数理解析研究所講究録
第 1682 巻 2010 年 25-32

Sum the Multiplicative Odds to One and Stop

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1 Introduction

Let n be a given positive integer, and suppose that n independent Bernoulli trials are performed one at a time, each of which results in a *success* or a *failure*. That is, if we let X_j equal 1 if the jth trial is a success and 0 if it is a failure, then $X_1, X_2, \ldots, X_n$ are independent Bernoulli random variables that are observed sequentially. When we seek an optimal stopping rule of this sequential observation problem with the objective of maximizing the probability of stopping on the last success, Bruss(2000) gave an elegant solution given below, where $p_j = P\{X_j = 1\}, q_j = 1 - p_j$, and $r_j = p_j/q_j$ represents the odds of success on the jth trial (if $p_j = 1$, r_j is taken to be ∞).

Bruss Theorem (Sum-the-Odds Theorem). For the above stopping problem, the optimal rule stops on the first success $X_k = 1$ with $k \geq s$, if any, where

$$s = \min\left\{k \geq 1 : \sum_{j=k+1}^{n} r_j \leq 1\right\}.$$

Moreover, the maximal probability of win (i.e., achieving the objective) is

$$v = \left(\prod_{j=s}^{n} q_j\right)\left(\sum_{j=s}^{n} r_j\right).$$

The optimal rule (1) has a nice interpretation, i.e., it stops on the first success for which the sum of the odds of success for the future trials is less than or equal to 1 (we are indifferent between stopping and continuing if the sum of the odds is equal to 1). We refer to this result as the *Sum-the-Odds Theorem* (called simply STOT) according to Ferguson (2008). Bruss (2000) started his argument, raising a question of guessing correctly the last "6" when a fair die is tossed a fixed nmber n of times. The STOT answers this question immediately. But what about the question of guessing correctly any one of the last two "6", i.e., guessing either the last "6" or the second last "6" ? This question seems a natural extension of the Bruss question, but the STOT cannot answer this question because the optimality criterion is different. This paper attempts to answer the problems of this kind. The criterion we adopt here is more generally described as maximizing the probability of stopping on any of the last m successes for a predetermined m (we assume $n > m$ unless otherwise specified, because, for $n \leq m$, the optimal rule evidently stops on the first success). The optimal rule of this problem also has a nice interpretation. That is, it can be shown that the optimal rule stops on the first success for which the sum of the m-fold multiplicative odds of success for the future trials is less than or equal to 1, if we define the *j-fold multiplicative odds* of successes on the *kth* trial by

$$R_{k,j} = \sum_{k \leq i_1 < i_2 < \cdots < i_j \leq n} r_{i_1} r_{i_2} \cdots r_{i_j}$$

for $1 \leq j \leq n-k+1$ and $R_{k,j} = 0$ for $j > n-k+1$. More explicitly we have

Theorem 1.1. (Sum-the-Multiplicative-Odds Theorem). For the stopping problem of maximizing the probability of stopping on any of the last m successes in n independent Bernoulli trials, the optimal rule stops on the first success $X_k = 1$ with $k \geq s_m$, if any, where

$$s_m = \min\{k \geq 1 : R_{k+1,m} \leq 1\}.$$

Moreover, the maximal probability of win is

$$v_m = \left(\prod_{j=s_m}^{n} q_j\right)\left(\sum_{j=1}^{m} R_{s_m,j}\right).$$

We call Theorem 1.1 *Sum-the-Multiplicative-Odds Theorem* (called simply STMOT) whose proof will be given in a more generality in Section 2. The STOT has been extended into other directions by Ferguson (2008). We show in Section 2 that the Ferguson extension can be also made to the STMOT. In Section 3, we apply the STMOT to the celebrated secretary problem which corresponds to the special case $p_i = 1/i, 1 \leq i \leq n$. In Section 4, we consider the full-information analogue of the secretary problem.

2 The General Model

In the STMOT, Bernoulli random variables $X_1, X_2, \ldots, X_n$ are assumed to be independent for a finite n and the payoff for not stopping is assumed to be zero. In this section, we attempt to extend the STMOT into the following directions: First, an infinite number of Bernoulli trials is allowed. Second, the payoff for not stopping is ω, which may be different from zero. Third, the Bernoulli random variables are allowed to be dependent. Fourth, at stage i, in addition to observing X_i, other dependent random variables are allowed to be observed that may influence the assessment of the probability of success at future stages. The method we use here is to change the original problem into a monotone stopping problem by not allowing stopping on a failure and then apply a simple result that gives conditions for the one-stage look-ahead rule (the 1-sla) to be optimal in a monotone problem (see, e.g., Ferguson (2006, Chapter 5) for a 1-sla and a monotone problem). This method is exactly the same that Ferguson (2008, Section 2) used to extend the STOT, so we mimic his argument.

We modify the original problem by not allowing stopping on a failure (this modification does not change the problem). When stopping on a failure is forbidden, we must change the notion of a "stage". A stage is defined to contain all the observations up to and including the next success if any. We model this as follows. For $i = 1, 2, \ldots$, let Z_i denote the set of random variables observed after success $i-1$ up to and including success i. If there are less than i successes, we let $Z_i = 0$, where "0" is a special absorbing state. Thus we treat the following general model. Let $Z_1, Z_2, \ldots$ be a stochastic process on an arbitrary space with an absorbing state called 0. We make the assumption that with probability one the process will eventually be absorbed at 0. We observe the process sequentially and must predict within m stages in advance when the state 0 will first be hit. If we predict correctly, we win 1, if we predict uncorrectly, we win nothing, and if the process hits 0 before we predict, we win ω (it is assumed here that $\omega < 1$, because, if $\omega \geq 1$, it is clearly optimal never to stop). This is a stopping rule problem in which stopping at stage k yields the payoff

$$\begin{aligned} Y_k &= \omega I(Z_k = 0) + I(Z_k \neq 0) P\{Z_{k+m} = 0 \mid \mathcal{G}_k\}, k = 1, 2, \ldots \\ Y_\infty &= \omega, \end{aligned} \tag{1}$$

where $\mathcal{G}_k = \sigma(Z_1, \ldots, Z_k)$ is the $\sigma-$field generated by $Z_1, \ldots, Z_k$ and $I(E)$ represents the indicator function of an event E. The assignment $Y_\infty = \omega$ means that if we never stop, we win ω. It is easy to see that the 1-sla is

$$N_m = \min\{k : Z_k = 0 \text{ or } (Z_k \neq 0 \text{ and } W_k/V_k \leq 1 - \omega)\}, \tag{2}$$

where

$$\begin{aligned} V_k &= P\{Z_{k+1} = 0 \mid \mathcal{G}_k\}, \\ W_k &= P\{Z_{k+m} \neq 0, Z_{k+m+1} = 0 \mid \mathcal{G}_k\}. \end{aligned}$$

and that a sufficient condition for the problem to be monotone is

$$W_k/V_k \text{ is a.s. non-increasing in } k. \tag{3}$$

We have the following result.

Theorem 2.1. Suppose that the process $Z_1, Z_2, \ldots$ has an absorbing state 0 such that $P\{Z_k \text{ is absorbed at } 0\} = 1$ and that the stopping problem with reward sequence (1) satisfies the condition (3). Then the 1-sla (2) is optimal.

Proof. Omitted.

The following result is immediate from Theorem 2.1.

Corollary 2.2. Suppose that n Bernoulli random variables $X_1, X_2, \ldots, X_n$ are observed sequentially. Let $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n$ be an increasing sequence of $\sigma-$fields such that $\{X_j = 1\}$ is in $\mathcal{F}_j$ for all $1 \leq j \leq n$. Let

$$\begin{aligned} V_k &= P\{X_{k+1} + \cdots + X_n = 0 \mid \mathcal{F}_k\}, \\ W_k &= P\{X_{k+1} + \cdots + X_n = m \mid \mathcal{F}_k\}. \end{aligned}$$

Then the optimal rule is described as

$$N_m = \min\{k \geq 1 : X_k = 1 \text{ and } W_k/V_k \leq 1\},$$

provided that the following condition is satisfied:

$$W_k/V_k \text{ is a.s. non-increasing in } k.$$

3 Application to the Secretary Problem

The *secretary problem* can be described as follows. A known number n of rankable applicants (1 being the best and n the worst) appear one at a time in random order with all $n!$ permutations equally likely. That is, each of the successive ranks of n applicants constitutes a random permutation. Suppose that all that can be observed are the relative ranks of the applicants as they appear. If Y_j denotes the relative rank of the jth applicant among the first j applicants, the sequentially observed random variables are $Y_1, Y_2, \ldots, Y_n$. It is well known that

(a) $Y_1, Y_2, \ldots, Y_n$ are independent random variables.
(b) $P\{Y_j = i\} = 1/j, \quad 1 \le i \le j, 1 \le j \le n.$

The jth applicant is called a *candidate* if he/she is relatively best, i.e., $Y_j = 1$. The problem we consider here is to stop on any of the last m successes, that is, any of the last m candidates (stopping is identified with selection of an applicant in the secretary problem). The independent random variables of Section 1 are therefore $X_1, X_2, \ldots, X_n$, where $X_j = I(Y_j = 1)$ from (a). Since $p_j = P\{X_j = 1\} = 1/j$ and so $r_j = 1/(j-1)$ from (b), we immediately have from the STMOT

Lemma 3.1 For the secretary problem, the optimal rule passes up the first $s_m - 1$ applicants and then selects the first candidate, if any, where

$$s_m = \min\left\{k \ge 1 : \sum_{k+1 \le i_1 < i_2 < \cdots < i_m \le n} \prod_{j=1}^{m} \left(\frac{1}{i_j - 1}\right) \le 1\right\}.$$

The maximal probability of win is

$$v_m = \left(\frac{s_m - 1}{n}\right) \sum_{k=1}^{m} \left[\sum_{s_m \le i_1 < i_2 < \cdots < i_k \le n} \prod_{j=1}^{k} \left(\frac{1}{i_j - 1}\right)\right].$$

Let n tend to infinity. Then asymptotically

$$(i) \quad s_m^* = \lim_{n \to \infty} \frac{s_m}{n} = \exp\left\{-(m!)^{1/m}\right\},$$

$$(ii) \quad v_m^* \;=\; \lim_{n\to\infty} v_m = \exp\left\{-(m!)^{1/m}\right\}\sum_{j=1}^{m}\frac{(m!)^{j/m}}{j!}.$$

Proof. Omitted.

Table 1

Values of s_m^* and v_m^* for several m

m	1	2	3	4	5	10
s_m^*	0.3679	0.2431	0.1625	0.1093	0.0739	0.0108
v_m^*	0.3679	0.5869	0.7260	0.8167	0.8767	* * * * *

Table 1 presents some numerical values of s_m^* and v_m^* for given m. For later use, we return to the finite problem and review the distribution of the number of candidates. Define

$$M_n = X_1 + X_2 + \cdots + X_n,$$

where $X_j = I(Y_j = 1)$ is defined as before. Then M_n denotes the total number of candidates. It is well known that the probability mass function of M_n, expressed as

$$p_n(k) = P\{M_n = k\}$$

satisfies the following recursion with $p_1(1) = 1$ and $p_n(k) = 0$ for $k = 0$ or $k > n$

$$p_n(k) = \frac{1}{n}p_{n-1}(k-1) + \left(1 - \frac{1}{n}\right)p_{n-1}(k), \quad 1 \le k \le n, \ 2 \le n.$$

4 Full-information Analogue

In contrast to the no-information problem considered in Section 3, the full-information analogue is the problem in which the observations are the true values of n applicants $Y_1, Y_2, \ldots, Y_n$, assumed to be i.i.d. random variables from a known continuous distribution, taken without loss of generality to be the uniform distribution on the interval $[0, 1]$. Let $L_k = \max\{Y_1, Y_2, \ldots, Y_k\}$ be the maximum of the first k observations and call the kth observation or the kth applicant a *record* if $L_k = Y_k$. It is desired to obtain a stopping rule that maximizes the probability of stopping on any of the last m successes, that is, the last m records. The case $m = 1$ is the full-information best-choice problem solved by

Gilbert and Mosteller (1966). For ease of description, let $a_k = P\{M_k < m\}, k \geq 0$ be the probability that the number of candidates is less than m when the total number of applicants is k in the secretary problem, namely, $a_k = \sum_{i=1}^{m-1} p_k(i)$ for $k \geq m$ and $a_k = 1$ for $k < m$ ($a_0 = 1$ for convenience). The main results can be summarized as follows.

Theorem 4.1 *(a) Optimal stopping rule: For a given positive integer m, there exists a non-decreasing sequence of the thresholds $\{b_j(m), 1 \leq j\}$ defined as $b_j(m) = 0$ for $1 \leq j < m$ and as a unique solution $x \in (0,1)$ to the equation*

$$\sum_{i=m}^{j} p_i(m) \binom{j}{i} \left(\frac{1-x}{x}\right)^i = 1$$

for $j \geq m$, such that the optimal rule is to choose the first record $Y_k(= L_k)$ that exceeds the threshold $b_{n-k}(m)$. Henceforth, we simply write b_j for $b_j(m)$ unless otherwise specified.

(b) Optimal probability: Let $P^*_{n,m}$ denote the optimal probability of win as a function of n and m. Then

$$P^*_{n,m} = \sum_{r=1}^{n} P(r),$$

where

$$P(1) = \frac{1}{n} \sum_{k=0}^{n-1} a_k \binom{n-1}{k} \sum_{j=k+1}^{n} \binom{n}{j} (1-b_{n-1})^j b_{n-1}^{n-j}$$

and

$$P(r) = \frac{1}{r-1} \sum_{k=0}^{n-r} a_k \binom{n-r}{k} \sum_{i=1}^{r-1} [P_1(i,k) + P_2(i,k)]$$

for $2 \leq r \leq n$, where

$$P_1(i,k) = \sum_{j=k+1}^{n} \frac{\binom{n}{j}}{\binom{n}{k}} \left[\frac{(1-b_{n-r})^j b_{n-r}^{n-j} - (1-b_{n-i})^j b_{n-i}^{n-j}}{n-k}\right]$$

and

$$P_2(i,k) = \sum_{j=k+1}^{n-r+1} \frac{\binom{n-r+1}{j}}{\binom{n-r+1}{k}} \left[\frac{(1-b_{n-i})^j b_{n-i}^{n-j}}{n-r-k+1}\right].$$

(c) Asymptotics: *Let c_m be the unique root t to the equation*

$$\sum_{i=m}^{\infty} p_i(m)\frac{t^i}{i!} = 1.$$

Then, as $n \to \infty$,

$$P^*_{n,m} \to P^*_m = e^{-c_m} J_m(c_m) + \{K_m(c_m) - c_m J_m(c_m)\} I(c_m),$$

where

$$\begin{aligned} I(t) &= \int_1^{\infty} \frac{e^{-tx}}{x} dx \\ J_m(t) &= \sum_{j=0}^{\infty} a_j \frac{t^j}{j!} = \sum_{j=0}^{m-1} \frac{t^j}{j!} + \sum_{j=m}^{\infty} a_j \frac{t^j}{j!} \\ K_m(t) &= \sum_{i=1}^{\infty} \min(i,m)\frac{t^i}{i!} + \sum_{i=m+1}^{\infty} \left(\sum_{j=m}^{i-1} a_j\right) \frac{t^i}{i!}. \end{aligned}$$

Proof. Omitted.

Table 2 presents some numerical values of c_m and P^*_m.

Table 2

Values of c_m and P^*_m for several m

m	1	2	3	4	5	10
c_m	0.8044	1.5151	2.3731	3.3573	4.4523	11.243
P^*_m	0.5802	0.8424	0.9465	0.9834	0.9953	0.9999

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