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On ϵ-Optimality Theorems and ϵ-Duality Theorems for Convex Semidefinite Optimization Problems with Conic Constraints

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Abstract

In this paper, we review results in Lee [13] and Lee et al. [14] without proofs. We consider ϵ-approximate solutions for a convex semidefinite optimization problem (SDP) with conic constraints and present ϵ-optimality theorems and ϵ-saddle point theorems for such solutions which hold under a weakened constraint qualification or which hold with out any constraint qualification. We give a Wolfe type dual problem for (SDP), and then we present ϵ-duality results, which hold under a weakened constraint qualification.

1 Introduction and Preliminaries

Convex semidefinite optimization problem is to optimize an objective convex function over a linear matrix inequality. When the objective function is linear and the corresponding matrices are diagonal, this problem become a linear optimization problem. So, this problem is an extension of a linear optimization problem. For convex semidefinite optimization problem, strong duality without constraint qualification [17], complete dual characterization conditions of solutions ([7, 11]), saddle point theorems [1] and characterizations of optimal solution sets [9] have been investigated.

To get the ϵ-approximate solution, many authors have established ϵ-optimality conditions, ϵ-saddle point theorems and ϵ-duality theorems for several kinds of optimization problems ([2, 3, 4, 15, 16, 18, 19]). Recently, Jeyakumar et al. [7] estabilished sequential optimality conditions for exact solutions of convex optimization problem which holds without any constraint qualification. Jeyakumar et al. [6] gave ϵ-optimality conditions for convex optimization problems, which hold without any constraint qualification. Yokoyama et al. [19] gave a special case of convex optimization problem which ϵ-optimality conditions. Kim et al. [12] proved sequential ϵ-saddle point theorems and ϵ-saddle point theorems for convex semidefinite optimization problems which have not conic constraints. Recently, Lee [13] and Lee et al. [14] extended results in Kim et al. [12] to convex semidefinite optimization problems with conic constraints.

In this paper, we review the results in Lee [13] and Lee et al. [14] without proofs. We consider ϵ-approximate solutions for a convex semidefinite optimization problem with conic constraints and present ϵ-optimality theorems and ϵ-saddle point theorems for such solutions which hold under a weakened constraint qualification or which hold with out any constraint qualification. We give a Wolfe type dual problem for the convex semidefinite optimization problem with conic constraint and then we present ϵ-duality results, which hold under a weakened constraint qualification.

Consider the following convex semidefinite optimization problem:

$$\textbf{(SDP)} \qquad \begin{array}{ll} \text{Minimize} & f(x) \\ \text{subject to} & F_0 + \sum_{i=1}^{m} x_i F_i \succeq 0, \ (x_1, x_2, \cdots, x_m) \in C, \end{array}$$

where $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ is a convex function, the space of $n \times n$ real symmetric matrices, C is a closed convex cone of $\mathbb{R}^m$, and for $i = 0, 1, \cdots, m$, $F_i \in S_n$. The space S_n is partially ordered by the Löwner order, that is, for $M, N \in S_n$, $M \succeq N$ if and only if $M - N$ is positive semidefinite. The inner product in S_n is defined by $(M, N) = Tr[MN]$, where $Tr[\cdot]$ is the trace operation.

Let $S := \{M \in S_n \mid M \succeq 0\}$. Then S is self-dual, that is,

$$S^+ := \{\theta \in S_n \mid (\theta, Z) \geqq 0, \text{ for any } Z \in S\} = S.$$

Let $F(x) := F_0 + \sum_{i=1}^{m} x_i F_i$, $\hat{F}(x) := \sum_{i=1}^{m} x_i F_i$, $x = (x_1 \cdots, x_m) \in \mathbb{R}^m$. Then $\hat{F}$ is a linear operator from $\mathbb{R}^m$ to S_n and its dual is defined by

$$\hat{F}^*(Z) = (Tr[F_1 Z], \cdots, Tr[F_m Z])$$

for any $Z \in S_n$. Clearly, $A := \{x \in C \mid F(x) \in S\}$ is the feasible set of **(SDP)**.

We define the ϵ-approximate solution of **(SDP)** as follows:

Definition 1.1 *Let $\epsilon \geqq 0$. Then $\bar{x}$ is called an ϵ-approximate solution of* **(SDP)** *if for any $x \in A$,*

$$f(x) \geqq f(\bar{x}) - \epsilon.$$

Now we give the definitions of subdifferential and ϵ-subdifferential of convex function in [5].

Definition 1.2 *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a convex function.*

(1) *The subdifferential of g at a is given by*

$$\partial g(a) = \{v \in \mathbb{R}^n \mid g(x) \geqq g(a) + < v, x - a >, \quad \text{for all } x \in \mathbb{R}^n\},$$

where $\langle \cdot, \cdot \rangle$ is the scalar product on $\mathbb{R}^n$

(2) *The ϵ-subdifferential of g at a is given by*

$$\partial_\epsilon g(a) = \{v \in \mathbb{R}^n \mid g(x) \geqq g(a) + < v, x - a > -\epsilon, \quad \text{for all } x \in \mathbb{R}^n\}.$$

Definition 1.3 *Let C be a closed convex set in $\mathbb{R}^n$ and $x \in C$.*

(1) *Let $N_C(x) = \{v \in \mathbb{R}^n \mid < v, y - x > \leqq 0, \text{ for all } y \in C\}$.*
Then $N_C(x)$ is called the normal cone to C at x.

(2) *Let $\epsilon \geqq 0$. Let $N_C^\epsilon(x) = \{v \in \mathbb{R}^n \mid < v, y - x > \leqq \epsilon, \text{ for all } y \in C\}$.*
Then $N_C^\epsilon(x)$ is called the $\epsilon -$ normal set to C at x.

(3) *When C is a closed convex cone in $\mathbb{R}^n$, $N_C(0)$ is denoted by C^* and called the negative dual cone of C.*

Following the proof of Lemma 2.2 in [8], we can obtain the following lemma.

Lemma 1.1 [13, 14] *Let $F_i \in S_n$, $i = 0, 1, \cdots, m$. Suppose that $A \neq \emptyset$. Let $u \in \mathbb{R}^m$ and $\alpha \in \mathbb{R}$. Then the following are equivalent:*

(*i*) $$\{x \in C \mid F_0 + \sum_{i=1}^{m} F_i x_i \succeq 0\} \subset \{x \in \mathbb{R}^m \mid \langle u, x \rangle \geqq \alpha\}.$$

$$(ii) \qquad \begin{pmatrix} u \\ \alpha \end{pmatrix} \in cl \left(\bigcup_{(Z,\delta)\in S\times\mathbb{R}_+} \left\{ \begin{pmatrix} \hat{F}^*(Z) \\ -Tr[ZF_0]-\delta \end{pmatrix} \right\} - C^* \times \mathbb{R}_+ \right).$$

Using the above Lemma 1.1, we can obtain the following lemma:

Lemma 1.2 [13, 14] *Suppose that $A \neq \emptyset$. Let $\bar{x} \in A$ and $\epsilon \geqq 0$. Then $\bar{x}$ is an ϵ-approximate solution of* **(SDP)** *if and only if there exist $\epsilon_0, \epsilon_1 \geqq 0$, $v \in \partial_{\epsilon_0} f(\bar{x})$ such that $\epsilon_0 + \epsilon_1 = \epsilon$, and*

$$\begin{pmatrix} v \\ \langle v, \bar{x} \rangle - \epsilon_1 \end{pmatrix} \in cl \left(\bigcup_{(Z,\delta)\in S\times\mathbb{R}_+} \left\{ \begin{pmatrix} \hat{F}^*(Z) \\ -Tr[ZF_0]-\delta \end{pmatrix} \right\} - C^* \times \mathbb{R}_+ \right).$$

2 ϵ-Optimality Conditions

Now, using the above Lemma 1.2, we can give the following two ϵ-optimality conditions for **(SDP)**.

Theorem 2.1 [13] *Let $\bar{x} \in A$ and $\bigcup_{(Z,\delta)\in S\times\mathbb{R}_+} \left\{ \begin{pmatrix} \hat{F}^*(Z) \\ -Tr[ZF_0]-\delta \end{pmatrix} \right\} - C^* \times \mathbb{R}_+$ is closed in $\mathbb{R}^m \times \mathbb{R}$. Then $\bar{x} \in A$ is an ϵ-approximate solution of* **(SDP)** *if and only if there exist ϵ_0, $\epsilon_1 \geqq 0$, $v \in \partial_{\epsilon_0} f(\bar{x})$, $Z \in S$, $c^* \in C^*$ such that $\epsilon_0 + \epsilon_1 = \epsilon$,*

$$v = \hat{F}^*(Z) - c^*$$

and

$$0 \leqq Tr[ZF(\bar{x})] \leqq \epsilon_1.$$

Theorem 2.2 [13] *Let $\bar{x} \in A$. Then $\bar{x}$ is an ϵ-approximate solution of* **(SDP)** *if and only if there exist ϵ_0, $\epsilon_1 \geqq 0$, $v \in \partial_{\epsilon_0} f(\bar{x})$, $c_n^* \in C^*$, $Z_n \in S$, $\delta_n \geqq 0$ such that $\epsilon_0 + \epsilon_1 = \epsilon$,*

$$v = \lim_{n\to\infty} (\hat{F}^*(Z_n) - c_n^*)$$

and

$$< v, \bar{x} > -\epsilon_1 = \lim_{n\to\infty} (-Tr[Z_n F_0] - \delta_n).$$

3 ϵ-Saddle Point Theorems and ϵ-Duality Theorem

Now we give ϵ-saddle point theorems and ϵ-duality theorems for **(SDP)**. Using Lemma 1.1, we can obtain the following two lemmas which are useful in proving our ϵ-saddle point theorems for **(SDP)**.

Lemma 3.1 [13] *Let* $\bar{x} \in A$. *Then* $\bar{x} \in A$ *is an* ϵ*-approximate solution of* **(SDP)** *if and only if there exists a sequence* $\{Z_n\}$ *in* S *such that*

$$f(x) - \liminf_{n\to\infty} Tr[Z_n F(x)] \geqq f(\bar{x}) - \epsilon, \quad \textit{for any } x \in C.$$

Lemma 3.2 [13, 14] *Let* $\bar{x} \in A$. *Suppose that* $\bigcup_{(Z,\delta)\in S\times\mathbb{R}_+} \left\{ \begin{pmatrix} \hat{F}^*(Z) \\ -Tr[ZF_0] - \delta \end{pmatrix} \right\} - C^* \times \mathbb{R}_+$ *is closed. Then* $\bar{x}$ *is an* ϵ*-approximate solution of* **(SDP)** *if and only if there exists* $Z \in S$ *such that for any* $x \in C$,

$$f(x) - Tr[ZF(x)] \geqq f(\bar{x}) - \epsilon.$$

Let $\epsilon \geqq 0$. Consider the following sequential ϵ-saddle point problem and ϵ-saddle point problem:

(SSP)$_\epsilon$ Find $\bar{x} \in C$ and a sequence $\{\bar{Z}_n\} \subset S$ such that

$$\begin{aligned} f(\bar{x}) - \liminf_{n\to\infty} Tr[Z_n F(\bar{x})] - \epsilon &\leqq f(\bar{x}) - \liminf_{n\to\infty} Tr[\bar{Z}_n F(\bar{x})] \\ &\leqq f(x) - \liminf_{n\to\infty} Tr[\bar{Z}_n F(x)] + \epsilon \end{aligned}$$

for any $x \in C$ and any sequence $\{Z_n\} \subset S$.

(SP)$_\epsilon$ Find $\bar{x} \in C$ and $\bar{Z} \in S$ such that

$$\begin{aligned} f(\bar{x}) - Tr[ZF(\bar{x})] - \epsilon &\leqq f(\bar{x}) - Tr[\bar{Z}F(\bar{x})] \\ &\leqq f(x) - Tr[\bar{Z}F(x)] + \epsilon \end{aligned}$$

for any $x \in C$ and any $Z \in S$.

Now we give a useful characterization of solution of **(SSP)$_\epsilon$**.

Lemma 3.3 [13] *Suppose that $A \neq \emptyset$. Let $(\bar{x}, \{\bar{Z}_n\}) \in C \times S$, $n = 1, 2, \cdots$. Then $(\bar{x}, \{\bar{Z}_n\})$ is a solution of* **(SSP)**$_\epsilon$ *if and only if*

$$f(\bar{x}) - \liminf_{n\to\infty} Tr[\bar{Z}_n F(\bar{x})] \leqq f(x) - \liminf_{n\to\infty} Tr[\bar{Z}_n F(x)] + \epsilon$$

for any $x \in C$,

$$0 \leqq \liminf_{n\to\infty} Tr[\bar{Z}_n F(\bar{x})] \leqq \epsilon$$

and $F(\bar{x}) \in S$.

Using Lemma 3.1 and 3.3, we can give a sequential ϵ-saddle point theorem which holds between **(SDP)** and **(SSP)**$_\epsilon$.

Theorem 3.1 (Sequential ϵ-Saddle Point Theorem) [13]

(1) *If $\bar{x} \in A$ is an ϵ-approximate solution of* **(SDP)**, *then there exists a sequence $\{\bar{Z}_n\}$ such that $(\bar{x}, \{\bar{Z}_n\})$ is a solution of* **(SSP)**$_\epsilon$

(2) *If $A \neq \emptyset$ and $(\bar{x}, \{\bar{Z}_n\})$ is a solution of* **(SSP)**$_\epsilon$, *then $\bar{x}$ is an 2ϵ-approximate solution of* **(SDP)**.

Using Lemma 3.2, we can give an ϵ-saddle point theorem which holds between **(SDP)** and **(SP)**$_\epsilon$.

Theorem 3.2 [13] **(ϵ- Saddle Point Theorem)** *Suppose that*

$$\bigcup_{(Z,\delta)\in S\times\mathbb{R}_+} \left\{ \begin{pmatrix} \hat{F}^*(Z) \\ -Tr[ZF_0] - \delta \end{pmatrix} \right\} - C^* \times \mathbb{R}_+$$

is closed. If $\bar{x} \in A$ is an ϵ-approximate solution of **(SDP)**, *then there exists $\bar{Z} \in S$ such that $(\bar{x}, \bar{Z})$ is a solution of* **(SP)**$_\epsilon$.

Theorem 3.3 [13] *If $(\bar{x}, \bar{Z})$ is a solution of* **(SP)**$_\epsilon$, *then $\bar{x}$ is an 2ϵ-approximate solution of* **(SDP)**.

Now we can formulate dual problem **(SDD)** of **(SDP)** as follows:

(SDD)	Maximize	$f(x) - Tr[ZF(x)]$
	subject to	$0 \in \partial_{\epsilon_0} f(x) - \hat{F}^*(Z) + N_C^{\epsilon_1}(x)$,
		$Z \succeq 0$,
		$\epsilon_0 + \epsilon_1 \in [0, \epsilon]$.

We can prove ϵ-weak and ϵ-strong duality theorems which hold between **(SDP)** and **(SDD)**.

Theorem 3.4 **(ϵ-Weak Duality)** [13, 14] *For any feasible x of* **(SDP)** *and any feasible (y, Z) of* **(SDD)**,

$$f(x) \geqq f(y) - Tr[ZF(y)] - \epsilon.$$

Theorem 3.5 **(ϵ-Strong Duality)** [13, 14] *Suppose that*

$$\bigcup_{(Z,\delta)\in S\times\mathbb{R}_+} \left\{ \begin{pmatrix} \hat{F}^*(Z) \\ -Tr[ZF_0] - \delta \end{pmatrix} \right\} - C^* \times \mathbb{R}_+ \text{ is closed.}$$

If $\bar{x}$ is an ϵ-approximate solution of **(SDP)**, *then there exists $\bar{Z} \in S$ such that $(\bar{x}, \bar{Z})$ is an 2ϵ-approximate solution of* **(SDD)**.

References

[1] N. Dinh, V. Jeyakumar and G. M. Lee, Sequential Lagrangian conditions for convex programs with applications to semidefinite programming, *J. Optim. Th. Appl.*, **125** (2005), 85-112.

[2] M. G. Govil and Mehra, ϵ-Optimality for multiobjective programming on a banach space, European. *J. Oper. Res.*, **157** (2004), 106-112.

[3] C. Gutiérrez, B. Jiménez and V. Novo, Multiplier rules and saddle-point theorems for Helbig's approximate solutions in convex pareto problems, *J. Global. Optim.*, **32** (2005), 367-383.

[4] A. Hamel, An ϵ-Lagrange multiplier rule for a mathematical programming problem on Banach spaces, *Optimization,* **49** (2001), 137-149.

[5] J. B. Hiriart-Urruty and C. Lemarechal, *Convex Analysis and Minimization Algorithms*, Volumes I and II, Springer-Verlag, Berlin, Heidelberg, 1993.

[6] V. Jeyakumar and B. M. Glover, Characterizing global optimality for DC optimization problems under convex inequality constraints, *J. Global. Optim.*, **8** (1996), 171-187.

[7] V. Jeyakumar, G. M. Lee and N. Dinh, New sequential Lagrange multiplier conditions characterizing optimality without constraint qualification for convex programs, *SIAM J. Optim.*, **14** (2003), 534-547.

[8] V. Jeyakumar and N. Dinh, Avoiding Duality Gaps in Convex Semidefinite Programming without Slater's Condition, Applied Mathematics Report AMR04/6, University of New South Wales, Sydney, Australia, 2004.

[9] V. Jeyakumar, G. M. Lee and N. Dinh, Lagrange multiplier conditions characterizing the optimal solution sets of cone-constrained convex programs, *J. Optim. Th. Appl.*, **1** (2004), 83-103.

[10] V. Jeyakumar, G. M. Lee and N. Dinh, Characterization of solution sets of convex vector minimization problems, *European J. Oper. Res.*, **174** (2006), 1380-1395.

[11] V. Jeyakumar, M. J. Nealon, Complete dual characterizations of optimality for convex semidefinite programming, *Canadian Mathematical Society Conference Proceeding*, **27** (2000), 165-173.

[12] G. S. Kim and G. M. Lee, On ϵ-Approximate solutions for convex semidefinite optimization problems, *Taiwanese. J. Math.*, **11** (2007), 765-784.

[13] J. H. Lee, On Epsilon Approximate Solutions of Convex Semidefinite Optimization Problems with Conic Constraints, *Thesis for the Degree of Master of Science*, Pukyong National University, Department of Applied Mathematics, Feb. 2010.

[14] G. M. Lee and J. H. Lee, ϵ-Duality Theorems for Convex Semidefinite Optimization Problems with Conic Constraints, *Journal of Inequalities and Applications* , to be appeared (doi:10.1155/2010/363012).

[15] J. C. Liu, ϵ-Duality theorem of nondifferentiable nonconvex multiobjective programming *J. Optim. Th. Appl.*, **69** (1991), 153-167.

[16] J. C. Liu, ϵ-Pareto optimality for nondifferentiable multiobjective programming via penalty function, *J. Math. Anal. Appl.*, **198** (1996), 248-261.

[17] M. V. Ramana, L. Tuncel and H. Wolkowicz, Strong duality for semidefinite programming, *SIAM J. Optim.*, **7** (1997), 641-662.

[18] J. J. Strodiot, V. H. Nguyen and N. Heukemes, ϵ-optimal solutions in nondifferentiable convex programming and some related questions, *Math. Programming*, **25** (1983), 307-328.

[19] K. Yokoyama and S. Shiraishi, An ϵ-Optimality Condition for Convex Programming Problems without Slater's Constraint Qualification, *preprint.*