Nonlinear Operators and Fixed Point Theorems
in Hilbert Spaces

Abstract. In this article, we first consider new classes of nonlinear mappings containing the class of firmly nonexpansive mappings which can be deduced from an equilibrium problem in a Hilbert space. Further, we deal with fixed point theorems and ergodic theorems for these nonlinear mappings.

1 Introduction

Let $H$ be a real Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Then a mapping $T : C \to H$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We know that if $C$ is a bounded closed convex subset of $H$ and $T : C \to C$ is nonexpansive, then the set $F(T)$ of fixed points of $T$ is nonempty. Further, from Baillon [1] we know the first nonlinear ergodic theorem in a Hilbert space: Let $C$ be a nonempty bounded closed convex subset of $H$ and let $T : C \to C$ be nonexpansive. Then, for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to an element $z \in F(T)$. An important example of nonexpansive mappings in a Hilbert space is a firmly nonexpansive mapping. A mapping $T$ is said to be firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle, \quad \forall x, y \in C;$$

see, for instance, Goebel and Kirk [8]. It is also known that a firmly nonexpansive mapping $T$ can be deduced from an equilibrium problem in a Hilbert space as follows: Let $C$ be a nonempty closed convex subset of $H$ and let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the following conditions:

(A1) $f(x, x) = 0$, $\forall x \in C$;

(A2) $f$ is monotone, i.e., $f(x, y) + f(y, x) \leq 0$, $\forall x, y \in C$;

(A3) $\lim_{t \downarrow 0} f(tz + (1 - t)x, y) \leq f(x, y)$, $\forall x, y, z \in C$;

(A4) for each $x \in C$, $y \mapsto f(x, y)$ is convex and lower semicontinuous.

We know the following lemma; see, for instance, [2] and [7].
Lemma 1.1. Let $C$ be a nonempty closed convex subset of $H$ and let $f$ be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying $(A1)$, $(A2)$, $(A3)$ and $(A4)$. Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that

$$f(z, y) + \frac{1}{r}(y - z, z - x) \geq 0, \quad \forall y \in C.$$ 

Further, if $T_r x = \{z \in C : f(z, y) + \frac{1}{r}(y - z, z - x) \geq 0, \forall y \in C\}$ for all $x \in H$, then the following hold:

1. $T_r$ is single-valued;
2. $T_r$ is firmly nonexpansive, i.e.,

$$\|T_r x - T_r y\|^2 \leq (T_r x - T_r y, x - y), \quad \forall x, y \in H.$$ 

Recently, Kohsaka and Takahashi [12] introduced the following nonlinear mapping: Let $E$ be a smooth, strictly convex and reflexive Banach space, let $J$ be the duality mapping of $E$ and let $C$ be a nonempty closed convex subset of $E$. Then, a mapping $S : C \to E$ is said to be nonspreading if

$$\phi(Sx, Sy) + \phi(Sy, Sx) \leq \phi(Sx, y) + \phi(Sy, x), \quad \forall x, y \in C,$$

where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$ for all $x, y \in E$. They considered such a mapping to study the resolvents of a maximal monotone operator in the Banach space. In the case when $E$ is a Hilbert space, we know that $\phi(x, y) = \|x - y\|^2$ for all $x, y \in E$. So, a nonspreading mapping $S$ in a Hilbert space is defined as follows:

$$2\|Sx - Sy\|^2 \leq \|Sx - y\|^2 + \|Sy - x\|^2, \quad \forall x, y \in C.$$ 

On the other hand, Takahashi [29] found another new nonlinear mapping called a hybrid mapping which is also deduced from a firmly nonexpansive mapping.

In this paper, we first discuss new classes of nonlinear mappings containing the class of firmly nonexpansive mappings which can be deduced from a firmly nonexpansive mapping in a Hilbert space. Further, we deal with fixed point theorems and ergodic theorems for these nonlinear mappings.

2 Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. In a Hilbert space, it is known that for all $x, y \in H$ and $\alpha \in \mathbb{R}$,

$$\|\alpha x + (1 - \alpha)y\|^2 = \alpha \|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2; \quad (2.1)$$

see, for instance, [28]. Further, in a Hilbert space, we have that for all $x, y, z, w \in H$,

$$2\langle x - y, z - w \rangle = \|x - w\|^2 + \|y - z\|^2 - \|x - z\|^2 - \|y - w\|^2. \quad (2.2)$$

Let $C$ be a nonempty subset of $H$ and let $T$ be a mapping of $C$ into $H$. We denote by $F(T)$ the set of all fixed points of $T$, that is, $F(T) = \{z \in C : Tz = z\}$. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. The following lemma is in [20].
Lemma 2.1. Let $C$ be a nonempty closed convex subset of $H$ and let $f : C \to (-\infty, \infty]$ be a proper convex lower semicontinuous function such that $f(z_m) \to \infty$ as $\|z_m\| \to \infty$. Then there exists an element $z_0 \in C$ such that

$$f(z_0) = \min\{f(z) : z \in C\}.$$ 

Let $\mathbb{N}$ be the set of positive integers and let $l^\infty$ be the Banach space of bounded sequences with supremum norm. Let $\mu$ be an element of $(l^\infty)^*$ (the dual space of $l^\infty$). Then, we denote by $\mu(f)$ the value of $\mu$ at $f = (x_1, x_2, x_3, \ldots) \in l^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional $\mu$ on $l^\infty$ is called a mean if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \ldots)$. A mean $\mu$ is called a Banach limit on $l^\infty$ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on $l^\infty$; see [20] for more details.

3 Nonlinear Mappings

Let $H$ be a Hilbert space. Let $C$ be a nonempty closed convex subset of $H$ and let $T$ be a mapping of $C$ into $H$. Then, from [29], we have the following equality:

$$\|Tx - Ty\|^2 = \|x - y -(Tx - Ty)\|^2 - \|x - y\|^2 + 2\langle x - y, Tx - Ty \rangle$$

(3.1)

for all $x, y \in C$. We have also from (2.2) that

$$2\langle x - y, Tx - Ty \rangle = \|x - Ty\|^2 + \|y - Tx\|^2 - \|x - Tx\|^2 - \|y - Ty\|^2.$$ 

(3.2)

Further, we have that

$$\|x - y -(Tx - Ty)\|^2 = \|x - Tx\|^2 + \|y - Ty\|^2 - 2\langle x - Tx, y - Ty \rangle.$$ 

(3.3)

If $T : C \to H$ is firmly nonexpansive, then

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle, \quad \forall x, y \in C.$$ 

So, we have from (3.1) that for all $x, y \in C$,

$$2\|Tx - Ty\|^2 \leq 2\langle x - y, Tx - Ty \rangle = \|Tx - Ty\|^2 - \|x - y -(Tx - Ty)\|^2 + \|x - y\|^2 \leq \|Tx - Ty\|^2 + \|x - y\|^2.$$ 

Then, we have $\|Tx - Ty\|^2 \leq \|x - y\|^2$ and hence $\|Tx - Ty\| \leq \|x - y\|$. Such a mapping is nonexpansive. We know that $T : C \to H$ is nonexpansive if and only if

$$2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Ts\|^2 - 2\langle x - Tx, y - Ty \rangle, \quad \forall x, y \in C;$$

see [29]. Thus, we can get new classes of nonlinear operators which contain the class of firmly nonexpansive mappings in a Hilbert space. For example, Kohsaka and Takahahi [12] obtained a nonspraying mapping. Let $T : C \to H$ be a firmly nonexpansive mapping. Then, we have that for all $x, y \in C$,

$$2\|Tx - Ty\|^2 \leq 2\langle x - y, Tx - Ty \rangle.$$

From (3.2), we obtain
\[ 2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2 - \|x - Tx\|^2 - \|y - Ty\|^2 \]
\[ \leq \|x - Ty\|^2 + \|y - Tx\|^2. \]
So, we have
\[ 2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2. \]
This is a nonspreading mapping. From Iemoto and Takahashi [10], we know the following lemma.

Lemma 3.1. Let $C$ be a nonempty closed convex subset of $H$. Then a mapping $S : C \to H$ is nonspreading if and only if
\[ \|Sx - Sy\|^2 \leq \|x - y\|^2 + 2\langle x - Sx, y - Sy \rangle \]
for all $x, y \in C$.

Further, from a firmly nonexpansive mapping, i.e.,
\[ \|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y \rangle, \quad \forall x, y \in C, \]
we have
\[ \|Tx - Ty\|^2 \leq 2\langle Tx - Ty, x - y \rangle, \quad \forall x, y \in C. \]
Such a mapping $T : C \to H$ is called $\frac{1}{2}$-inverse strongly monotone. Takahashi [29] also defined the following hybrid mapping, i.e.,
\[ 3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|y - Tx\|^2 + \|x - Ty\|^2, \quad \forall x, y \in C. \]
From Takahashi [29], we know the following lemma.

Lemma 3.2. Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Then a mapping $T : C \to H$ is hybrid if and only if
\[ \|Tx - Ty\|^2 \leq \|x - y\|^2 + \langle x - Tx, y - Ty \rangle, \quad \forall x, y \in C. \]
So, a hybrid mapping $T : C \to H$ is different from a nonspreading mapping. Further, we define a new nonlinear operator from a firmly nonexpansive mapping. We have that for any $x, y \in C$,
\[ 2\|Tx - Ty\|^2 \leq 2\langle x - y, Tx - Ty \rangle \]
\[ \iff \|Tx - Ty\|^2 + \|Tx\|^2 + \|Ty\|^2 - 2\langle Tx, Ty \rangle \leq 2\langle x - y, Tx - Ty \rangle \]
\[ \iff \|Tx - Ty\|^2 - 2\langle Tx, Ty \rangle \leq 2\langle x - y, Tx - Ty \rangle \]
\[ \iff \|Tx - Ty\|^2 \leq 2\langle Tx, Ty \rangle + 2\langle x - y, Tx - Ty \rangle. \]
So, we can define a new mapping called a metric mapping, i.e.,
\[ \|Tx - Ty\|^2 \leq 2\langle Tx, Ty \rangle + 2\langle x - y, Tx - Ty \rangle, \quad \forall x, y \in C. \]
Let $T : C \rightarrow H$ be a nonexpansive mapping and put $A = I - T$. Then, we have from [28] that $A$ is $1/2$-inverse strongly monotone, i.e.,

$$\frac{1}{2}\|Ax - Ay\|^2 \leq \langle x - y, Ax - Ay \rangle, \quad \forall x, y \in C.$$

Let $T : C \rightarrow H$ be a nonspreading mapping and put $A = I - T$. Then, we have from Lemma 3.1 and (3.1) that for any $x, y \in C$,

$$\|Ax - Ay\|^2 = \|x - y - (Ax - Ay)\|^2 - \|x - y\|^2 + 2\langle x - y, Ax - Ay \rangle$$

$$\leq \|x - y\|^2 + 2\langle x - Tx, y - Ty \rangle - \|x - y\|^2 + 2\langle x - y, Ax - Ay \rangle$$

$$= 2\langle Ax, Ay \rangle + 2\langle x - y, Ax - Ay \rangle.$$

This implies that $A$ is a metric mapping.

4 Generalized Fixed Point Theorem and its Applications

In this section, we obtain a generalized fixed point theorem in a Hilbert space. Before proving the theorem, we can obtain the following lemma from Lemma 2.1.

**Lemma 4.1.** Let $C$ be a nonempty closed convex subset of a Hilbert space $H$, let $\{x_n\}$ be a bounded sequence in $H$ and let $\mu$ be a Banach limit. If $g : C \rightarrow \mathbb{R}$ is defined by

$$g(z) = \mu_n\|x_n - z\|^2, \quad \forall z \in C,$$

then there exists a unique $z_0 \in C$ such that

$$g(z_0) = \min\{g(z) : z \in C\}.$$

Using Lemma 4.1, we can prove the following generalized fixed point theorem [31].

**Theorem 4.2.** Let $H$ be a Hilbert space, let $C$ be a nonempty closed convex subset of $H$ and let $T$ be a mapping of $C$ into itself. Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded and

$$\mu_n\|T^n x - Ty\|^2 \leq \mu_n\|T^n x - y\|^2, \quad \forall y \in C$$

for some Banach limit $\mu$. Then, $T$ has a fixed point in $C$.

**Proof.** Using a Banach limit $\mu$ on $l^\infty$, we can define $g : C \rightarrow \mathbb{R}$ as follows:

$$g(z) = \mu_n\|T^n x - z\|^2, \quad \forall z \in C.$$

From Lemma 4.1, there exists a unique $z_0 \in C$ such that

$$g(z_0) = \min\{g(z) : z \in C\}.$$

So, we have

$$g(Tz_0) = \mu_n\|T^n x - Tz_0\|^2 \leq \mu_n\|T^n x - z_0\|^2 = g(z_0).$$
Since $Tz_0$ is in $C$ and $z_0 \in C$ is a unique element such that
\[ g(z_0) = \min \{ g(z) : z \in C \}, \]
we have $Tz_0 = z_0$. This completes the proof. \qed

Using Theorem 4.2, we can obtain some fixed point theorems. The following is the well-known fixed point theorem for nonexpansive mappings in a Hilbert space; see, for instance, [28].

**Theorem 4.3.** Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $T : C \rightarrow C$ be a nonexpansive mapping, i.e.,
\[ \| Tx - Ty \| \leq \| x - y \|, \quad \forall x, y \in C. \]
Suppose that there exists an element $x \in C$ such that $\{ T^n x \}$ is bounded. Then, $T$ has a fixed point in $C$.

**Proof.** Let $\mu$ be a Banach limit on $l^\infty$. For any $n \in \mathbb{N}$ and $y \in C$, we have
\[ \| T^{n+1} x - Ty \|^2 \leq \| T^n x - y \|^2. \]
So, we have
\[ \mu_n \| T^n x - Ty \|^2 = \mu_n \| T^{n+1} x - Ty \|^2 \leq \mu_n \| T^n x - y \|^2 \]
for all $y \in C$. By Theorem 4.2, $T$ has a fixed point in $C$. \qed

The following is a fixed point theorem for nonexpansive mappings in a Hilbert space.

**Theorem 4.4 ([12]).** Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $T : C \rightarrow C$ be a nonexpansive mapping, i.e.,
\[ 2 \| Tx - Ty \|^2 \leq \| Tx - y \|^2 + \| Ty - x \|^2, \quad \forall x, y \in C. \]
Suppose that there exists an element $x \in C$ such that $\{ T^n x \}$ is bounded. Then, $T$ has a fixed point in $C$.

**Proof.** Let $\mu$ be a Banach limit on $l^\infty$. For any $n \in \mathbb{N}$ and $y \in C$, we have
\[ 2 \| T^{n+1} x - Ty \|^2 \leq \| T^{n+1} x - y \|^2 + \| T^n x - Ty \|^2. \]
So, we have
\[ 2 \mu_n \| T^n x - Ty \|^2 = 2 \mu_n \| T^{n+1} x - Ty \|^2 \leq \mu_n \| T^{n+1} x - y \|^2 + \mu_n \| T^n x - Ty \|^2 \]
and hence
\[ \mu_n \| T^n x - Ty \|^2 \leq \mu_n \| T^n x - y \|^2. \]
By Theorem 4.2, $T$ has a fixed point in $C$. \qed

The following is a fixed point theorem for hybrid mappings in a Hilbert space.
**Theorem 4.5** ([29]). Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $T : C \to C$ be a hybrid mapping, i.e.,
\[
\|Tx - Ty\|^2 \leq \|x - y\|^2 + (x - Tx, y - Ty), \quad \forall x, y \in C.
\]
Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded. Then, $T$ has a fixed point in $C$.

**Proof.** Let $\mu$ be a Banach limit on $l^\infty$. We know from Lemma 3.2 that a mapping $T : C \to C$ is hybrid if and only if
\[
3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.
\]
So, for any $n \in \mathbb{N}$ and $y \in C$, we have
\[
3\|T^{n+1}x - Ty\|^2 \leq \|T^nx - y\|^2 + \|T^{n+1}x - y\|^2 + \|T^nx - Ty\|^2.
\]
So, we have
\[
3\mu_n\|T^n x - Ty\|^2 = 3\mu_n\|T^{n+1}x - Ty\|^2 \\
\leq 2\mu_n\|T^nx - y\|^2 + \mu_n\|T^nx - Ty\|^2
\]
and hence
\[
\mu_n\|T^n x - Ty\|^2 \leq \mu_n\|T^nx - y\|^2.
\]
By Theorem 4.2, $T$ has a fixed point in $C$. $\square$

We can also prove the following two fixed point theorems in a Hilbert space.

**Theorem 4.6.** Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $T : C \to C$ be a mapping such that
\[
2\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2, \quad \forall x, y \in C.
\]
Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded. Then, $T$ has a fixed point in $C$.

**Proof.** Let $\mu$ be a Banach limit on $l^\infty$. For any $n \in \mathbb{N}$ and $y \in C$, we have
\[
2\|T^{n+1}x - Ty\|^2 \leq \|T^nx - y\|^2 + \|T^{n+1}x - y\|^2.
\]
So, we have
\[
2\mu_n\|T^n x - Ty\|^2 = 2\mu_n\|T^{n+1}x - Ty\|^2 \\
\leq 2\mu_n\|T^nx - y\|^2
\]
and hence
\[
\mu_n\|T^n x - Ty\|^2 \leq \mu_n\|T^nx - y\|^2.
\]
By Theorem 4.2, $T$ has a fixed point in $C$. $\square$
Theorem 4.7. Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $T : C \rightarrow C$ be a mapping such that

$$3\|Tx - Ty\|^2 \leq 2\|Tx - y\|^2 + \|Ty - x\|^2, \ \forall x, y \in C.$$  

Suppose that there exists an element $x \in C$ such that $\{T^nx\}$ is bounded. Then, $T$ has a fixed point in $C$.

Proof. Let $\mu$ be a Banach limit on $l^\infty$. For any $n \in \mathbb{N}$ and $y \in C$, we have

$$3\|T^{n+1}x - Ty\|^2 \leq 2\|T^{n+1}x - y\|^2 + \|T^nx - Ty\|^2.$$  

So, we have

$$3\mu_n\|T^nx - Ty\|^2 \leq 2\mu_n\|T^nx - y\|^2 + \mu_n\|T^nx - Ty\|^2$$  

and hence

$$\mu_n\|T^nx - Ty\|^2 \leq \mu_n\|T^nx - y\|^2.$$  

By Theorem 4.2, $T$ has a fixed point in $C$.  

We also know the following theorem by Ray [15].

Theorem 4.8. Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Then, the following are equivalent:

(i) Every nonexpansive mapping of $C$ into itself has a fixed point in $C$;

(ii) $C$ is bounded.

Using Ray's theorem, we can prove the following theorem [29].

Theorem 4.9. Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Then, the following are equivalent:

(i) Every hybrid mapping of $C$ into itself has a fixed point in $C$;

(ii) $C$ is bounded.

Proof. From Theorem 4.5, we know that (ii) implies (i). Let us show that (i) implies (ii). We know that every firmly nonexpansive mapping is a hybrid mapping. So, the class of hybrid mappings of $C$ into itself contains the class of firmly nonexpansive mappings of $C$ into itself. To show (i) $\implies$ (ii), it is sufficient to show that if every firmly nonexpansive mapping in $C$ into itself has a fixed point in $C$, then every nonexpansive mapping of $C$ into itself has a fixed point in $C$. Let $T$ be a nonexpansive mapping of $C$ into itself. Then, $S = \frac{1}{2}I + \frac{1}{2}T$ is a firmly nonexpansive mapping. Further, it is not difficult to show $F(T) = F(S)$. So, every firmly nonexpansive mapping in $C$ into itself has a fixed point in $C$ if and only if every nonexpansive mapping of $C$ into itself has a fixed point in $C$. This completes the proof.  

Similarly, we have the following theorem.

Theorem 4.10. Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Then, the following are equivalent:

(i) Every nonspraying mapping of $C$ into itself has a fixed point in $C$;

(ii) $C$ is bounded.
5 Nonlinear Ergodic Theorems

Baillon [1] proved the first nonlinear ergodic theorem for nonexpansive mappings in a Hilbert space.

Theorem 5.1. Let $H$ be a Hilbert space, let $C$ be a nonempty closed convex subset of $H$ and let $T$ be a nonexpansive mapping of $C$ into itself such that $F(T)$ is nonempty. Then, for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to an element $z \in F(T)$.

We can also prove the following nonlinear ergodic theorem [31] for our nonlinear operators in a Hilbert space. Before proving it, we need, for example, the following result [31].

Lemma 5.2. Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $T : C \rightarrow C$ be a mapping such that

$$2 \|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2, \quad \forall x, y \in C.$$

Then $T$ is demiclosed, i.e., $x_n \rightarrow u$ and $x_n - Tx_n \rightarrow 0$ imply $u \in F(T)$.

Proof. Let $\{x_n\} \subset C$ be a sequence such that $x_n \rightarrow u$ and $x_n - Tx_n \rightarrow 0$ as $n \rightarrow \infty$. Then the sequences $\{x_n\}$ and $\{Tx_n\}$ are bounded. Suppose that $u \neq Tu$. From Opial’s theorem [14], we have

$$\liminf_{n \rightarrow \infty} \|x_n - u\|^2 < \liminf_{n \rightarrow \infty} \|x_n - Tu\|^2$$

$$= \liminf_{n \rightarrow \infty} \|x_n - Tx_n + Tx_n - Tu\|^2$$

$$= \liminf_{n \rightarrow \infty} \|Tx_n - Tu\|^2$$

$$\leq \liminf_{n \rightarrow \infty} \frac{1}{2} (\|x_n - u\|^2 + \|Tx_n - u\|^2)$$

$$= \liminf_{n \rightarrow \infty} \frac{1}{2} (\|x_n - u\|^2 + \|Tx_n - x_n + x_n - u\|^2)$$

$$= \liminf_{n \rightarrow \infty} \|x_n - u\|^2.$$

This is a contradiction. Hence we get the conclusion.

Theorem 5.3. Let $H$ be a Hilbert space, let $C$ be a nonempty closed convex subset of $H$ and let $T$ be a mapping of $C$ into itself such that $F(T)$ is nonempty. Suppose that $T$ satisfies one of the following conditions:

(i) $T$ is nonspreading;
(ii) $T$ is hybrid;
(iii) $2 \|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2$, $\forall x, y \in C$;
(iv) $3 \|Tx - Ty\|^2 \leq 2 \|Tx - y\|^2 + \|Ty - x\|^2$, $\forall x, y \in C$. 
Then, for any \( x \in C \),
\[
S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x
\]
converges weakly to an element \( z \in F(T) \).

**Proof.** Let us prove the case of (iii) by using Lemma 5.2. We first show that \( F(T) \) is closed and convex. It follows from Lemma 5.2 that \( F(T) \) is closed. In fact, let \( \{ x_n \} \subset F(T) \) and \( x_n \rightarrow z \). Then, we have \( x_n \rightarrow z \) and \( x_n - Tx_n = 0 \). So, from Lemma 5.2 we have \( z = Tz \). This implies that \( F(T) \) is closed. Let us show that \( F(T) \) is convex. Let \( x, y \in F(T) \) and let \( \alpha \in [0, 1] \). Put \( z = \alpha x + (1 - \alpha) y \). Then, we have from (2.1) that
\[
\| z - Tz \|^2 = \| \alpha x + (1 - \alpha) y - Tz \|^2
\]
\[
= \alpha \| x - Tz \|^2 + (1 - \alpha) \| y - Tz \|^2 - \alpha (1 - \alpha) \| x - y \|^2
\]
\[
= \alpha \| T x - Tz \|^2 + (1 - \alpha) \| T y - Tz \|^2 - (1 - \alpha) \| x - y \|^2
\]
\[
\leq \alpha (1 - \alpha)^2 \| x - y \|^2 + (1 - \alpha) \alpha^2 \| x - y \|^2 - (1 - \alpha) \| x - y \|^2
\]
\[
= (1 - \alpha^2 + (1 - \alpha) \alpha - 1) \| x - y \|^2
\]
\[
= 0.
\]
So, we have \( Tz = z \). This implies that \( F(T) \) is convex. Let \( x \in C \) and let \( P \) be the metric projection of \( H \) onto \( F(T) \). Then, we have
\[
\| P T^n x - T^n x \| \leq \| P T^{n-1} x - T^{n-1} x \|
\]
\[
= \| T P T^{n-1} x - T^{n-1} x \|
\]
\[
\leq \| P T^{n-1} x - T^{n-1} x \|.
\]
This implies that \( \{ \| P T^n x - T^n x \| \} \) is nonincreasing. We also know that for any \( v \in C \) and \( u \in F(T) \),
\[
\langle v - P v, P v - u \rangle \geq 0
\]
and hence
\[
\| v - P v \|^2 \leq \langle v - P v, v - u \rangle.
\]
So, we get
\[
\| P v - u \|^2 = \| P v - v + v - u \|^2
\]
\[
= \| P v - v \|^2 - 2 \langle P v - v, u - v \rangle + \| v - u \|^2
\]
\[
\leq \| v - u \|^2 - \| P v - v \|^2.
\]
Let \( m, n \in \mathbb{N} \) with \( m \geq n \). Putting \( v = T^m x \) and \( u = P T^n x \), we have
\[
\| P T^m x - P T^n x \|^2 \leq \| T^m x - P T^n x \|^2 - \| P T^m x - T^m x \|^2
\]
\[
\leq \| T^m x - P T^n x \|^2 - \| P T^m x - T^m x \|^2.
\]
So, \( \{ P T^n x \} \) is a Cauchy sequence. Since \( F(T) \) is closed, \( \{ P T^n x \} \) converges strongly to an element \( p \) of \( F(T) \). Take \( u \in F(T) \). Then we obtain that for any \( n \in \mathbb{N} \),
\[
\| S_n x - u \| \leq \frac{1}{n} \sum_{k=0}^{n-1} \| T^k x - u \| \leq \| x - u \|. 
\]
So, $\{S_n x\}$ is bounded and hence there exists a weakly convergent subsequence $\{S_{n_i} x\}$ of $\{S_n x\}$. If $S_{n_i} x \rightarrow v$, then we have $v \in F(T)$. In fact, for any $y \in C$ and $k \in \mathbb{N} \cup \{0\}$, we have that

$$2\|T^{k+1}x - Ty\|^2 \leq \|T^k x - y\|^2 + \|T^{k+1} x - y\|^2$$

$$= \|T^k x - Ty\|^2 + 2\langle T^k x - Ty, Ty - y \rangle + \|Ty - y\|^2$$

$$+ \|T^{k+1} x - Ty\|^2 + 2\langle T^{k+1} x - Ty, Ty - y \rangle + \|Ty - y\|^2.$$ 

So, we obtain that

$$\|T^{k+1} x - Ty\|^2 \leq \|T^k x - Ty\|^2 + 2\langle T^k x - Ty, Ty - y \rangle + 2\|Ty - y\|^2.$$ 

Summing these inequalities with respect to $k = 0, 1, \ldots, n - 1$, we have

$$\|T^n x - Ty\|^2 \leq \|x - Ty\|^2 + 2\sum_{k=0}^{n-1} \langle T^k x - nTy, Ty - y \rangle$$

$$+ 2\sum_{k=0}^{n-1} \langle T^{k+1} x - nTy, Ty - y \rangle + 2n\|Ty - y\|^2$$

$$= \|x - Ty\|^2 + 4\sum_{k=0}^{n-1} \langle T^k x - nTy, Ty - y \rangle$$

$$+ 2\langle T^n x - x, Ty - y \rangle + 2n\|Ty - y\|^2.$$ 

Deviding this inequality by $n$, we have

$$\frac{1}{n}\|T^n x - Ty\|^2 \leq \frac{1}{n}\|x - Ty\|^2 + 4\langle S_n x - Ty, Ty - y \rangle$$

$$+ \frac{2}{n}\langle T^n x - x, Ty - y \rangle + 2\|Ty - y\|^2,$$

where $S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$. Replacing $n$ by $n_i$ and letting $n_i \rightarrow \infty$, we obtain from $S_{n_i} x \rightarrow v$ that

$$0 \leq 2\|Ty - y\|^2 + 4\langle v - Ty, Ty - y \rangle.$$ 

Putting $y = v$, we have $0 \leq 2\|Tv - v\|^2 + 4\langle v - Tv, Tv - v \rangle$ and hence

$$0 \leq \|Tv - v\|^2 + 2\langle v - Tv, Tv - v \rangle.$$ 

So, we have $0 \leq -\|Tv - v\|^2$ and hence $Tv = v$. To complete the proof of (iii), it is sufficient to show that if $S_n x \rightarrow v$ and $p = \lim_{n \rightarrow \infty} PT^n x$, then $v = p$. We have that for any $u \in F(T)$,

$$\langle T^k x - PT^k x, PT^k x - u \rangle \geq 0.$$ 

Since $\{\|T^k x - PT^k x\|\}$ is nonincreasing, we have

$$\langle u - p, T^k x - PT^k x \rangle \leq \langle PT^k x - p, T^k x - PT^k x \rangle$$

$$\leq \|PT^k x - p\| \cdot \|T^k x - PT^k x\|$$

$$\leq \|PT^k x - p\| \cdot \|x - Px\|. $$
Adding these inequalities from $k = 0$ to $k = n - 1$ and dividing $n$, we have
\[
\langle u - p, S_{n}x - \frac{1}{n} \sum_{k=0}^{n-1} PT^{k}x \rangle \leq \frac{\|x - Px\|}{n} \sum_{k=0}^{n-1} \|PT^{k}x - p\|.
\]
Since $S_{n}x \rightarrow v$ and $PT^{k}x \rightarrow p$, we have
\[
\langle u - p, v - p \rangle \leq 0.
\]
We know $v \in F(T)$. So, putting $u = v$, we have $(v - p, v - p) \leq 0$ and hence $\|v - p\|^2 \leq 0$. So, we obtain $v = p$. This completes the proof of (iii). See [31] for the proofs of (1), (ii) and (iv).

References


W. Takahashi, Fixed point theorems for new nonlinear mappings in a Hilbert space, to appear.

