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Parametric Duality for Nondifferentiable Minimax Fractional Programming Problems *

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Abstract

We consider a nondifferentiable fractional programming problem as the form:

$$(P) \qquad \min_{x\in X} \max_{y\in Y} \frac{f(x,y)}{g(x,y)}$$

$$\text{subject to} \quad X=\{x\in\mathbb{R}^n \mid h_j(x)\le 0, \quad j=1,2,\ldots,p\}\subset\mathbb{R}^n$$

$$Y\subset\mathbb{R}^m, \text{ a compact subset,}$$

where $f\ge 0$ and $g>0$ are continuous nondifferentiable but subdifferentiable on $X\times Y\subset\mathbb{R}^n\times\mathbb{R}^m$. The necessary and sufficient optimality conditions for problem (P) are established. Subsequently, we apply the optimality conditions to formulate a one-parameter dual programming and prove the weak duality, strong duality, and strict converse duality theorems.

1 Introduction

Minimax fractional programming is an interesting subject, which features in several types of optimization problems. For example, it can be used in engineering and economics to measure a ratio of functions between a given period of time and utilised resource to measure sufficiency or productivity of a system as well as in game theory. In many situations the considered functions are

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continuous but nonsmooth and/or nonconvex. In order to smooth their theories, functions are often assumed to be subdifferentiable and/or generalized convexity as well as generalized invexity. For example, see Chen and Lai [2] for (F, ρ, θ)-invexity and Lai et al. [6] for generalized convexity. The problem (P) is motivated by the previous paper: Lai et al. [5] in which the authors considered a nondifferentiable minimax fractional programming problem as the following form:

$$(P_0) \qquad \min_{x \in X} \sup_{y \in Y} \frac{\phi(x,y) + (x^T Ax)^{1/2}}{\psi(x,y) - (x^T Bx)^{1/2}}$$
$$\text{subject to} \quad h_j(x) \leq 0, \quad j = 1, 2, \ldots, p, \quad x \in \mathbb{R}^n$$

Here $Y \subset \mathbb{R}^m$ is a compact subset, two functions ϕ and ψ are C^1 functions defined on $\mathbb{R}^n \times \mathbb{R}^m$, and $h_j(\cdot) : \mathbb{R}^n \to \mathbb{R}$ for $j = 1, 2, \ldots, p$ are C^1. The matrices A and B are positive semidefinite $n \times n$ matrices. $\phi(x,y) + (x^T Ax)^{1/2} \geq 0$ and $\psi(x,y) - (x^T Bx)^{1/2} > 0$ for each $(x,y) \in X \times Y$, where X is the set of all feasible solutions of problem (P_0); in the other words,

$$X = \{x \in \mathbb{R}^n \mid h_j(x) \leq 0, \quad j = 1, 2, \ldots, p\}.$$

In the case $A = B = 0$, the problem (P_0) becomes a differentiable minimax fractional programming problem (P). Several authors have investigated the problem (P_0) by setting $A = B = 0$. See also Yadav and Mukherjee [12], Chandra and Kumar [1], and Lai et al. [6-10]. Nonfractional minimax problem was first considered by Schmitendorf [11] with $\psi \equiv 1, A = 0 = B$ for (P_0)

2 Necessary and Sufficient Conditions

Since $Y \subset \mathbb{R}^m$ is a compact subset and for any $x \in X$, $f(x, \cdot)$ and $g(x, \cdot)$ are continuous on Y, there exists finite y_i, $i = 1, 2, \ldots, k$, such that the $sup_{y \in Y}$ is attained its maximum. Now for each $x \in X$, we denote

$$\begin{aligned} K_\lambda(x) = \{ (k, \mu, y) \in \mathbb{N} \times \mathbb{R}^k_+ \times \mathbb{R}^{mk} \mid 1 \leq k \leq n+1; \mu = (\mu_1, \ldots, \mu_k) \in \mathbb{R}^k_+, \\ \sum_{i=1}^k \mu_i = 1 \quad \text{and} \quad \mu_i > 0; \\ y = (y_1, \ldots, y_k) \quad \text{with} \quad y_i \in Y_\lambda(x) \quad \text{for all} \quad i = 1, \ldots, k\}, \end{aligned}$$

where the set

$$Y_\lambda(x) = \{y \in Y \mid f(x,y) - \lambda g(x,y) = \max_{z \in Y} (f(x,z) - \lambda g(x,z))\}$$

is a closed subset of Y. We define

$$F_\lambda(x) = \max_{y \in Y} (f(x,y) - \lambda g(x,y))$$

with the parameter λ, which is defined by

$$\max_{y \in Y} \frac{f(x,y)}{g(x,y)} = \lambda, \quad \frac{f(x,y)}{g(x,y)} \leq \lambda \quad \text{for all} \quad y \in Y.$$

So that λ is dependent on x, and $f(x,y) - \lambda g(x,y) \leq 0$.

We can show that the following nonfractional parametric problem

$$(P_\lambda) \qquad v(\lambda) = \min_{x \in X} \max_{y \in Y} (f(x,y) - \lambda g(x,y)) \quad (\leq 0)$$

where $\lambda \in \mathbb{R}_+ = [0, \infty)$ is a parameter,

is equivalent to problem (P) with optimal value λ^* and $v(\lambda^*) = 0$. That is, (P_{λ^*}) and (P) have the same optimal solution. It is known that if the functions in (P) are nonsmooth, nonconvex on $\mathbb{R}^n$ but subdifferentiable, then a point $x_0 \in \mathbb{R}^n$ is an optimal solution of the programming

$$(\widetilde{P}) \qquad \text{minimize } \Phi(x)$$

$$\text{subject to} \quad h_j(x) \leq 0, \quad j = 1, 2, \ldots, p \quad x \in \mathbb{R}^n$$

if and only if there is $\mu^* \in \mathbb{R}^p_+$ such that

$$0 \in \partial\Phi(x) + \sum_{j=1}^{p} \mu_j^* \partial h_j(x_0) + N(x_0/X) \quad \text{and} \quad \sum_{j=1}^{p} \mu_j^* h_j(x_0) = 0$$

where X is a compact set. It follows that if x_0 is (P)-optimal and λ^* is (P)-optimal value, then from (P_{λ^*}), we have

$$v(\lambda^*) = \max_{y \in Y} (f(x_0,y) - \lambda^* g(x_0,y))$$

$$= f(x_0, y_i^*) - \lambda^* g(x_0, y_i^*)$$

for $\max_{y \in Y} \phi(x_0,y)$ is attained at some y_i when $\phi(x_0,y)$ is continuous on the compact set Y. Hence the necessary and sufficient optimality conditions for problem (P) can be stated as the following theorem. Precisely, we state the following theorem of optimality condition.

Theorem 2.1 (Necessary and Sufficient Conditions). *Suppose that problem (P) satisfies a regularity condition. That is, there exists $x' \in X$ such that $h_j(x') < 0$ for all $j = 1,2,\ldots,p$. Then x_0 is an optimal solution of (P) if and only if there exist $\lambda^* \in \mathbb{R}_+$, $(k^*, \mu^*, y^*) \in K_{\lambda^*}(x_0)$ and $\beta^* \in \mathbb{R}^p_+$ such that*

$$0 \in \sum_{i=1}^{k^*} \mu_i^* \left(\partial f(x_0,y_i^*) + \lambda^* \partial \left(-g(x_0,y_i^*) \right) \right) + \sum_{j=1}^{p} \beta_j^* \partial h_j(x_0) + N(x_0/X); \tag{2.1}$$

$$f(x_0,y_i^*) - \lambda^* g(x_0,y_i^*) = 0 \quad \textit{for all} \quad i = 1,2,\ldots,k^*, \tag{2.2}$$

$$\sum_{j=1}^{p} \beta_j^* h_j(x_0) = 0, \tag{2.3}$$

and $\lambda^* = \max_{y \in Y} f(x_0,y)/g(x_0,y)$, *the optimal value of* (P).

Remark

1. A convex function is subdifferentiable, but a subdifferentiable function may not be convex.
2. Usually, if we establish the necessary optimality conditions, then we can establish the sufficient optimality condition from the converse of necessary condition with extra assumptions.
3. Since the extra assumptions are various, the sufficient optimality conditions are also various.
4. Duality models are usually formulated from the sufficient optimality conditions, so duality models are various.
5. A fractional programming (P) can be reduced to an equivalence of nonfractional parametric programming (P_λ) so we will formulate here a parametric dual model in the next section.

3 Parametric Dual Model

From the optimality conditions for problem (P) in the preceeding section, we can formulate the following parametric maximization problem which is a dual problem to the minimax problem (P) as follows:

$$(D_P) \qquad \max_{(k,\mu,y) \in K_\lambda(u)} \; \max_{(u,\beta,\lambda) \in H(k,\mu,y)} \lambda$$

where $H(k, \mu, y)$ denotes the set of all triples $(u, \beta, \lambda) \in \mathbb{R}^n \times \mathbb{R}^p_+ \times \mathbb{R}_+$ which satisfies

$$0 \in \sum_{i=1}^{k} \mu_i (\partial f(u,y_i) + \lambda\, \partial(-g(u,y_i))) + \sum_{j=1}^{p} \beta_j\, \partial h_j(u) + N(u/X), \tag{3.1}$$

$$\sum_{j=1}^{k} \mu_i (f(u,y_i) + \lambda(-g(u,y_i))) + \sum_{j=1}^{p} \beta_j h_j(u) \geq 0. \tag{3.2}$$

If for a triple $(k, \mu, y) \in K_\lambda(u)$, the set $H(k, \mu, y)$ is empty, then we define the supremum over it to be $-\infty$.

In a dual problem, we always need to establish three theroems: weak duality, strong duality, and strict converse duality theorem, which we state as follows.

Theorem 3.1 (Weak Duality). *Let x and $(u, \beta, \lambda, \mu, y)$ be (P)-feasible and (D_P)-feasible, respectively. Then*

$$\max_{y \in Y} f(x,y)/g(x,y) \geq \lambda.$$

Proof. This theorem can be proved by contrary. If the result is not true, we then deduce that

$$f(x,y) - \lambda g(x,y) < 0 \quad \text{for all} \quad y \in Y$$

and if there are k points, $y_i\,(1 \leq i \leq k)$, such that they are attained the maximum point in the compact set $Y_\lambda(u)$, then there corresponds $\mu_i > 0$ with $\sum_{i=1}^{k} \mu_i = 1$, and

$$\sum_{i=1}^{k} \mu_i (f(x,y_i) + \lambda(-g(x,y_i)) < 0). \tag{3.3}$$

By the optimality conditions (3.1) and (3.2), it will reduce the inequality

$$\sum_{i=1}^{k} \mu_i (f(x,y_i) + \lambda(-g(x,y_i)) \geq 0). \tag{3.4}$$

This condtradicts the inequality (3.3), and the proof is complete. □

Theorem 3.2 (Strong Duality). *Suppose that problem (P) satisfies a regularity constraint. Let x_0 be an optimal solution of (P). Then there exist $\lambda^* \in \mathbb{R}_+$, $(k^*, \mu^*, y^*) \in K_{\lambda^*}(x_0)$ and $\beta_j^* \in \mathbb{R}^p_+$ such that $(x_0, \beta^*, \lambda^*, k^*, \mu^*, y^*)$ is an optimal solution of (D_P), and the optimal values of (P) and (D_P) are equal, that is,* $\min(P) = \max(D_P)$.

Proof. It follows from the Theorem 3.1(Weak Duality) that if x_0 is a (P)-optimal solution with optimal value λ^*, then there exist $(k^*,\mu^*,y^*) \in K_{\lambda^*(x_0)}$ and $\beta_j^* \in \mathbb{R}_+^P$ such that $(x_0,\beta^*,\lambda^*,k^*,\mu^*,y^*)$ is a feasible solution of (D_P). As $\lambda^* = \max_{y\in Y} f(x_0,y)/g(x_0,y)$, we see that $(x_0,\beta^*,\lambda^*,k^*,\mu^*,y^*) \in$ (D_P)-optimal and λ^* is also the (D_P) optimal value. □

Theorem 3.3 (Strict Converse Duality). *Suppose that problem (P) satisfies a regularity constraint. Let x' and $(x_0, \beta^*, \lambda^*, k^*, \mu^*, y^*)$ be optimal solutions of (P) and (D_P), respectively. If one of the functions $f(\cdot,y)$, $-g(\cdot,y)$, and $h_j(\cdot)$ for $j = 1,2,\dots,p$ is strictly convex at x_0, then $x' = x_0$, that is, x_0 is an opotimal solution of (P) and* $\max_{y\in Y} f(x',y)/g(x',y) = \lambda^*$.

Proof. Assume that the (P)-optimal $x' \neq x_0$, then by Theorem 3.2, there is $(k',\mu',y') \in K_{\lambda'}(x')$ such that $\lambda' = \max_{y\in Y} f(x',y)/g(x',y) = \lambda^*$, and hence $(x',\beta',\lambda',k',\mu',y')$ is (D_P)-optimal. Thus $(x_0,\beta',\lambda',k',\mu',y')$ is (D_P)-optimal with value $\lambda' = \lambda^*$, and it must reduce $x' = x_0$

□

4 Remarks for Further Development

We can see two possible ways to extend this work. One can try to relax convexity to some kind of generalized convexity; for example, pseudoconvexity, quasiconvexity, or (F,ρ)-convexity. In addition, one can study problem (P) in complex variables (see Lai et al. [8] and [9]) or set variables (cf Lai et al.[6] and [10]). For minimax fractional programming problem

$$\min_{\zeta\in X}\max_{\eta\in Y} \frac{Ref(\zeta,\eta)}{Reg(\zeta,\eta)}$$
$$\text{subjet to} \quad \zeta \in S^0 = \{\, \zeta \in C^{2n} \mid -h(\zeta) \in S \,\},$$

the general study for nondifferentiable functions is still open.

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