数理解析研究所講究録
第 1681 巻 2010 年 79-84

A CHARACTARIZATION OF M-MAPS

YOSHIFUMI KONAMI
(SHIMANE UNIVERSTY)

TAKUO MIWA
(SHIMANE UNIVERSITY)

1. Introduction

In this paper, all spaces are topological spaces and all maps are continuous. Throughout this paper, (B, τ) is a base space, $N(b)$ is the family of all open neighborhoods of $b \in B$. For undefined terminology and notations, see the papers of references.

Following theorems are known for M-spaces in the category TOP.

Theorem 1.1. Let X be a paracompact Hausdorff space. X is an M-space if and only if X is a p-space.

Theorem 1.2. Let X be a paracompact Hausdorff space. X is a p-space if and only if X is closedly embeddable to a product of a metric space and a compact Hausdorff space.

Theorem 1.3. A topological space X is an M-space if and only if there exist a metric space Y and a quasi-perfect surjection $f : X \to Y$.

Bai and Miwa [1] tried to extend these theorems to the fibrewise category TOP_B and had the following results:

Theorem 1.4 ([1] Theorem 5.1). A paracompact map $f : X \to B$ is an M-map if and only if f is a p-map.

As a partial result corresponds to Theorem 1.2, they have the following:

Theorem 1.5 ([1] Theorem 6.6). If $f : X \to B$ is a map such that a preimage-map of an MT-map $g : Y \to B$ under a perfect morphism $\lambda : f \to g$, then f is closedly embeddable to a product of g and a T_2-compactification $f' : X' \to B$ of f.

In their paper, the following problem was posed:

Problem 1.6 ([1] Problem 6.4). Let $f : X \to B$ be an M-map. Does there exist an MT-map $g : Y \to B$ and a quasi-perfect morphism $\lambda : f \to g$?

In this paper, we define a notion of strong M-maps and we show an answer (Theorem 3.6) to this problem for strong M-maps..

2. Preliminaries

First, we recall the definition of MT-maps (see Definition 2.3) from [3].

Definition 2.1 ([3] Definition 2.5). A map $f : X \to B$ is *collectionwise prenormal* if for any discrete collection $\{F_s | s \in \mathcal{S}\}$ of closed subsets of X and every $b \in B$, there exist $W \in N(b)$ and a discrete collection $\{U_s | s \in \mathcal{S}\}$ of open subsets of X_W such that $F_s \cap X_W \subset U_s$.

The map f is said to be *collectionwise normal* if for every $W \in \tau$, the map $f|_{X_W} : X_W \to W$ is collectionwise prenormal.

Definition 2.2 ([3] Definition 2.8). Let $\mathcal{W}_1, \mathcal{W}_2, \cdots$ be a sequence of open (in X) covers of X_b $(b \in B)$.

$\{\mathcal{W}_n\}$ is a *b-development* if for each $x \in X_b$ and every neighborhood $U(x)$ of x in X, there exist $n \in \mathbb{N}$ and $W \in N(b)$ such that $x \in \mathrm{st}(x, \mathcal{W}_n) \cap X_W \subset U(x)$.

f has an *f-development* if f has a b-development for every $b \in B$.

Definition 2.3 ([3] Definition 2.9). Let $f : X \to B$ be a closed map. f is a *metrizable type map* (shortly, *MT-map*) if f is collectionwise normal and has an f-development.

In [3] many characterizations of MT-maps are given. One of them is as follow:

Theorem 2.4 ([3] Theorem 2.15). For a continuous map $f : X \to B$ the following are equivalent:

(1) f is an MT-map,
(2) f is a closed T_0-map with a normal f-development.

3. Main Theorem

To show our main result (Theorem 3.6), we introduce a notion of "strong MT-maps".

Definition 3.1. Let $f : X \to B$ be a closed T_0-map. f is a *strong MT-map* if f has a normal f-development $\{\{\mathcal{U}_n(b)\}_{n \in \mathbb{N}} | b \in B\}$ satisfying following conditions:

(*): for each $b \in B$ and each $n \in \mathbb{N}$, there exists $W \in N(b)$ such that

(1) $\mathcal{U}_n(b)$ is a covering of X_W,
(2) for any $b' \in W$, there exist $W' \in N(b)$ and $n' \in \mathbb{N}$ such that
  (a) $\mathcal{U}_{n'}(b')$ is a covering of $X_{W'}$,
  (b) $W' \subset W$ and $\mathcal{U}_{n'}(b')|_{X_{W'}} < \mathcal{U}_n(b)$.

The following is trivial.

Lemma 3.2. Strong MT-maps are MT-maps.

Next, we recall the notion of M-maps from [1]. We notice that in the definition of an M-map, the condition "T_2-compactifiable" does not need.

Definition 3.3 ([1] Definition 4.1). Let $f : X \to B$ be a map. f is an *M-map* if for each $b \in B$, there exists $\{\mathcal{U}_n(b)\}_{n\in\mathbb{N}}$ which is a sequence of open (in X) covers of X_b satisfying:

(M1) If $x \in X_b$ and $x_n \in \mathrm{st}(x, \mathcal{U}_n(b)) \cap X_b$ for every $n \in \mathbb{N}$, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ has an accumulation point in X_b.
(M2) For every $n \in \mathbb{N}$, $\mathcal{U}_{n+1}(b)$ is a b-star refinement of $\mathcal{U}_n(b)$.

We introduce a slightly strengthened notion of M-map as follow.

Definition 3.4. $f : X \to B$ is a *strong M-map* if for each $b \in B$, there exists $\{\{\mathcal{U}_n(b)\}_{n\in\mathbb{N}} | b \in B\}$ where $\{\mathcal{U}_n(b)\}_{n\in\mathbb{N}}$ is a sequence of open (in X) coverings of X_b which satisfies following conditions:

(sM1) If for each $x \in X_b$ and $x_{nW} \in \mathrm{st}(x, \mathcal{U}_n(b)) \cap X_W$ for every $n \in \mathbb{N}$ and $W \in N(b)$, then $\{x_{nW}\}$ has an accumulation point in X_b.
(sM2) For every $n \in \mathbb{N}$, $\mathcal{U}_{n+1}(b)$ is a b-star refinement of $\mathcal{U}_n(b)$.
(sM3) For each $b \in B$ and $n \in \mathbb{N}$, there exists $W \in N(b)$ such that $\mathcal{U}_n(b)$ is a covering of X_W and for every $b' \in W$ there exist $W' \in N(b')$ and $n' \in \mathbb{N}$ such that $\mathcal{U}_{n'}(b')$ is a covering of $X_{W'}$, $W' \subset W$ and
$$\mathcal{U}_{n'}(b')|_{X_{W'}} < \mathcal{U}_n(b)$$

The following is trivial in a sense of the Definition 3.3.

Lemma 3.5. Strong M-maps are M-maps.

For strong M-maps, we have a characterization as follow:

Theorem 3.6. A closed T_1-map $f : X \to B$ is a strong M-map if and only if there exist a strong MT-map $g : Y \to B$ and a quasi-perfect surjective morphism $\lambda : f \to g$.

Proof. Let λ be a quasi-perfect morphism from f onto a strong MT-map $g : Y \to B$. Let $\{\{\mathcal{V}_i(b) | i \in \mathbb{N}\} | b \in B\}$ be a normal g-development on g satisfying the condition (*) of Definition 3.1.

Let $\mathcal{U}_i(b) := \lambda^{-1}(\mathcal{V}_i), i \in \mathbb{N}$, then $\{U_i(b)|i \in \mathbb{N}\}$ forms a normal sequence of open (in X) coverings of X_b satisfying the conditions (sM2) and (sM3).

To prove the condition (sM1), let $x \in X_b$ and $x_{iW} \in \mathrm{st}(x, \mathcal{U}_i(b)) \cap X_W$ for every $i \in \mathbb{N}$ where $W \in N(b)$. Since $\lambda(x_{iW}) \in \mathrm{st}(\lambda(x), \mathcal{V}_i) \cap Y_W$, $\{\lambda(x_{iW})\}$ has an accumulation point $y \in Y_b$. Assume that $\overline{\{x_{iW}\}} \cap \lambda^{-1}(y) = \emptyset$. Since λ is closed, there exists $V \in N(y)$ such that $\overline{\{x_{iW}\}} \cap \lambda^{-1}(V) = \emptyset$. Thus $V \cap \{\lambda(x_{iW})\} = \emptyset$. This contradicts to $y \in \overline{\{\lambda(x_{iW})\}}$. Therefore $\{x_{iW}\}$ has an accumulation point in X_b, which shows (sM1).

Conversely, let f be a strong M-map and $\{\mathcal{U}_i(b)|i \in \mathbb{N}\}$ a normal sequence of open coverings of X_b satisfying (sM1)–(sM3).

We put $\Phi = \{\{\mathcal{U}_i(b)\}_{n\in\mathbb{N}}|b \in B\}$. We shall denote that (X, Φ) is a topological space with a topology defined by taking the following system $\mathcal{B}(x)$ as a neighborhood basis for $x \in X_b$:

$$\mathcal{B}(x) = \{\mathrm{st}(x, \mathcal{U}_i(b)) \cap X_W|i \in \mathbb{N}, W \in N(b)\}.$$

We shall show that $\mathcal{B}(x), x \in X_b(b \in B)$ is a neighborhood basis. First, it is clear that if $x \in X_b(b \in B)$, then $\mathcal{B}(x) \neq \emptyset$ and $x \in U$ for every $U \in \mathcal{B}(x)$. Next, let $U_1, U_2 \in \mathcal{B}(x)$. Then $U_1 = \mathrm{st}(x, \mathcal{U}_i(b)) \cap X_W, U_2 = \mathrm{st}(x, \mathcal{U}_j(b)) \cap X_{W'}$ for some $i, j \in \mathbb{N}$ and $W, W' \in N(b)$. We can assume $i < j$. If so, $\mathcal{U}_j(b) <^* \mathcal{U}_i(b)$. Put $W'' = W \cap W'$. Then we have that $\mathrm{st}(x, \mathcal{U}_j(b)) \cap X_{W''} \in \mathcal{B}(x)$ and $\mathrm{st}(x, \mathcal{U}_j(b)) \cap X_{W''} \subset U_1 \cap U_2$. Last, let $U = \mathrm{st}(x, \mathcal{U}_i(b)) \cap X_W \in \mathcal{B}(x)$ for some $i \in \mathbb{N}$ and $W \in N(b)$. Let $V = \mathrm{st}(x, \mathcal{U}_{i+1}(b)) \cap X_W$. It is clear $V \in \mathcal{B}(x)$. We shall show that for every $y \in V$ there exists $V' \in \mathcal{B}(y)$ such that $V' \subset U$. Let $b' = f(y)$, then $b' \in W$. From the conditions (sM2) and (sM3) there exist $W' \in N(b')$ and $j \in \mathbb{N}$ such that $W' \subset W$ and $\mathcal{U}_j(b')|_{X_{W'}} <^* \mathcal{U}_{i+1}(b)$. Let $V' = \mathrm{st}(y, \mathcal{U}_j(b')) \cap X_{W'}$. We shall show $V' \subset U$. If $z \in V'$, then there exists $V_1 \in \mathcal{U}_j(b')$ such that $y, z \in V_1 \cap X_{W'}$. Hence $y, z \in \mathrm{st}(V_1 \cap X_{W'}, \mathcal{U}_j(b')|_{X_{W'}}) \subset V_2$ for some $V_2 \in \mathcal{U}_{i+1}(b)$. Since $y \in V$, there exists $V_3 \in \mathcal{U}_{i+1}(b)$ such that $x, y \in V_3 \cap X_W$. Noting $V_2 \cap V_3 \neq \emptyset$, we have $x, z \in \mathrm{st}(V_3, \mathcal{U}_{i+1}(b)) \subset U_2$ for some $U_2 \in \mathcal{U}_i(b)$. Hence $z \in \mathrm{st}(x, \mathcal{U}_i(b)) \cap X_W = U$.

Thus we obtain a new topological space (X, Φ). Let define a function $\tilde{f} : (X, \Phi) \to B, x \mapsto f(x)$, then it is clear that $\tilde{f}$ is continuous.

For any $A \subset X$ we put

$$\mathrm{Int}(A; \Phi) = \{x \in X|\exists i \in \mathbb{N}, \exists W \in N(f(x)); (\mathrm{st}(x, \mathcal{U}_i(f(x))) \cap X_W \subset A)\}.$$

Then $\mathrm{Int}(A; \Phi)$ is open in (X, Φ).

Define a relation $\sim$ by

$x \sim x'$ if and only if $x, x' \in X_b$ for some $b \in B$ and $x' \in \bigcap_{i=1}^{\infty} \mathrm{st}(x, \mathcal{U}_i(b))$.

Then it is easy to see that $\sim$ is an equivalence relation on $\tilde{f}$. Let Y be a fibrewise quotient space $X/\sim$ obtained from (X, Φ) and ν be the quotient map of (X, Φ) onto Y (Note that Y has the projection $g : Y \to B$ which is defined as $\nu(x) \mapsto \tilde{f}(x)$). Then we shall show that for $A \subset X$

$$(**) \quad \nu^{-1}(\nu(\mathrm{Int}(A; \Phi))) = \mathrm{Int}(A; \Phi).$$

In fact, if $x \in \nu^{-1}(\nu(\mathrm{Int}(A; \Phi)))$, then $\nu(x) \in \nu(\mathrm{Int}(A; \phi))$. Therefore there exists $x' \in \mathrm{Int}(A; \Phi)$ such that $x \sim x'$. Because $x' \in \mathrm{Int}(A; \Phi)$, there exist $i \in \mathbb{N}$ and $W \in N(b)$ such that $\mathrm{st}(x', \mathcal{U}_i(b)) \cap X_W \subset A$ where $b = \tilde{f}(x')$. Since $x \sim x'$, $x \in \mathrm{st}(x', \mathcal{U}_{i+1}(b))$ thus

$$\begin{aligned} \mathrm{st}(x, \mathcal{U}_{i+1}(b)) \cap X_W &\subset \mathrm{st}(\mathrm{st}(x', \mathcal{U}_{i+1}(b)), \mathcal{U}_{i+1}(b)) \cap X_W \\ &\subset \mathrm{st}(x', \mathcal{U}_i(b)) \cap X_W \\ &\subset A. \end{aligned}$$

This shows $x \in \mathrm{Int}(A; \Phi)$. Converse inclusion is clear.

The equality (**) means ν is an open continuous morphism from (X, Φ) onto Y. Let ι be an identity map from X onto (X, Φ). It is clear that ι is continuous. We put $\lambda = \nu \circ \iota$. Now we shall show that (1) $g : Y \to B$ is a strong MT-map, where $g(\nu(x)) = f(x)$, and (2) $\lambda : f \to g$ is quasi-perfect.

(1) First, since f is closed, so is g. From the condition (sM3) of the map f it is easy to show that g satisfies the condition (*) of Definition 3.1.

Next, let $\mathcal{V}_n(b) := \{\nu(U) | U \in \mathcal{U}_n(b)\}$, then it is clear that $\{\mathcal{V}_n(b) | n \in \mathbb{N}\}$ is a normal sequence of open (in X) coverings of Y_b. For $y = \nu(x) \in Y_b$ and a neighborhood $V(y)$ of y in Y, there exist $i \in \mathbb{N}$ and $W \in N(b)$ such that $\mathrm{st}(x, \mathcal{U}_i(b)) \cap X_W \subset \nu^{-1}(V(y))$. We shall show that $\mathrm{st}(y, \mathcal{V}_i(b)) \cap Y_W \subset V(y)$. In fact, let $y' \in \mathrm{st}(y, \mathcal{V}_i(b)) \cap Y_W$, then there exists $U \in \mathcal{U}_i(b)$ such that $y, y' \in \nu(U)$. There exists $x' \in U$ such that $\nu(x') = y'$. Hence $x, x' \in U$. Thus $x' \in \mathrm{st}(x, \mathcal{U}_i(b)) \subset \nu^{-1}(V(y))$ and we have $y' = \nu(x') \in V(y)$. Therefore $\mathrm{st}(y, \mathcal{V}_i(b)) \cap Y_W \subset V(y)$. Thus $\{\mathcal{V}_i(b)\}$ is a normal b-development.

Last, g is a T_1-map. Because if $\nu(x) \neq \nu(x')$ and $x, x' \in X_b$, then there exists i such that $x' \notin \mathrm{st}(x, \mathcal{U}_i(b))$. Thus $\nu(x') \notin \mathrm{st}(\nu(x), \mathcal{V}_i(b))$. Hence g is an MT-map by [3] Theorem 2.15.

(2) First, we shall prove closedness of λ.

Let $A \subset X$ be closed and $y_0 \in \overline{\lambda(A)}$, $y_0 \in Y_b$. Let $x_0 \in \lambda^{-1}(y_0)$. Since $\mathrm{st}(\mathrm{st}(x_0, \mathcal{U}_{i+1}(b)), \mathcal{U}_{i+1}(b)) \subset \mathrm{st}(x_0, \mathcal{U}_i(b))$, we have

$$\mathrm{st}(x_0, \mathcal{U}_{i+1}(b)) \subset \mathrm{Int}(\mathrm{st}(x_0, \mathcal{U}_i(b)); \Phi) \subset \mathrm{st}(x_0, \mathcal{U}_i(b)).$$

Hence $\nu(\mathrm{Int}(\mathrm{st}(x_0, \mathcal{U}_i(b)); \Phi)) \cap Y_{W_j} \cap \lambda(A) \neq \emptyset$. Thus

$$\nu^{-1}(\nu(\mathrm{Int}(\mathrm{st}(x_0, \mathcal{U}_i(b)); \Phi))) \cap X_{W_j} \cap A \neq \emptyset.$$

This means $\mathrm{st}(x_0, \mathcal{U}_i(b)) \cap X_{W_j} \cap A \neq \emptyset$ for $i = 1, 2, \cdots$. Now we select $x_{iW} \in \mathrm{st}(x_0, \mathcal{U}_i(b)) \cap X_W \cap A$ for $i = 1, 2, \cdots, W \in N(b)$. By (sM1) there exists an accumulation point $x' \in X_b$. Since A is closed, we have $x' \in A$ and $x' \in \mathrm{st}(x_0, \mathcal{U}_i(b)), i = 1, 2, \cdots$. Thus $x_0 \sim x'$. This shows $y_0 = \nu(x_0) = \nu(x') \in \lambda(A)$. Hence λ is closed.

Next, let $\{x_i\}$ be a sequence in $\nu^{-1}(y_0)$, $y_0 \in Y_b$. Let $x_0 \in \nu^{-1}(y_0)$. Then $x_i \in \mathrm{st}(x_0, \mathcal{U}_j(b)) \cap X_b$ for $i, j = 1, 2, \cdots$. Hence $\{x_i\}$ has an accumulation point in $\nu^{-1}(y_0)$ by (sM1). This shows $\nu^{-1}(y_0)$ is countably compact. Therefore λ is quasi-perfect morphism. This completes the proof. $\square$

References

[1] Y.F.Bai and T.Miwa, On p-maps and M-maps, Tsukuba J. Math., 32(2008), 179–196.

[2] D.Buhagiar, Paracompact maps, Q and A in General Topology, 15(1997), 203–223.

[3] D.Buhagiar, T.Miwa, B.A.Pasynkov, On metrizable type (MT-)maps and spaces, Topology and its Appl., 96(1999), 31–51.

[4] R.Engelking, *General Topology*, Heldermann, Berlin, rev. ed., 1989.

[5] I.M.James, *Fibrewise Topology*, Cambridge Univ. Press, Cambridge, 1989.

Yoshifumi Konami, Shimane University, Post Doctorial Research.
E-mail address: `yoshi@math-konami.com`

Takuo Miwa, Shimane University
E-mail address: `takuo-miwa@mable.ne.jp`