Probabilistic Interpretation Beyond Completely Monotone Capacities

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Abstract

A family $H$ of subsets is called increasing if $B \in H$ whenever $A \subseteq B$ for some $A \in H$. The fundamental theorem of random sets, known as Choquet theorem, will be discussed in the light of random increasing family $H$ of subsets. The distribution of $H$ is determined by the completely monotone capacity $\varphi(A) = \mathbb{P}(A \in H)$ if $\mathbb{P}(A, B \in H) = \mathbb{P}(A \cap B \in H)$ for every pair $(A, B)$. Similarly the completely alternating capacity $\varphi$ characterizes $H$ for which $\mathbb{P}(A, B \in H) = \mathbb{P}(A \in H) + \mathbb{P}(B \in H) - \mathbb{P}(A \cup B \in H)$. This paper presents our ongoing investigation in an effort of searching an extension of probabilistic interpretation of Choquet capacities.

1 Probabilistic interpretation of capacities

By $B$ we denote the Boolean algebra of a finite set $S$, and introduce a natural partial ordering by the inclusion $\subseteq$. This poset $B$ has the minimum element $\emptyset$ and the maximum element $S$, denoted respectively by $\hat{0}$ and $\hat{1}$. We call a nonnegative function $\varphi$ on $B$ a capacity if $a \subseteq b$ implies $\varphi(a) \leq \varphi(b)$ with $\varphi(\hat{0}) = 0$ and $\varphi(\hat{1}) = 1$ [i.e., $\varphi(\emptyset) = 0$ and $\varphi(S) = 1$].

Here we introduce successive difference functionals on capacities. For any $a \in B$ and any sequence $b_1, b_2, \ldots$ of $B$, we can define the following functionals recursively by

$$\nabla_{b_1}^a \varphi = \varphi(a) - \varphi(a \cap b_1)$$

$$\nabla_{b_1, \ldots, b_{n+1}}^a \varphi = \left( \nabla_{b_1, \ldots, b_n}^a - \nabla_{b_1, \ldots, b_{n+1}}^a \right) \varphi, \quad n = 1, 2, \ldots$$

The definition above does not depend on the order of $b_1$'s, and $\nabla_{b_1, \ldots, b_{n+1}}^a = \nabla_{b_1, \ldots, b_n}^a$ if $b_i = b_{n+1}$ for some $i \leq n$. Therefore, we can only define a distinct successive difference functional for every nonempty subset $B = \{b_1, \ldots, b_n\}$ of $B$, and denote it simply by $\nabla_B^a$. We call a capacity $\varphi$ completely monotone if $\nabla_B^a \varphi \geq 0$ for any...
Let $a \in B$ and any nonempty subset $B$ of $B$. Choquet [1] showed that if $\varphi$ is completely monotone then a $B_0$-valued random variable $X$ satisfies

\begin{equation}
\varphi(a) = \mathbb{P}(X \subseteq a) = \sum_{b \subseteq a} f(b) \quad \text{for} \ a \in B,
\end{equation}

where $f(b) = \mathbb{P}(X = b)$ is the probability mass function of $X$. This representation is unique up to the probability mass function, and also suffices the property of complete monotonicity.

A subset $H$ of $B$ is called an up-set (or a hereditary set to the right) if $a \subseteq b$ and $a \in H$ imply $b \in H$. By $\mathcal{H}_0$ we denote the class of nonempty up-sets which do not contain the minimum element $\hat{0}$ of $B$; that is, $\mathcal{H}_0$ is the class of all up-sets except for $\emptyset$ nor $B$. We define the capacity $\chi_H$ by

\begin{equation}
\chi_H(a) = \begin{cases} 1 & \text{if } a \in H; \\ 0 & \text{if } a \not\in H, \end{cases}
\end{equation}

with $H \in \mathcal{H}_0$, and call it an extreme capacity. The entire class of capacities is a convex polytope (i.e., a bounded polyhedron) on the vector space of real-valued functions on $B$, and that it consists of extreme points of the form $\chi_H$ (see [1]). Therefore, for any capacity $\varphi$ we can find the representation

\begin{equation}
\varphi(a) = \sum_{H \in \mathcal{H}_0} g(H) \chi_H(a), \quad a \in B
\end{equation}

where $g$ is a probability mass function on $\mathcal{H}_0$.

Murofushi [2] pioneered the following greedy algorithm to construct the weight $g$ in (1.2). Define a map $H(t) = \{a \in B : \varphi(a) > t\}$ from $[0, 1)$ to $\mathcal{H}_0$, and observe that $H(t) = H_i$ for $t \in [r_{i-1}, r_i)$ for $i = 1, \ldots, m$ with an increasing sequence $0 = r_0 < r_1 < \cdots < r_m = 1$ and a decreasing sequence $H_1 \supset \cdots \supset H_m$. Then set $g(H_i) = r_i - r_{i-1}$ for $i = 1, \ldots, m$, and $g(K) = 0$ for every other $K \in \mathcal{H}_0$. It is easily checked that $g$ satisfies (1.2).

We introduce an $\mathcal{H}_0$-valued random variable $H$, and call it a random up-set. We can restate (1.2) by

\begin{equation}
\varphi(a) = \sum_{H \in \mathcal{H}_0} g(H) \chi_H(a), \quad a \in B
\end{equation}

where $H$ has the probability mass function $g(K) = \mathbb{P}(H = K)$. However, as the following example shows, such a probabilistic interpretation is no longer unique in general.

**Example 1.1.** Let $B = \{\hat{0}, \alpha, \beta, \gamma, \alpha \beta, \alpha \gamma, \beta \gamma, \hat{1}\}$ be the Boolean algebra with $\hat{0} = \emptyset$ and $\hat{1} = \alpha \beta \gamma$, and let $\varphi$ be a capacity on $B$ with $\varphi(\alpha \beta) = \varphi(\alpha \gamma) = \varphi(\beta \gamma) = \frac{1}{3}$ and $\varphi(\alpha) = \varphi(\beta) = \varphi(\gamma) = 0$; here we write $\alpha \beta$ for a subset $\{\alpha, \beta\}$. Then $\varphi$ is completely monotone, and (1.1) holds for $f(\alpha \beta) = f(\alpha \gamma) = f(\beta \gamma) = \frac{1}{3}$. Let $\langle B \rangle = \{a \in B : b \subseteq a \text{ for some } b \in B \}$.
be the up-set generated by a subset $B$ of $\mathcal{B}$. The probability mass function $g$ on $\mathcal{H}_0$ with $g(\langle \alpha \beta \rangle) = g(\langle \alpha \gamma \rangle) = g(\langle \beta \gamma \rangle) = \frac{1}{3}$ satisfies (1.3). On the other hand, the Murofushi’s greedy algorithm determines $g(\langle 1 \rangle) = \frac{1}{3}$ and $g(\langle \{\alpha \beta, \alpha \gamma, \beta \gamma\} \rangle) = \frac{2}{3}$ for which (1.3) also holds.

2 Möbius functions and successive differences

Let $\mathcal{P}$ be a finite poset. The Möbius function $\mu_\mathcal{P}$ is the unique function defined for every pair $(a, b)$ such that $a \leq b$, and satisfies $\mu_\mathcal{P}(a, a) = 1$ and $\sum_{b \leq a} \mu_\mathcal{P}(b, a) = 0$ for each $a \in \mathcal{P}$. Then we have

\[(2.4) \quad f(a) = \sum_{b \leq a} \varphi(b) \mu_\mathcal{P}(b, a)\]

if and only if $\varphi(a) = \sum_{b \leq a} f(b)$. Here $f$ is called the Möbius inversion of $\varphi$.

Example 2.1. The results here and other arguments of this section are either taken from or inspired by Stanley [3]. Let $\mathcal{B}$ be a Boolean algebra of a finite set $S$. For $a \subseteq b$ we obtain the Möbius function $\mu_\mathcal{B}(a, b) = (-1)^{|b \setminus a|}$, where $|b \setminus a|$ denotes the number of elements in the set difference $b \setminus a$.

Let $a \in \mathcal{B}$ and a nonempty subset $B \subseteq \mathcal{B}$ be fixed. Then we can introduce the subposet

$\mathcal{P}_B^a = \{\cap B' \cap a : B' \subseteq B\}$

with the partial order $\leq$ by inclusion, where

$$\cap B' = \begin{cases} \cap_{b \in B'} b & \text{if } B' \neq \emptyset; \\ 1 & \text{if } B' = \emptyset. \end{cases}$$

In what follows we simply write $\mathcal{P}$ for $\mathcal{P}_B^a$ if there is no confusion with other posets in discussion. It is not difficult to see (e.g., Stanley [3]) that

$$\sum_{b' \leq b} \varphi(b) \mu_\mathcal{P}(b, a) = \sum_{B' \subseteq B} (-1)^{|B'|} \varphi(\cap B' \cap a) = \nabla_B^a \varphi$$

Here the element $a$ is the maximum of $\mathcal{P}$.

Let $\varphi$ be a capacity, and let $f$ be the Möbius inversion of $\varphi$. We can argue the following interesting connection to the Möbius inversion from the successive difference $\nabla_B^a$ (or equivalently from the Möbius function $\mu_\mathcal{P}$). For $b \in \mathcal{P}$ we set $F(b)$ to be the summation of $f(x)$ over all $x$’s satisfying $x \subseteq b$ and $x \not\subset b'$ for all $b' < b$ in $\mathcal{P}$. Then we have

$$\sum_{b' \leq b} F(b') = \sum_{x \subseteq b} f(x) = \varphi(b)$$
and therefore,
\begin{equation}
F(a) = \sum_{b \leq a} \varphi(b) \mu_{\mathcal{P}}(b, a) = \nabla_{B}^{a} \varphi
\end{equation}

As a special case of (2.5) we obtain \( f(a) = \nabla_{B}^{a} \varphi \) when \( a = \{\alpha_{1}, \ldots, \alpha_{k}\} \) and \( B = \{S \setminus \{\alpha_{1}\}, \ldots, S \setminus \{\alpha_{k}\}\} \). Hence we have shown the result of Choquet.

**Proposition 2.2.** A capacity \( \varphi \) is completely monotone if and only if the Möbius inversion \( f \) is nonnegative.

The Möbius inversion \( f \) of a capacity \( \varphi \) satisfies \( f(\hat{0}) = 0 \) and \( \sum_{a \subseteq 1} f(a) = 1 \), and it can be viewed as a probability mass function on \( \mathcal{B}_{0} \) when \( f \) is nonnegative.

For any capacity \( \varphi \) we can define the dual capacity \( \varphi^{*} \) by \( \varphi^{*}(a) = 1 - \varphi(\hat{1} \setminus a) \) for \( a \in \mathcal{B} \). Then we can introduce the functional \( \Delta_{a}^{B} \) by
\begin{equation}
\Delta_{a}^{B} \varphi = -\nabla_{B^{*}}^{\hat{1} \setminus a} \varphi^{*}
\end{equation}
where \( B^{*} = \{\hat{1} \setminus b : b \in B\} \). We can formulate it recursively by
\[
\Delta_{a}^{b_{1}} \varphi = \varphi(a) - \varphi(a \cup b_{1})
\]
\[
\Delta_{a}^{b_{1}, \ldots, b_{n+1}} \varphi = \left( \Delta_{a}^{b_{1}, \ldots, b_{n}} - \Delta_{a \cup b_{n+1}}^{b_{1}, \ldots, b_{n}} \right) \varphi, \quad n = 1, 2, \ldots
\]

We call \( \varphi \) completely alternating if \( \Delta_{a}^{B} \varphi \leq 0 \) for any \( a \in B \) and any nonempty subset \( B \subseteq \mathcal{B} \). It is clear from (2.6) that \( \varphi \) is completely alternating if and only if \( \varphi^{*} \) is completely monotone. By Proposition 2.2 we can see that if \( \varphi \) is completely alternating then a \( \mathcal{B}_{0} \)-valued random variable \( X \) satisfies
\begin{equation}
\varphi(a) = \mathbb{P}(X \cap a \neq \hat{0}) = \sum_{b \cap a \neq \hat{0}} f^{*}(b)
\end{equation}
where \( f^{*}(b) = \mathbb{P}(X = b) \) is the Möbius inversion of \( \varphi^{*} \).

### 3 An extension of probabilistic interpretation

We introduce a partial order \( \preceq \) on the class \( \mathcal{H} \) of nonempty up-sets \( H \) as follows: For \( K, H \in \mathcal{H} \) we define \( K \preceq H \) if and only if \( K \supseteq H \). Recall that a subset \( A \) of \( \mathcal{B} \) is an antichain if none of pairs of \( A \) are comparable. We can obtain the Möbius function on \( \mathcal{H} \) as follows; see Stanley [3]. For \( K \preceq H \) we have
\[
\mu_{\mathcal{H}}(K, H) = \begin{cases} (-1)^{|K \setminus H|} & \text{if } K \setminus H \text{ is an antichain;} \\
0 & \text{otherwise.}
\end{cases}
\]

Let \( \varphi \) be a capacity. A nonempty up-set \( H \) is called feasible with respect to \( \varphi \) if \( \Delta_{b \cap b'}^{b, b'} \varphi < 0 \) for every pair \( (b, b') \) of \( H \) satisfying \( b \cap b' \notin H \).
Definition 3.1. If a nonempty up-set $H$ is not feasible, we can find a feasible up-set $K \preceq H$ greedily. First, set $K = H$, and repeat the following steps until $K$ becomes feasible.

1. Choose a pair $(b, b')$ of $K$ such that $b \cap b' \not\in K$ and $\Delta_{b \cap b'}^{(b, b')} \varphi \geq 0$.

2. Set $K = \langle b \cap b' \rangle \cup K$. Here $\langle b \rangle = \{a \in \mathcal{B} : b \subseteq a\}$ is the up-set generated by $b \in \mathcal{B}$. Thus, $\langle b \cap b' \rangle \cup K$ is also an up-set.

Lemma 3.2. For any nonempty up-set $H$, Definition 3.1 generates the maximum feasible up-set $K \preceq H$ (in the poset $\mathcal{H}$) satisfying $K \preceq H$.

Let $H^o$ denote the antichain of all the minimal elements of the unique feasible up-set $K \preceq H$ generated via Definition 3.1. For $H \in \mathcal{H}$ we define $\Phi(H) = -\Delta_H^{H^o} \varphi$, and introduce the M"obius inversion $g$ of $\Phi$ by

$$g(H) = \sum_{K \preceq H} \Phi(K) \mu_{\mathcal{H}}(K, H) \quad (3.8)$$

If $g$ is nonnegative then the probability mass function $\mathbb{P}(\textbf{H} = K) = g(K)$ on $\mathcal{H}_0$ satisfies

$$\mathbb{P}(a, b \in \textbf{H}) = \varphi(a \cap b) - \left[\Delta_{a \cap b}^{a, b} \varphi\right]_- \quad \text{for} \ a, b \in \mathcal{B}_0, \quad (3.9)$$

where $[x]_- = \min\{x, 0\}$.

Theorem 3.3. If an $\mathcal{H}_0$-valued random up-set $\textbf{H}$ satisfies (3.9) with some capacity $\varphi$ then the M"obius inversion $g$ in (3.8) is nonnegative and uniquely determines the probability mass function of $\textbf{H}$.

If $\varphi$ is completely monotone and satisfies (1.1) with some $\mathcal{B}_0$-valued random variable, then $\textbf{H} = \langle X \rangle$ satisfies (3.9). If $\varphi$ is completely alternating and satisfies (2.7) then $\textbf{H} = \{a \in \mathcal{B} : X \cap a \neq \emptyset\}$ satisfies (3.9). Theorem 3.3 indicates that the random up-set $\textbf{H}$ can be directly constructed via (3.8).

Theorem 3.3 also implies that there is a class of capacities capable of characterizing $\textbf{H}$ uniquely in a way to extend Choquet theorem. Due to the nature of our research in progress we omit the proofs for Lemma 3.2 and Theorem 3.3.

References

