

The other explanation of the Gray subbase is through inversions of the digits of the ordinary binary sequences, instead of changing the IFS. We define an embedding φ'_G as follows.

$$\varphi'_G(x) = \begin{cases} 0 : \varphi'_G(f_0^{-1}(x)) = 0 : \varphi'_G(2x) & (x < 1/2) \\ 1 : (\text{not } 0)\varphi'_G(f_1^{-1}(x)) = 1 : (\text{not } 0)\varphi'_G(2x-1) & (x > 1/2) \\ \perp : 1 : 0^\omega (= \perp : \varphi'_G(f_0^{-1}(x)) = \perp : (\text{not } 0)\varphi'_G(f_1^{-1}(x))) & (x = 1/2) \end{cases}$$

Here, (not $n_0, \dots, n_{k-1}$) is a function from $\mathbb{T}^\omega$ to $\mathbb{T}^\omega$ to invert values at indices n_i $(i < k)$ if they are digits. With the IFS (f_0, f_1), $\varphi_{f_0(G)}(1/2) = 10^\omega$ and $\varphi_{f_1(G)}(1/2) = 0^\omega$ differ at the first digit. However, for any pair (S, T) of independent subbases of $[0, 1/2]$ and $[1/2, 1]$, we can invert the values of T to obtain an independent subbase of $[1/2, 1]$ so that the two independent subbases agree on $1/2$. Therefore, we can construct an independent subbase of $[0, 1]$ by inverting some of the digits of T and adding one more digit which assigns 0 to $[0, 1/2)$ and 1 to $(1/2, 1]$. Note that $1/2$ is an endpoint of $[0, 1/2]$ (and $[1/2, 1]$) and it cannot be the boundary of a regular open set, and therefore the code sequences $\varphi_S(1/2)$ (and $\varphi_T(1/2)$) do not contain a $\perp$. Therefore, we can add a new $\perp$ to the front to generate the code sequence of $1/2$.

In order to make a similar definition for the Sierpinski Gasket, there are two difficulties. One is that the Sierpinski Gasket is defined through three similarity maps but we are allowed to use only two digits 0 and 1 for an independent subbase.

The other one is that the three components are connected at three points. Therefore, it is impossible to combine any triple of independent subbases of the three components by inverting some of the digits, and the three independent subbases should satisfy some condition.

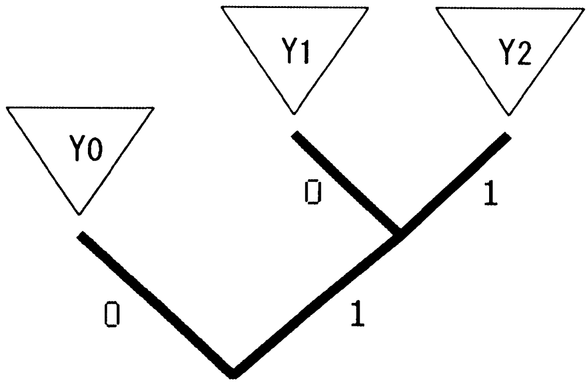

図 3: The coding of the three components.

3 Recursively Defined Independent Subbases of the Sierpinski Gasket

A similarity map from Y to Y_i is expressed as the dilation composed with one of the six self-congruence maps of Y_i $(i = 0, 1, 2)$. Since a self-congruence map of Y (and also Y_i) can be expressed as a permutation of the vertices 0, 1, and 2 in Figure 1, we express a similarity map from Y to Y_i as a permutation of (0,1,2).

For the Gray subbase of $\mathbb{I}$, we gave two explanations; through another set of similarity maps and through inversions of values of some of the digits. For the Gasket, we consider an independent subbase which is defined through the combination of both of the two.

As we mentioned above, our first problem is that there are three components Y_0, Y_1, Y_2 but we can use only two digits 0 and 1. In order to distinguish them, we use one digit 0 for Y_0 and sequences 10 and 11 for Y_1 and Y_2 as Figure 3 shows. Therefore, we adopt the following form of definition.

$$\varphi(x) = \begin{cases} 0 : h_0(\varphi(f_0^{-1}(x)) & (x \in Y_0) \\ 1 : 0 : h_1(\varphi(f_1^{-1}(x))) & (x \in Y_1) \\ 1 : 1 : h_2(\varphi(f_2^{-1}(x))) & (x \in Y_2) \end{cases}$$

Here, f_i is a similarity map from Y to Y_i selected from the six possibilities, and h_i is a function from $\mathbb{T}^\omega$ to $\mathbb{T}^\omega$ which inverts values of some of the elements. Therefore, in Y_i $(i = 0, 1, 2)$, we have the same code as Y rotated and flipped according to f_i and inverted according to h_i, and shifted for one character in Y_0 and two characters in Y_1 and Y_2, and the first digit for Y_0 is 0 whereas the first two digits of Y_1 and Y_2 are 10 and 11.

On the three boundary points q_0, q_1, and q_2, we can apply two definitions. We require that the two definitions on both sides agree except for one digit. For example, on q_2 which is the boundary between Y_0 and Y_1, we have two sequences $0 : h_0(\varphi(f_0(q_2)))$ and

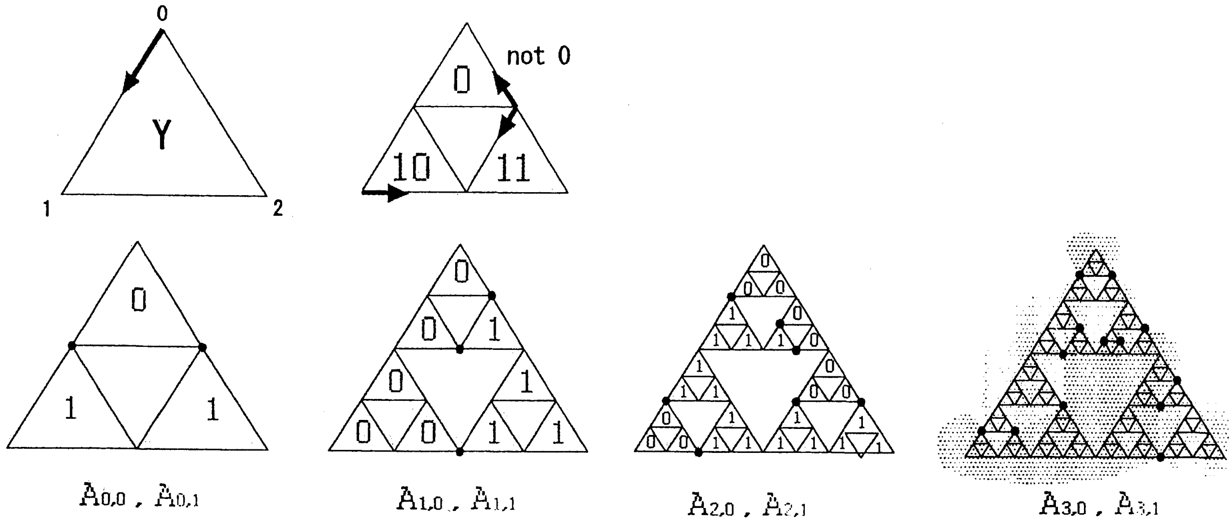

図 4: A: Recursively defined independent subbase #1. As the figures on the first line shows, Y is rotated 120 degree to the right on Y_0 and to the left on Y_1, and not rotated on Y_2 and the 0-th value is inverted on Y_0.

$1 : 0 : h_1(\varphi(f_1(q_2)))$ on the two sides. Since the first digit disagree, we require that $h_0(\varphi(f_0(q_2))) = 0 : h_1(\varphi(f_1(q_2)))(= p)$, and we define the code of q_2 as $\perp : p$.

There are three recursive equations of this form. They are first found by a computer program, and checked through hand calculation as we explain later.

A: Recursively defined independent subbase #1.

$$f_0^{-1} = (1,2,0), f_1^{-1} = (2,0,1), f_2^{-1} = (0,1,2),$$
$$h_0 = (\text{not } 0), h_1 = id, h_2 = id.$$

B: Recursively defined independent subbase #2.

$$f_0^{-1} = (2,1,0), f_1^{-1} = (0,1,2), f_2^{-1} = (0,2,1),$$
$$h_0 = (\text{not } 0), h_1 = (\text{not } 1), h_2 = (\text{not } 1).$$

C: Recursively defined independent subbase #3.

$$f_0^{-1} = (2,1,0), f_1^{-1} = (1,0,2), f_2^{-1} = (1,2,0),$$
$$h_0 = (\text{not } 0), h_1 = (\text{not } 0,1), h_2 = (\text{not } 0,1).$$

3.1 A: Recursively defined independent subbase #1

Figure 4 shows the first four components of the subbase A. With the definition, we can calculate as follows;

$$\begin{aligned}
\varphi_A(0) &= 0 : (\text{not } 0)\varphi_A(1) = 0 : (\text{not } 0)1 : 0 : \varphi_A(0) = 0^\omega, \\
\varphi_A(1) &= 1 : 0 : \varphi_A(0) = 10^\omega, \\
\varphi_A(2) &= 1 : 1 : \varphi_A(2)1^\omega.
\end{aligned}$$

For q_i $(i = 0, 1, 2)$, let $\varphi_A(q_i(\leftarrow j))$ the value of $\varphi_A(q_i)$ when approached from Y_j.

$$\begin{aligned}
\varphi_A(q_2(\leftarrow 0)) &= 0 : (\text{not } 0)\varphi_A(2) = 0 : (\text{not } 0)1^\omega = 001^\omega, \\
\varphi_A(q_2(\leftarrow 1)) &= 1 : 0 : \varphi_A(2) = 101^\omega, \\
\varphi_A(q_0(\leftarrow 1)) &= 1 : 0 : \varphi_A(1) = 1010^\omega, \\
\varphi_A(q_0(\leftarrow 2)) &= 1 : 1 : \varphi_A(1) = 1110^\omega, \\
\varphi_A(q_1(\leftarrow 2)) &= 1 : 1 : \varphi_A(0) = 110^\omega, \\
\varphi_A(q_1(\leftarrow 0)) &= 0 : (\text{not } 0)\varphi_A(0) = 010^\omega.
\end{aligned}$$

Thus, since $\varphi_A(q_i(\leftarrow j))$ for $j \neq i$ differ only at one digit, it forms an independent subbase of dimension 1 with the following definition.

$$\begin{aligned}
\varphi_A(q_2) &= \perp 01^\omega, \\
\varphi_A(q_0) &= 1\perp 10^\omega, \\
\varphi_A(q_1) &= \perp 10^\omega.
\end{aligned}$$

3.2 B: Recursively defined independent subbase #2

For the second one, we similarly have the following calculation.

$$\begin{aligned}
\varphi_B(0) &= 0 : (\text{not } 0)\varphi_B(2) = 001^\omega, \\
\varphi_B(1) &= 1 : 0 : (\text{not } 1)\varphi_B(1) = 101^\omega, \\
\varphi_B(2) &= 1 : 1 : (\text{not } 1)\varphi_B(1) = 1^\omega.
\end{aligned}$$

$$\begin{aligned}
\varphi_B(q_2(\leftarrow 0)) &= 0 : (\text{not } 0)\varphi_B(1) = 0001^\omega, \\
\varphi_B(q_2(\leftarrow 1)) &= 1 : 0 : (\text{not } 1)\varphi_B(0) = 1001^\omega, \\
\varphi_B(q_0(\leftarrow 1)) &= 1 : 0 : (\text{not } 1)\varphi_B(2) = 10101^\omega, \\
\varphi_B(q_0(\leftarrow 2)) &= 1 : 1 : (\text{not } 1)\varphi_B(2) = 11101^\omega, \\
\varphi_B(q_1(\leftarrow 2)) &= 1 : 1 : (\text{not } 1)\varphi_B(0) = 1101^\omega, \\
\varphi_B(q_1(\leftarrow 0)) &= 0 : (\text{not } 0)\varphi_B(0) = 0101^\omega.
\end{aligned}$$

Thus, it also forms an independent subbase of dimension 1 with the following definition.

$$\begin{aligned}
\varphi_B(q_2) &= \perp 001^\omega, \\
\varphi_B(q_0) &= 1\perp 101^\omega, \\
\varphi_B(q_1) &= \perp 101^\omega.
\end{aligned}$$

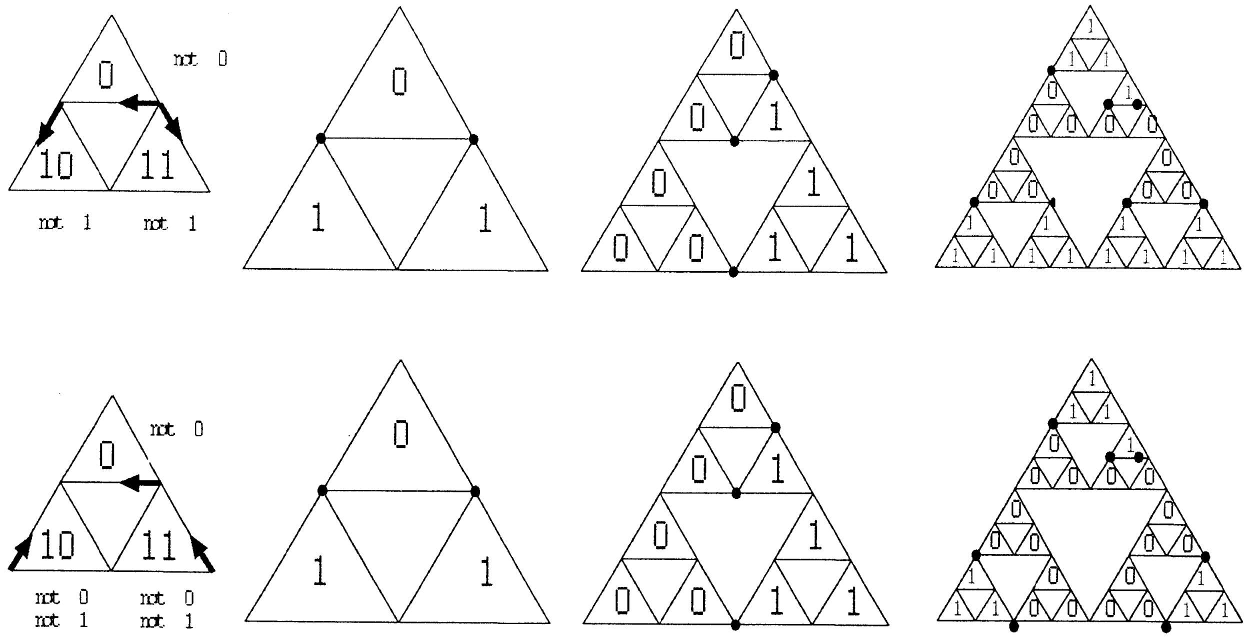

図 5: The subbases B (upper line) and C (lower line).

3.3 C: Recursively defined independent subbase #3

$$\varphi_C(0) = 0 : (\text{not } 0)\varphi_C(2) = 001^\omega,$$
$$\varphi_C(1) = 1 : 0 : (\text{not } 0, 1)\varphi_C(0) = 101^\omega,$$
$$\varphi_C(2) = 1 : 1 : (\text{not } 0, 1)\varphi_C(0) = 1^\omega.$$

$$\varphi_C(q_2(\leftarrow 0)) = 0 : (\text{not } 0)\varphi_C(1) = 0001^\omega,$$
$$\varphi_C(q_2(\leftarrow 1)) = 1 : 0 : (\text{not } 0, 1)\varphi_C(1) = 1001^\omega,$$
$$\varphi_C(q_0(\leftarrow 1)) = 1 : 0 : (\text{not } 0, 1)\varphi_C(2) = 10001^\omega,$$
$$\varphi_C(q_0(\leftarrow 2)) = 1 : 1 : (\text{not } 0, 1)\varphi_C(2) = 11001^\omega,$$
$$\varphi_C(q_1(\leftarrow 2)) = 1 : 1 : (\text{not } 0, 1)\varphi_C(1) = 1101^\omega,$$
$$\varphi_C(q_1(\leftarrow 0)) = 0 : (\text{not } 0)\varphi_C(0) = 0101^\omega.$$

Thus, it also forms an independent subbase of dimension 1 with the following definition.

$$\varphi_C(q_2) = \perp 001^\omega,$$
$$\varphi_C(q_0) = 1\perp 001^\omega,$$
$$\varphi_C(q_1) = \perp 101^\omega.$$

We can show that an independent subbase of dimension 1 always satisfies the following condition.

Condition A: For each level $n < \omega$, we have even number of $\perp$ along the inner triangle of the Gasket.

At level-0, we have $\perp$ on q_1 and q_2. At level-1, we have a $\perp$ on q_0 and there should be one $\perp$ on the edge between q_1 and q_2. Therefore, f_0^{-1} should have the form $(*, *, 0)$ or $(*, 0, *)$. For symmetricity, we only consider the case $(*, *, 0)$. that is, $(1, 2, 0)$ and $(2, 1, 0)$. Note that f_i $(i = 0, 1, 2)$ determine all the positions of $\perp$ at all the level. Conversely, if condition A is satisfied by an IFS (f_0, f_1, f_2), then by fixing a sequence for $\varphi(0)$, an independent subbase is uniquely determined by inserting appropriate functions h_i $(i = 0, 1, 2)$.

Therefore, we only need to find an IFS (f_0, f_1, f_2) which satisfies Condition A. With the above arguments, we only need to consider $2 \times 6 \times 6 = 72$ cases. We checked that all the other 69 cases do not satisfy condition A, both with hand calculation and through computer calculation. Therefore, there are only three recursively defined independent subbases of the Sierpinski Gasket, if we identify rotationally and reflectively equivalent ones and equivalent ones through switching the role of $S_{n,0}$ and $S_{n,1}$ for $n < \omega$.

[1] Hideki Tsuiki. Computational dimension of topological spaces. In *Computability and Complexity in Analysis*, *LNCS 2064*, 323–335, 2001.

[2] Hideki Tsuiki. Real number computation through gray code embedding. *Theoretical Computer Science*, 284(2):467–485, 2002.

[3] Hideki Tsuiki. Compact metric spaces as minimal-limit sets in domains of bottomed sequences. *Mathematical Structure in Computer Science*, 14:853–878, 2004.

[4] Hideki Tsuiki, Dyadic subbases and efficiency properties of the induced $\{0, 1, \perp\}^{\omega}$-representations. *Topology Proceedings*,28(2):673–687, 2004.

[5] Haruto Ohta, Hideki Tsuiki, and Shuji Yamada, Independent subbases and non-redundant codings of separable metrizable spaces. submitted.

[6] 立木秀樹 and 山田修司,「力学系により生成される二分的部分基」力学系の研究 — トポロジーと計算機による新展開, 京都大学数理解析研究所講究録 1485, pp. 110-119, 2006.

[7] Klaus Weihrauch, *Computable analysis, an Introduction.* Springer-Verlag, Berlin, 2000.