A CHARACTERIZATION OF M-MAPS

YOSHIFUMI KONAMI
(SHIMANE UNIVERSITY)

TAKUO MIWA
(SHIMANE UNIVERSITY)

1. INTRODUCTION

In this paper, all spaces are topological spaces and all maps are continuous. Throughout this paper, \((B, \tau)\) is a base space, \(N(b)\) is the family of all open neighborhoods of \(b \in B\). For undefined terminology and notations, see the papers of references.

Following theorems are known for M-spaces in the category \(TOP\).

**Theorem 1.1.** Let \(X\) be a paracompact Hausdorff space. \(X\) is an M-space if and only if \(X\) is a p-space.

**Theorem 1.2.** Let \(X\) be a paracompact Hausdorff space. \(X\) is a p-space if and only if \(X\) is closely embeddable to a product of a metric space and a compact Hausdorff space.

**Theorem 1.3.** A topological space \(X\) is an M-space if and only if there exist a metric space \(Y\) and a quasi-perfect surjection \(f : X \to Y\).

Bai and Miwa [1] tried to extend these theorems to the fibrewise category \(TOP_B\) and had the following results:

**Theorem 1.4** ([1] Theorem 5.1). A paracompact map \(f : X \to B\) is an M-map if and only if \(f\) is a p-map.

As a partial result corresponds to Theorem 1.2, they have the following:

**Theorem 1.5** ([1] Theorem 6.6). If \(f : X \to B\) is a map such that a preimage-map of an MT-map \(g : Y \to B\) under a perfect morphism \(\lambda : f \to g\), then \(f\) is closely embeddable to a product of \(g\) and a \(T_2\)-compactification \(f' : X' \to B\) of \(f\).

In their paper, the following problem was posed:

**Problem 1.6** ([1] Problem 6.4). Let \(f : X \to B\) be an M-map. Does there exist an MT-map \(g : Y \to B\) and a quasi-perfect morphism \(\lambda : f \to g\)?
In this paper, we define a notion of strong $M$-maps and we show an answer (Theorem 3.6) to this problem for strong $M$-maps.

2. Preliminaries

First, we recall the definition of $MT$-maps (see Definition 2.3) from [3].

**Definition 2.1** ([3] Definition 2.5). A map $f : X \to B$ is *collectionwise prenormal* if for any discrete collection $\{F_s | s \in S\}$ of closed subsets of $X$ and every $b \in B$, there exist $W \in N(b)$ and a discrete collection $\{U_s | s \in S\}$ of open subsets of $X_W$ such that $F_s \cap X_W \subset U_s$.

The map $f$ is said to be *collectionwise normal* if for every $W \in \tau$, the map $f|_{X_W} : X_W \to W$ is collectionwise prenormal.

**Definition 2.2** ([3] Definition 2.8). Let $W_1, W_2, \cdots$ be a sequence of open (in $X$) covers of $X_b$ ($b \in B$).

$\{W_n\}$ is a *$b$-development* if for each $x \in X_b$ and every neighborhood $U(x)$ of $x$ in $X$, there exist $n \in \mathbb{N}$ and $W \in N(b)$ such that $x \in \text{st}(x, W_n) \cap X_W \subset U(x)$.

$f$ has an *$f$-development* if $f$ has a $b$-development for every $b \in B$.

**Definition 2.3** ([3] Definition 2.9). Let $f : X \to B$ be a closed map. $f$ is a *metrizable type map* (shortly, *$MT$-map*) if $f$ is collectionwise normal and has an $f$-development.

In [3] many characterizations of $MT$-maps are given. One of them is as follow:

**Theorem 2.4** ([3] Theorem 2.15). For a continuous map $f : X \to B$ the following are equivalent:

1. $f$ is an $MT$-map,
2. $f$ is a closed $T_0$-map with a normal $f$-development.

3. Main Theorem

To show our main result (Theorem 3.6), we introduce a notion of “strong $MT$-maps”.

**Definition 3.1.** Let $f : X \to B$ be a closed $T_0$-map. $f$ is a *strong $MT$-map* if $f$ has a normal $f$-development $\{\{U_n(b)\}_{n \in \mathbb{N}}| b \in B\}$ satisfying following conditions:

(*): for each $b \in B$ and each $n \in \mathbb{N}$, there exists $W \in N(b)$ such that
(1) $\mathcal{U}_n(b)$ is a covering of $X_W$.
(2) for any $b' \in W$, there exist $W' \in N(b)$ and $n' \in \mathbb{N}$ such that
   (a) $\mathcal{U}_{n'}(b')$ is a covering of $X_{W'}$, 
   (b) $W' \subset W$ and $\mathcal{U}_{n'}(b')|_{X_{W'}} < \mathcal{U}_n(b)$.

The following is trivial.

**Lemma 3.2.** Strong $MT$-maps are $MT$-maps.

Next, we recall the notion of $M$-maps from [1]. We notice that in the definition of an $M$-map, the condition “$T_2$-compactifiable” does not need.

**Definition 3.3** ([1] Definition 4.1). Let $f : X \to B$ be a map. $f$ is an $M$-map if for each $b \in B$, there exists $\{\mathcal{U}_n(b)\}_{n \in \mathbb{N}}$ which is a sequence of open (in $X$) covers of $X_b$ satisfying:

(M1) If $x \in X_b$ and $x_n \in \text{st}(x, \mathcal{U}_n(b)) \cap X_b$ for every $n \in \mathbb{N}$, then the sequence $\{x_n\}_{n \in \mathbb{N}}$ has an accumulation point in $X_b$.

(M2) For every $n \in \mathbb{N}$, $\mathcal{U}_{n+1}(b)$ is a $b$-star refinement of $\mathcal{U}_n(b)$.

We introduce a slightly strengthened notion of $M$-map as follow.

**Definition 3.4.** $f : X \to B$ is a strong $M$-map if for each $b \in B$, there exists $\{\{\mathcal{U}_n(b)\}_{n \in \mathbb{N}}\}_{b \in B}$ where $\{\mathcal{U}_n(b)\}_{n \in \mathbb{N}}$ is a sequence of open (in $X$) coverings of $X_b$ which satisfies following conditions:

(sM1) If for each $x \in X_b$ and $x_{nW} \in \text{st}(x, \mathcal{U}_n(b)) \cap X_W$ for every $n \in \mathbb{N}$ and $W \in N(b)$, then $\{x_{nW}\}$ has an accumulation point in $X_b$.

(sM2) For every $n \in \mathbb{N}$, $\mathcal{U}_{n+1}(b)$ is a $b$-star refinement of $\mathcal{U}_n(b)$.

(sM3) For each $b \in B$ and $n \in \mathbb{N}$, there exists $W \in N(b)$ such that $\mathcal{U}_n(b)$ is a covering of $X_W$ and for every $b' \in W$ there exist $W' \in N(b')$ and $n' \in \mathbb{N}$ such that $\mathcal{U}_{n'}(b')$ is a covering of $X_{W'}$, $W' \subset W$ and $\mathcal{U}_{n'}(b')|_{X_{W'}} < \mathcal{U}_n(b)$.

The following is trivial in a sense of the Definition 3.3.

**Lemma 3.5.** Strong $M$-maps are $M$-maps.

For strong $M$-maps, we have a characterization as follow:

**Theorem 3.6.** A closed $T_1$-map $f : X \to B$ is a strong $M$-map if and only if there exist a strong $MT$-map $g : Y \to B$ and a quasi-perfect surjective morphism $\lambda : f \to g$.

**Proof.** Let $\lambda$ be a quasi-perfect morphism from $f$ onto a strong $MT$-map $g : Y \to B$. Let $\{\mathcal{V}_i(b) : i \in \mathbb{N}\}_{b \in B}$ be a normal $g$-development on $g$ satisfying the condition (*) of Definition 3.1.
Let $\mathcal{U}_i(b) := \lambda^{-1}(\mathcal{V}_i), i \in \mathbb{N}$, then $\{\mathcal{U}_i(b)|i \in \mathbb{N}\}$ forms a normal sequence of open (in $X$) coverings of $X_b$ satisfying the conditions (sM2) and (sM3).

To prove the condition (sM1), let $x \in X_b$ and $x_{iw} \in \text{st}(x, \mathcal{U}_i(b)) \cap X_W$ for every $i \in \mathbb{N}$ where $W \in N(b)$. Since $\lambda(x_{iw}) \in \text{st}(\lambda(x), \mathcal{V}_i) \cap Y_W$, $\{\lambda(x_{iw})\}$ has an accumulation point $y \in Y_b$. Assume that $\{x_{iw}\} \cap \lambda^{-1}(y) = \emptyset$. Since $\lambda$ is closed, there exists $V \in N(y)$ such that $\{x_{iw}\} \cap \lambda^{-1}(V) = \emptyset$. Thus $V \cap \{\lambda(x_{iw})\} = \emptyset$. This contradicts to $y \in \{\lambda(x_{iw})\}$. Therefore $\{x_{iw}\}$ has an accumulation point in $X_b$, which shows (sM1).

Conversely, let $f$ be a strong $M$-map and $\{\mathcal{U}_i(b)|i \in \mathbb{N}\}$ a normal sequence of open coverings of $X_b$ satisfying (sM1)–(sM3).

We put $\Phi = \{\mathcal{U}_i(b)|_{n\in \mathbb{N}}|b \in B\}$. We shall denote that $(X, \Phi)$ is a topological space with a topology defined by taking the following system $B(x)$ as a neighborhood basis for $x \in X_b$:

$B(x) = \{\text{st}(x, \mathcal{U}_i(b)) \cap X_W|i \in \mathbb{N}, W \in N(b)\}$.

We shall show that $B(x), x \in X_b(b \in B)$ is a neighborhood basis. First, it is clear that if $x \in X_b(b \in B)$, then $B(x) \neq \emptyset$ and $x \in U$ for every $U \in B(x)$. Next, let $U_1, U_2 \in B(x)$. Then $U_1 = \text{st}(x, \mathcal{U}_i(b)) \cap X_W, U_2 = \text{st}(x, \mathcal{U}_j(b)) \cap X_W$, for some $i, j \in \mathbb{N}$ and $W, W' \in N(b)$. We can assume $i < j$. If so, $\mathcal{U}_j(b) \subset^* \mathcal{U}_i(b)$. Put $W'' = W \cap W'$. Then we have that $\text{st}(x, \mathcal{U}_j(b)) \cap X_{W''} \subset B(x)$ and $\text{st}(x, \mathcal{U}_i(b)) \cap X_{W''} \subset U_1 \cap U_2$. Let $V = \text{st}(x, \mathcal{U}_{i+1}(b)) \cap X_W$. It is clear $V \in B(x)$. We shall show that for every $y \in V$ there exists $V' \in B(y)$ such that $V' \subset U$. Let $b' = f(y)$, then $b' \in W$. From the conditions (sM2) and (sM3) there exist $W' \in N(b')$ and $j \in \mathbb{N}$ such that $W' \subset W$ and $\mathcal{U}_j(b')|_{X_{W'}} \subset^* \mathcal{U}_{i+1}(b)$. Let $V' = \text{st}(y, \mathcal{U}_j(b')) \cap X_{W'}$. We shall show $V' \subset U$. If $z \in V'$, then there exists $V_1 \in \mathcal{U}_j(b')$ such that $y, z \in V_1 \cap X_{W'}$. Hence $y, z \in \text{st}(V_1 \cap X_{W'}, \mathcal{U}_j(b')|_{X_{W'}}) \subset V_2$ for some $V_2 \in \mathcal{U}_{i+1}(b)$. Since $y \in V$, there exists $V_3 \in \mathcal{U}_{i+1}(b)$ such that $x, y \in V_3 \cap X_{W'}$. Note $V_2 \cap V_3 \neq \emptyset$, we have $x, z \in \text{st}(V_3, \mathcal{U}_{i+1}(b)) \subset U_2$ for some $U_2 \in \mathcal{U}_i(b)$. Hence $z \in \text{st}(x, \mathcal{U}_i(b)) \cap X_{W} = U$.

Thus we obtain a new topological space $(X, \Phi)$. Let define a function $\bar{f} : (X, \Phi) \rightarrow B, x \mapsto f(x)$, then it is clear that $\bar{f}$ is continuous.

For any $A \subset X$ we put

$\text{Int}(A; \Phi) = \{x \in X|\exists i \in \mathbb{N}, \exists W \in N(f(x)); (\text{st}(x, \mathcal{U}_i(f(x))) \cap X_W \subset A)\}$.

Then $\text{Int}(A; \Phi)$ is open in $(X, \Phi)$.

Define a relation $\sim$ by
$x \sim x'$ if and only if $x, x' \in X_b$ for some $b \in B$ and $x' \in \bigcap_{i=1}^{\infty} \text{st}(x, \mathcal{U}_i(b))$.

Then it is easy to see that $\sim$ is an equivalence relation on $\tilde{f}$. Let $Y$ be a fibrewise quotient space $X/\sim$ obtained from $(X, \Phi)$ and $\nu$ be the quotient map of $(X, \Phi)$ onto $Y$ (Note that $Y$ has the projection $g : Y \to B$ which is defined as $\nu(x) \mapsto \tilde{f}(x)$). Then we shall show that for $A \subset X$,

$$** \quad \nu^{-1}(\nu(\text{Int}(A; \Phi))) = \text{Int}(A; \Phi).$$

In fact, if $x \in \nu^{-1}(\nu(\text{Int}(A; \Phi)))$, then $\nu(x) \in \nu(\text{Int}(A; \phi))$. Therefore there exists $x' \in \text{Int}(A; \Phi)$ such that $x \sim x'$. Because $x' \in \text{Int}(A; \Phi)$, there exist $i \in \mathbb{N}$ and $W \in N(b)$ such that $\text{st}(x', \mathcal{U}_i(b)) \cap X_W \subset A$ where $b = \tilde{f}(x')$. Since $x \sim x'$, $x \in \text{st}(x', \mathcal{U}_{i+1}(b))$ thus

$$\text{st}(x, \mathcal{U}_{i+1}(b)) \cap X_W \subset \text{st}(\text{st}(x', \mathcal{U}_{i+1}(b)), \mathcal{U}_{i+1}(b)) \cap X_W$$
$$\subset \text{st}(x', \mathcal{U}_i(b)) \cap X_W$$
$$\subset A.$$

This shows $x \in \text{Int}(A; \Phi)$. Converse inclusion is clear.

The equality $(**)$ means $\nu$ is an open continuous morphism from $(X, \Phi)$ onto $Y$. Let $\iota$ be an identity map from $X$ onto $(X, \Phi)$. It is clear that $\iota$ is continuous. We put $\lambda = \nu \circ \iota$. Now we shall show that (1) $g : Y \to B$ is a strong MT-map, where $g(\nu(x)) = f(x)$, and (2) $\lambda : f \to g$ is quasi-perfect.

(1) First, since $f$ is closed, so is $g$. From the condition (sM3) of the map $f$ it is easy to show that $g$ satisfies the condition $(*)$ of Definition 3.1.

Next, let $\mathcal{V}_n(b) := \{\nu(U) | U \in \mathcal{U}_n(b)\}$, then it is clear that $\{\mathcal{V}_n(b) | n \in \mathbb{N}\}$ is a normal sequence of open (in $X$) coverings of $Y_b$. For $y = \nu(x) \in Y_b$ and a neighborhood $V(y)$ of $y$ in $Y$, there exist $i \in \mathbb{N}$ and $W \in N(b)$ such that $\text{st}(x, \mathcal{U}_i(b)) \cap X_W \subset \nu^{-1}(V(y))$. We shall show that $\text{st}(y, \mathcal{V}_i(b)) \cap Y_W \subset V(y)$. In fact, let $y' \in \text{st}(y, \mathcal{V}_i(b)) \cap Y_W$, then there exists $U \in \mathcal{U}_i(b)$ such that $y, y' \in \nu(U)$. There exists $x' \in U$ such that $\nu(x') = y'.$ Hence $x, x' \in U$. Thus $x' \in \text{st}(x, \mathcal{U}_i(b)) \subset \nu^{-1}(V(y))$ and we have $y' = \nu(x') \in V(y)$. Therefore $\text{st}(y, \mathcal{V}_i(b)) \cap Y_W \subset V(y)$. Thus $\{\mathcal{V}_i(b)\}$ is a normal $b$-development.

Last, $g$ is a $T_1$-map. Because if $\nu(x) \neq \nu(x')$ and $x, x' \in X_b$, then there exists $i$ such that $x' \notin \text{st}(x, \mathcal{U}_i(b))$. Thus $\nu(x') \notin \text{st}(\nu(x), \mathcal{V}_i(b))$. Hence $g$ is an MT-map by [3] Theorem 2.15.

(2) First, we shall prove closedness of $\lambda$. 


Let $A \subset X$ be closed and $y_0 \in \overline{\lambda(A)}$, $y_0 \in Y_b$. Let $x_0 \in \lambda^{-1}(y_0)$. Since $\text{st}(x_0, \mathcal{U}_i(b)) \subset \text{st}(x_0, \mathcal{U}_i(b))$, we have
\[ \text{st}(x_0, \mathcal{U}_{i+1}(b)) \subset \text{Int}(\text{st}(x_0, \mathcal{U}_i(b)); \Phi) \subset \text{st}(x_0, \mathcal{U}_i(b)). \]
Hence $\nu(\text{Int}(\text{st}(x_0, \mathcal{U}_i(b)); \Phi)) \cap Y_{W_j} \cap \lambda(A) \neq \emptyset$. Thus
\[ \nu^{-1}(\nu(\text{Int}(\text{st}(x_0, \mathcal{U}_i(b)); \Phi))) \cap X_{W_j} \cap A \neq \emptyset. \]
This means $\text{st}(x_0, \mathcal{U}_i(b)) \cap X_{W_j} \cap A \neq \emptyset$ for $i = 1, 2, \ldots$. Now we select $x_{iW} \in \text{st}(x_0, \mathcal{U}_i(b)) \cap X_W \cap A$ for $i = 1, 2, \ldots, W \in N(b)$. By (sM1) there exists an accumulation point $x' \in X_b$. Since $A$ is closed, we have $x' \in A$ and $x' \in \text{st}(x_0, \mathcal{U}_i(b)), i = 1, 2, \ldots$. Thus $x_0 \sim x'$. This shows $y_0 = \nu(x_0) = \nu(x') \in \lambda(A)$. Hence $\lambda$ is closed.

Next, let $\{x_i\}$ be a sequence in $\nu^{-1}(y_0)$, $y_0 \in Y_b$. Let $x_0 \in \nu^{-1}(y_0)$. Then $x_i \in \text{st}(x_0, \mathcal{U}_i(b)) \cap X_b$ for $i, j = 1, 2, \ldots$. Hence $\{x_i\}$ has an accumulation point in $\nu^{-1}(y_0)$ by (sM1). This shows $\nu^{-1}(y_0)$ is countably compact. Therefore $\lambda$ is quasi-perfect morphism. This completes the proof.

**REFERENCES**


YOSHIFUMI KONAMI, SHIMANE UNIVERSITY, POST DOCTORIAL RESEARCH.

E-mail address: yoshi@math-konami.com

TAKUO MIWA, SHIMANE UNIVERSITY

E-mail address: takuo-miwa@mable.ne.jp