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Chaotic continua of continuum-wise expansive homeomorphisms

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1 Introduction.

In this note, we consider the following problem:

Problem 1.1. *If $f : X \to X$ is an expansive (or a continuum-wise expansive) homeomorphism of a one-dimensional continuum X, does X contain an indecomposable subcontinuum? Moreover, what kinds of dynamical structures does such an indecomposable continuum admit? Is each chaotic continuum of f indecomposable?*

In this note, we will give some partial answers in the affirmative to the above problem. It is well known that every continuum X with $\dim X \geq 2$ contains an indecomposable subcontimuum. Also there is an expansive homeomorphism f on the 2-dimensional torus T^2 such that T^2 is the only chaotic continuum of f and hence T^2 is the decomposable chaotic continuum of f. In [5] and [6], we investigated chaotic continuum of homeomorphism. We proved the existence of (minimal) chaotic continuum of continuum-wise expansive homeomorphism and we also investigated the indecomposability of chaotic continua and their composants. In fact, we proved that if G is a finite graph and $f : X \to X$ of a G-like continuum X is a continuum-wise expansive homeomorphism, then there is an indecomposable chaotic continuum of f. In [9], Mouron proved the existence of an indecomposable subcontinuum of X for the case that X is a k-cyclic continuum ($k < \infty$) and X admits an expansive homeomorphism. In this note, we define the notion of *closed subset having uncountable handles* and we show that if $f : X \to X$ is a continumm-wise expansive homeomorphism of a continuum X and Z is a minimal chaotic continuum of f, then for each proper closed subset A of Z with $\mathrm{Int}_Z A \neq \phi$, A has uncountable handles in Z. As a corollary, we see that if $f : X \to X$ is a continumm-wise expansive homeomorphism and X does not contain any subcontinuum having uncountable handles, then each minimal chaotic continuum of f is indecomposable. This implies a stronger result than the Mouron's theorem above [9]. In fact, we obtain that if X is a k-cyclic continuum and X admits a continuum-wise expansive homeomorphism f, then each minimal chaotic continuum of f is indecomposable. The proof is different from the methods of the proof of Mouron [9].

2 Expansive homeomorphism and continuum-wise expansive homeomorphism.

All spaces considered in this note are assumed to be separable metric spaces. By a *compactum* we mean a compact metric space. A *continuum* is connected, nondegenerate compactum. A homeomorphism $f : X \to X$ of a compactum X with metric d is called *expansive* ([15]) if there is $c > 0$ such that for any $x, y \in X$ and $x \neq y$, then there is an integer $n \in \mathbb{Z}$ such that

$$d(f^n(x), f^n(y)) > c.$$

A homeomorphism $f : X \to X$ of a compactum X is *continuum-wise expansive* (resp. *positively continuum-wise expansive*) [4] if there is $c > 0$ such that if A is a nondegenerate subcontinuum of X, then there is an integer $n \in \mathbb{Z}$ (resp. a positive integer $n \in \mathbb{N}$) such that

$$\operatorname{diam} f^n(A) > c,$$

where $\operatorname{diam} B = \sup\{d(x,y) | x, y \in B\}$ for a set B. Such a positive number c is called an *expansive constant* for f. Note that each expansive homeomorphism is continuum-wise expansive, but the converse assertion is not true. There are many continuum-wise expansive homeomorphisms which are not expansive (see [4]). These notions have been extensively studied in the area of topological dynamics, ergodic theory and continuum theory (see [1]-[3],[8],[12]-[15]).

The hyperspace 2^X of X is the set all nonempty closed subsets of X with the *Hausdorff metric* d_H. Let

$$C(X) = \{A \in 2^X | \ A \text{ is connected}\}.$$

Note that 2^X and $C(X)$ are compact metric spaces (e.g., see [7] or [11]). For a homeomorphism $f : X \to X$, we define sets of stable and unstable nondegenerate subcontinua of X as follows (see [6]):

$$\mathbf{V}^s (= \mathbf{V}^s_f) = \{A | \ A \text{ is a nondegenerate subcontinuum of } X \text{ such that } \lim_{n\to\infty} \operatorname{diam} f^n(A) = 0\},$$

$$\mathbf{V}^u (= \mathbf{V}^u_f) = \{A | \ A \text{ is a nondegenerate subcontinuum of } X \text{ such that } \lim_{n\to\infty} \operatorname{diam} f^{-n}(A) = 0\}.$$

For each $0 < \delta < \epsilon$, put

$$\mathbf{V}^s(\delta; \epsilon) = \{A \in C(X) | \operatorname{diam} A \geq \delta, \text{ and } \operatorname{diam} f^n(A) \leq \epsilon \text{ for each } n \geq 0\}$$
$$\mathbf{V}^u(\delta; \epsilon) = \{A \in C(X) | \operatorname{diam} A \geq \delta, \text{ and } \operatorname{diam} f^{-n}(A) \leq \epsilon \text{ for each } n \geq 0\}.$$

Similarly, for each closed subset Z of X and $x \in Z$, the *continuum-wise σ-stable sets* $V^\sigma(x; Z)$ $(\sigma = s, u)$ of f are defined as follows:

$$V^s(x; Z) = \{y \in Z | \text{ there is } A \in C(Z) \text{ such that } x, y \in A \text{ and } \lim_{n\to\infty} \operatorname{diam} f^n(A) = 0\},$$

$$V^u(x; Z) = \{y \in Z | \text{ there is } A \in C(Z) \text{ such that } x, y \in A \text{ and } \lim_{n\to\infty} \operatorname{diam} f^{-n}(A) = 0\}.$$

A subcontinuum Z of X is called a *σ-chaotic continuum* of f (where $\sigma = s, u$) if

1. for each $x \in Z$, $V^\sigma(x; Z)$ is dense in Z, and
2. there is $\tau > 0$ such that for each $x \in Z$ and each neighborhood U of x in X, there is $y \in U \cap Z$ such that

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) \geq \tau \text{ in case } \sigma = s, \text{ or}$$

$$\liminf_{n\to\infty} d(f^{-n}(x), f^{-n}(y)) \geq \tau \text{ in case } \sigma = u.$$

A subcontinuum Z of X is called a *minimal σ-chaotic continuum* of f (where $\sigma = s, u$) if Z is a σ-chaotic continuum of f and Z does not contain any proper σ-chaotic continuum of f. In this note, we often abbreviate σ-chaotic continuum to chaotic continuum. Note that $\mathbf{V}^\sigma(\delta;\epsilon)(\sigma = u, s)$ is closed in $C(X)$. Also, note that if $f : X \to X$ is a continuum-wise expansive homeomorphism with an expansive constant $c > 0$, then (1) for each $0 < \delta < \epsilon < c$, $\mathbf{V}^\sigma(\delta;\epsilon) \subset \mathbf{V}^\sigma$, and $\mathbf{V}^\sigma$ is an F_σ-set in $C(X)$, and (2) $V^u(z; Z)$ is a connected F_σ-set containing z, because

$$V^u(z;Z) = \cup_{n=0}^{\infty}(\cup\{A \in C(Z)|\ z \in A,\ \operatorname{diam} f^{-i}(A) \leq \epsilon \text{ for } i \geq n\})(\text{see [4, (2.1)]}).$$

Similarly, $V^s(z; Z)$ is a connected F_σ-set containing z. In [5], we showed that if $f : X \to X$ is a continuum-wise expansive homeomorphism of a compactum X with dim $X > 0$, then there exists a minimal chaotic continuum of f (see [5, (3.6)]). In this case, if Z is a σ-chaotic continuum of f, then the decomposition $\{V^\sigma(z; Z)|\, z \in Z\}$ of Z is an uncountable family of mutually disjoint, dense connected F_σ-sets in Z.

A continuum X is *decomposable* if there are two proper subcontinua A and B of X such that $A \cup B = X$. A continuum X is *indecomposable* if it is not decomposable. Let X be a continuum and let $p \in X$. Then the set

$$c(p) = \{x \in X|\text{ there is a proper subcontinuum } A \text{ of } X \text{containing } p \text{ and } x\}$$

is called the *composant* of X containing p. Note that if X is an indecomposable continuum, then $\{c(p)|\, p \in X\}$ is an uncountable family of mutually disjoint, dense connected F_σ-sets in X. See [7] for some fundamental properties of indecomposable continua and composants. A closed subset A of X has *uncountable handles* if there is a family $\{H_\alpha|\alpha \in \Lambda\}$ of mutually disjoint nondegenerate subcontiua H_α (i.e., $H_\alpha \cap H_\beta = \phi$ for $\alpha \neq \beta$) of X such that each $A \cap H_\alpha (\neq \phi)$ has at least two components and Λ is an uncountable set. A continuum X is *k-cyclic* if for any $\epsilon > 0$, there is a finite open cover $\mathcal{U}$ of X such that mesh$(\mathcal{U}) < \epsilon$ and the nerve $N(\mathcal{U})$ of $\mathcal{U}$ is a one-dimensional polyhedron which has at most k distinct simple closed curves.

3 Results.

Proposition 3.1. *If a continuum X is k-cyclic for some $k < \infty$, then X contains no subcontinuum having uncountable handles.*

Remark. The converse assertion of the above proposition is not true. Hawaiian earring H contains no subcontinuum having uncountable handles and H is not k-cyclic for any $k < \infty$.

Lemma 3.2. (see [5, (3.2)]) *Let $f : X \to X$ be a continuum-wise expansive homeomorphism of a compactum X with an expansive constant $c > 0$, and let $0 < \epsilon < c/2$. Then there is $\epsilon > \delta > 0$ such that if A is a subcontinuum of X with* $\operatorname{diam} A \leq \delta$ *and* $\operatorname{diam} f^m(A) \geq \epsilon$ *for some $m \in \mathbb{Z}$, then one of the following two conditions holds:*

1. *If $m \geq 0$, for each $n \geq m$ and $x \in f^n(A)$, there is a subcontinuum B of A such that $x \in f^n(B), \operatorname{diam} f^j(B) \leq \epsilon$ for $0 \leq j \leq n$ and $\operatorname{diam} f^n(B) = \delta$.*

2. *If $m < 0$, for each $n \geq -m$ and $x \in f^{-n}(A)$, there is a subcontinuum B of A such that $x \in f^{-n}(B), \operatorname{diam} f^{-j}(B) \leq \epsilon$ for $0 \leq j \leq n$, and $\operatorname{diam} f^{-n}(B) = \delta$.*

Lemma 3.3. ([5, (3.3) and (3.4)]) *Let $f : X \to X$ be a continuum-wise expansive homeomorphism of a compactum X with $\dim X > 0$. Then the following are true.*

1. *$\mathbf{V}^u \neq \phi$ or $\mathbf{V}^s \neq \phi$.*

2. *If $\delta > 0$ is as in the above lemma, then for each $\gamma > 0$ there is a natural number $N(\gamma)$ such that if A is a subcontinuum of X with $\operatorname{diam} A \geq \gamma$, then either $\operatorname{diam} f^n(A) \geq \delta$ for each $n \geq N(\gamma)$ or $\operatorname{diam} f^{-n}(A) \geq \delta$ for each $n \geq N(\gamma)$ holds.*

Theorem 3.4. *If $f : X \to X$ is a continuum-wise expansive homeomorphism of a continuum X and Z is a minimal chaotic continuum of f, then for any proper closed subset A of Z with $Int_Z A \neq \phi$, A has uncountable handles. Moreover, Z is decomposable if and only if there exists a proper subcontinuum C of Z with $Int_Z C \neq \phi$ such that C has uncountable handles in Z.*

Corollary 3.5. *Suppose that a continuum X contains no subcontinuum having uncountable handles. If $f : X \to X$ is a continuum-wise expansive homeomorphism of X and Z is a minimal chaotic continuum of f, then Z is indecomposable.*

Corollary 3.6. *If X is a k-cyclic continuum for some $k < \infty$ and X admits a continuum-wise expansive homeomorphism f, then each minimal chaotic continuum of f is indecomposable.*

Next, we consider the following problem.

Problem 3.7. *Suppose that $f : X \to X$ is a continuum-wise expansive homeomorphism of a one-dimensional continuum X and Z is an indecomposable σ-chaotic continuum of f. Does the composant $c(z)$ of Z containing z coincide with $V^\sigma(z; Z)$ for each $z \in Z$?*

We give a partial answer in the affirmative to the problem. A subcontinuum A of X has *uncountable handlebars* if there is a family $\{H_\alpha | \alpha \in \Lambda\}$ of mutually disjoint nondegenerate subcontiua H_α of X such that $H_\alpha - A \neq \phi$, $A \cap H_\alpha (\neq \phi)$ for each $\alpha \in \Lambda$ and Λ is an uncountable set. Note that if a subcontinuum A of X has uncountable handles, then A has uncountable handlebars. Acontinuum X is *k-branched* ($k \in \mathbb{N}$) if for any $\epsilon > 0$, there is a finite open cover $\mathcal{U}$ of X such that mesh$(\mathcal{U}) < \epsilon$ and the nerve $N(\mathcal{U})$ is a one-dimensional polyhedron which has at most k distinct branch points. Note that Hawaiian earring H is a 1-branched continuum.

Proposition 3.8. *If a continuum X is k-branched for some $k < \infty$, then X contains no subcontinuum having uncountable handlebars.*

We need the following lemma.

Lemma 3.9. (Sum theorem of dimension) *If* X_i $(i \in \mathbb{N})$ *are closed subsets of a separable metric space* X *such that* $\dim X_i \leq n$ *and* $X = \bigcup_{i\in\mathbb{N}} X_i$, *then* $\dim X \leq n$.

Theorem 3.10. *Suppose that a continuum* X *contains no subcontinuum having uncountable handlebars. If* $f : X \to X$ *is a continuum-wise expansive homeomorphism, then there is a* σ*-chaotic continuum* Z *of* f *such that* Z *is an indecomposable continuum and for each* $z \in Z$, *the composant* $c(z)$ *of* Z *containing* z *coincides with* $V^\sigma(z; Z)$.

Corollary 3.11. *If* $f : X \to X$ *is a continuum-wise expansive homeomorphism of a* k*-branched continuum* X $(k < \infty)$, *then there is a* σ*-chaotic continuum* Z *of* f $(\sigma = s$ *or* $u)$ *such that* Z *is an indecomposable continuum such that for each* $z \in Z$, *the composant* $c(z)$ *of* Z *containing* z *coincides with* $V^\sigma(z; Z)$.

In [10], Mouron proved that if $f : X \to X$ is an expansive homeomorphism, then X is not tree-like. We will give a more precise result than Mouron's result. We need the following simple lemmas.

Lemma 3.12. *Let* (X, d) *be a metric space and let* $\delta > 0$. *Then for each positive integer* n, *there is a positive number* $\eta = \eta(\delta, n) > 0$ *such that if* A *is any connected subset* M *of* X *with* $\mathrm{diam}(M) \geq \delta$, *then there are distinct points* y_i $(i = 1, 2, ..., n)$ *in* M *such that* $d(y_i, y_j) \geq \eta$ *for* $i \neq j$.

Lemma 3.13. *Let* $f : X \to X$ *be an expansive homeomorphism of a compactum* X *with an expansive constant* $c > 0$. *For each* $\eta > 0$, *there is a positive integer* $n = n(\eta)$ *such that if* $x, y \in X$ *with* $d(x, y) \geq \eta$, *then* $\max\{d(f^i(x), f^i(y))| -n \leq i \leq n\} \geq c$.

Theorem 3.14. *Let* $f : X \to X$ *be an expansive homeomorphism of a continuum* X. *If a subcontinuum* Y *of* X *satisfies the condition* $P_\sigma(y; Y)$ *for some* $y \in Y$, *then* Y *is not a tree-like continuum. In particular, every chaotic continuum of* f *is not tree-like.*

Corollary 3.15. *Suppose that a continuum* X *contains no subcontinuum having uncountable handles. If* $f : X \to X$ *is an expansive homeomorphism of* X *and* Z *is a minimal chaotic continuum of* f, *then* Z *is an indecomposable continuum which is not tree-like.*

Corollary 3.16. *Suppose that a continuum* X *contains no subcontinuum having uncountable handlebars. If* $f : X \to X$ *is an expansive homeomorphism, then there is a* σ*-chaotic continuum* Z *of* f *such that* Z *is not tree-like,* Z *is indecomposable and for each* $z \in Z$, *the composant* $c(z)$ *of* Z *containing* z *coincides with* $V^\sigma(z; Z)$.

Remark. There are many tree-like chaotic continua of continuum-wise expansive homeomorphisms.

References

[1] N, Aoki, *Topological dynamics, in : Topics in general topology* (eds, K. Morita and J. Nagata), Elsevier Science Publishers B. V., (1989), 625-740.

[2] J. F. Jacobson and W. R. Utz, *The nonexistence of expansive homeomorphisms of a closed 2-cell*, Pacific J. Math., 10 (1960), 1319-1321.

[3] H. Kato, *Chaos of continuum-wise expansive homeomorphisms and dynamical properties of sensitive maps of graphs*, Pacific J. Math., 175 (1996), 93-116.

[4] H. Kato, *Continuum-wise expansive homeomorphisms*, Canad. J. Math., 45 (1993), 576-598.

[5] H. Kato, *Chaotic continua of (continuum-wise) expansive homeomorphisms and chaos in the sense of Li and Yorke*, Fund. Math., 145 (1994), 261-279.

[6] H. Kato, *On indecomposability and composants of chaotic continua*, Fund. Math., 150 (1996), 245-253.

[7] K. Kuratowski, *Topology*, Vol. II, Academic Press, New York, 1968.

[8] R. Mañé, *Expansive homeomorphisms and topological dimension*, Trans. Amer. Math. Soc., 252 (1979), 313-319.

[9] C. Mouron, *Expansive homeomorphisms and indecomposability*, Topology Appl. 126(2002), 13-28.

[10] C. Mouron, *Tree-like continua do not admit expansive homeomorphisms*, Proc. Amer. Math. Soc., 130(2002), 3409-3413.

[11] S. B. Nadler, Jr., *Hyperspaces of sets*, Pure and Appl. Math., 49 (Dekker, New York, 1978).

[12] T. O'Brien and W. Reddy, *Each compact orientable surface of positive genus admits an expansive homeomorphism*, Pacific J. Math., 35 (1970), 737-741.

[13] R. V. Plykin, *On the geometry of hyperbolic attractors of smooth cascades*, Russian Math. Surveys, 39 (1984), 85-131.

[14] W. Reddy, *The existence of expansive homeomorphisms of manifolds*, Duke Math. J., 32 (1965), 627-632.

[15] W. Utz, *Unstable homeomorphisms*, Proc. Amer. Math. Soc., 1 (1950), 769-774.