数理解析研究所講究録
第 1685 巻 2010 年 259-269

Properties and Examples of Unified Scalarizing Functions for Sets
(集合に対する統一的なスカラー化関数の性質といくつかの例)*

新潟大学大学院自然科学研究科　　孫田祐哉, 桑野一成, 田中　環
Yuuya Sonda, Issei Kuwano, Tamaki Tanaka†
Graduate School of Science and Technology,
Niigata University, Japan

Abstract

In the paper, we introduce two types of general nonlinear scalarizing functions for sets as images of set-valued maps and investigate how some kinds of cone-continuity for parent set-valued maps are inherited to these composite functions by unified types of scalarizing functions.

1 Introduction

Shimizu, Tanaka and Yamada [7] introduced twelve types of scalarizing functions for sets, and they called these functions unified types of scalarizing functions. In [4], they introduced some properties of these functions including the inheritance on convexity and semicontinuity of parent set-valued maps.

If a real-valued or vector-valued function has some kinds of convexity and continuity, then we can utilize such properties as a means to solve several types of equilibrium problems including variational inequalities, minimax problems, complementarity problems and optimization problems. In vector-valued case, we apply corresponding real-valued results as long as we find suitable scalarizing functions for objective vector-valued functions. However it is unclear whether any scalarizing function for set-valued maps plays such a similar role. Hence, it is important that we verify its inherited properties on cone-continuity of set-valued maps. The aim of this paper is to go ahead with the investigation into some inherited properties on cone-continuity of parent set-valued maps for unified types of scalarizing functions.

In the paper, we consider two types of composite functions of unified types of scalarizing functions and set-valued maps, proposed in [7], and we investigate the inheritance on cone-continuity of parent set-valued maps.

The organization of the paper is as follows. In Section 2, we introduce mathematical methodology on comparison between two sets in an ordered vector space and some definitions of continuity and cone-continuity for set-valued maps (see [3]). In Section 3, we introduce two types of nonlinear scalarizing functions for sets proposed by the unified approach in [7], and we investigate how some kinds of cone-continuity for set-valued maps are inherited to the new nonlinear scalarizing functions.

---

* This work is based on research 21540121 supported by Grant-in-Aid for Scientific Research (C) from Japan Society for the Promotion of Science.

† *E-mail:* {sonda_yuuya,kuwano}@m.sc.niigata-u.ac.jp, tamaki@math.sc.niigata-u.ac.jp

2000 *Mathematics Subject Classification.* 49J53, 54C60, 90C29.
Key words and phrases. Set-valued analysis, nonlinear scalarization, set-relation.

2 Mathematical Preliminaries

Let Y be a real topological vector space with the vector ordering $\leq_C$ induced by a nonempty convex cone C ($C + C = C$ and $\lambda C \subset C$ for all $\lambda \geq 0$) as follows:

$$x \leq_C y \text{ if } y - x \in C \text{ for } x, y \in Y.$$

It is well known that $\leq_C$ is reflexive and transitive where C is a convex cone, moreover, $\leq_C$ has invariable properties to vector space structure as translation and scalar multiplication. Then, the space Y is called an *ordered topological vector space*. In particular, if C is pointed, then $\leq_C$ is antisymmetric, and hence Y is a partially ordered topological vector space.

Throughout the paper, X is a real topological vector space, Y a real ordered topological vector space and F a set-valued map from X into $2^Y \setminus \{\emptyset\}$. Moreover, for any $A \subset Y$ we denote the interior, closure, complement of A by $\mathrm{int}\,(A)$, $\mathrm{cl}\,(A)$, $(A)^c$, respectively.

At first, we review some basic concepts of set-relation and some definitions of continuity and cone-continuity for set-valued maps.

Definition 2.1. ([3]) For nonempty sets A, $B \subset Y$ and convex cone C in Y, we write

$A \leq_C^{(1)} B$ by $A \subset \bigcap_{b \in B}(b - C)$, equivalently $B \subset \bigcap_{a \in A}(a + C)$;
$A \leq_C^{(2)} B$ by $A \cap \left(\bigcap_{b \in B}(b - C)\right) \neq \emptyset$;
$A \leq_C^{(3)} B$ by $B \subset (A + C)$;
$A \leq_C^{(4)} B$ by $\left(\bigcap_{a \in A}(a + C)\right) \cap B \neq \emptyset$;
$A \leq_C^{(5)} B$ by $A \subset (B - C)$;
$A \leq_C^{(6)} B$ by $A \cap (B - C) \neq \emptyset$, equivalently $(A + C) \cap B \neq \emptyset$.

Proposition 2.1. ([3]) *For nonempty sets $A, B \subset Y$, the following statements hold.*

$A \leq_C^{(1)} B$ *implies* $A \leq_C^{(2)} B$; $\quad A \leq_C^{(1)} B$ *implies* $A \leq_C^{(4)} B$;
$A \leq_C^{(2)} B$ *implies* $A \leq_C^{(3)} B$; $\quad A \leq_C^{(4)} B$ *implies* $A \leq_C^{(5)} B$;
$A \leq_C^{(3)} B$ *implies* $A \leq_C^{(6)} B$; $\quad A \leq_C^{(5)} B$ *implies* $A \leq_C^{(6)} B$.

Proposition 2.2. ([4]) *For nonempty sets $A, B \subset Y$, the following statements hold.*

i. For each $j = 1, \ldots, 6$,
$A \leq_C^{(j)} B$ *implies* $(A + y) \leq_C^{(j)} (B + y)$ *for* $y \in Y$, *and*
$A \leq_C^{(j)} B$ *implies* $\alpha A \leq_C^{(j)} \alpha B$ *for* $\alpha > 0$;
ii. For each $j = 1, \ldots, 5$, $\leq_C^{(j)}$ is transitive;
iii. For each $j = 3, 5, 6$, $\leq_C^{(j)}$ is reflexive.

Next, we recall some definitions of continuity and cone-continuity for set-valued maps. At first, we introduce the continuity for set-valued maps.

Definition 2.2. ([1]) A set-valued map F is called *lower continuous at* x_0 if for every open set $V \subset Y$ with $F(x) \cap V \neq \emptyset$, there exists an open neighborhood U of x_0 such that $F(x) \cap V \neq \emptyset$ for all $x \in U$. We shall say that F is *lower continuous on* X if it is lower continuous at every point $x \in X$.

Definition 2.3. ([1]) A set-valued map F is called *upper continuous at* x_0 if for every open set $V \subset Y$ with $F(x) \subset V$, there exists an open neighborhood U of x_0 such that $F(x) \subset V$ for all $x \in U$. We shall say that F is *upper continuous on* X if it is upper continuous at every point $x \in X$.

Finally, we introduce the definitions of cone-continuity for set-valued maps.

Definition 2.4. ([2]) Let C be a convex cone in Y with nonempty interior, a set-valued map F is called *C-lower continuous at* x_0 if for every $y_0 \in F(x_0)$ and open neighborhood G of $\theta_Y \in Y$, there exists an open neighborhood U of x_0 such that $F(x) \cap (y_0 + G + C) \neq \emptyset$ for all $x \in U \cap \mathrm{Dom}F$. We shall say that F is *C-upper continuous on X* if it is C-upper continuous at every point $x \in X$.

Definition 2.5. ([2]) Let C be a convex cone in Y with nonempty interior, a set-valued map F is called *C-upper continuous at x_0* if for every open neighborhood V of $F(x_0)$, there exists an open neighborhood U of x_0 such that $F(x) \subset V + C$ for all $x \in U \cap \mathrm{Dom}F$. We shall say that F is *C-lower continuous on X* if it is C-lower continuous at every point $x \in X$.

3 Inherited Properties on Cone-Continuity of Set-Valued Maps

At first, we introduce the definition of two types of nonlinear scalarizing functions for sets proposed by a unified approach in [7]

Let V and V' be nonempty subsets of Y, and direction $k \in C \setminus (-\mathrm{cl}\,(C))$. For each $j = 1, \ldots, 6$, $I^{(j)}_{k,V'} : 2^Y \setminus \{\emptyset\} \to \mathbb{R} \cup \{\pm\infty\}$ and $S^{(j)}_{k,V'} : 2^Y \setminus \{\emptyset\} \to \mathbb{R} \cup \{\pm\infty\}$ are defined by

$$I^{(j)}_{k,V'}(V) := \inf\left\{ t \in \mathbb{R} \,\middle|\, V \leq^{(j)}_C (tk + V') \right\},$$

$$S^{(j)}_{k,V'}(V) := \sup\left\{ t \in \mathbb{R} \,\middle|\, (tk + V') \leq^{(j)}_C V \right\},$$

respectively.

In this section, we introduce inherited properties on continuity and cone-continuity of set-valued maps. At first, we remark that the unified types of scalarizing functions have an important merit on the inheritance properties in contrast with the approach of [6].

Let $V' \in 2^Y \setminus \{\emptyset\}$ and direction $k \in C \setminus \ (-\mathrm{cl}\,(C))$. For any $x \in X$ and for each $j = 1, \ldots, 6$, we consider the following composite functions:

$$(I^{(j)}_{k,V'} \circ F)(x) := I^{(j)}_{k,V'}(F(x)),$$

$$(S^{(j)}_{k,V'} \circ F)(x) := S^{(j)}_{k,V'}(F(x)).$$

Then, we can directly discuss inherited properties on cone-continuity of parent set-valued map F to $I^{(j)}_{k,V'} \circ F$ and $S^{(j)}_{k,V'} \circ F$ in an analogous fashion to linear scalarizing function like inner product. For this end, we consider the following level sets;

$$\mathrm{lev}^l_r(f) := \{x \in X | f(x) \leq r\},$$

$$\mathrm{lev}^u_r(f) := \{x \in X | r \leq f(x)\},$$

where $f : X \to \mathbb{R} \cup \{\pm\infty\}$. Then, we show how some kinds of cone-continuity of parent set-valued maps are inherited to these composite functions by unified types of scalarizing functions.

Theorem 3.1. ([5]) *Let F be a set-valued map. Then, the following statements are hold.*

i. For each $j = 1, 4, 5$,
  - *(a) If $F$is lower continuous, then $I^{(j)}_{k,V'} \circ F$ is lower semicontinuous on X,*
  - *(b) If $F$is upper continuous, then $I^{(j)}_{k,V'} \circ F$ is upper semicontinuous on X.*

ii. For each $j = 2, 3, 6$,
  - *(a) If $F$is lower continuous, then $I^{(j)}_{k,V'} \circ F$ is upper semicontinuous on X,*
  - *(b) If $F$is upper continuous, then $I^{(j)}_{k,V'} \circ F$ is lower semicontinuous on X.*

Corollary 3.1. ([5]) *Let F be a set-valued map. Then, the following statements are hold.*

i. For each $j = 1, 2, 3$,
  - *(a) If $F$is lower continuous, then $S^{(j)}_{k,V'} \circ F$ is upper semicontinuous on X,*

(b) *If $F$is upper continuous, then $S^{(j)}_{k,V'} \circ F$ is lower semicontinuous on X.*

ii. *For each $j = 4, 5, 6$,*

(a) *If $F$is lower continuous, then $S^{(j)}_{k,V'} \circ F$ is lower semicontinuous on X,*

(b) *If $F$is upper continuous, then $S^{(j)}_{k,V'} \circ F$ is upper semicontinuous on X.*

Theorem 3.2. *Let F be a set-valued map and C be a convex cone in Y. Then, the following statements are hold.*

i. *For each $j = 1, 4, 5$,*

(a) *If $F$is C-lower continuous, then $I^{(j)}_{k,V'} \circ F$ is lower semicontinuous on X,*

(b) *If $F$is $(-C)$-upper continuous, then $I^{(j)}_{k,V'} \circ F$ is upper semicontinuous on X.*

ii. *For each $j = 2, 3, 6$,*

(a) *If $F$is $(-C)$-lower continuous, then $I^{(j)}_{k,V'} \circ F$ is upper semicontinuous on X,*

(b) *If $F$is C-upper continuous, then $I^{(j)}_{k,V'} \circ F$ is lower semicontinuous on X.*

Proof. First, we prove (i)-(a) and (ii)-(b). For each $j = 1, \ldots, 6$, we show that

$$\mathrm{lev}^l_r(I) := \{x \in X | (I^{(j)}_{k,V'} \circ F)(x) \le r\}$$

is closed for any $r \in \mathbb{R}$, that is, for any $\{x_\alpha\}_{\alpha \in J} \subset \mathrm{lev}^l_r(I)$,

$$x_\alpha \to \bar{x} \Rightarrow \bar{x} \in \mathrm{lev}^l_r(I),$$

where J is a directed set. Let $t_{\bar{x}} := I^{(j)}_{k,V'} \circ F(\bar{x})$, $t_\alpha := I^{(j)}_{k,V'} \circ F(x_\alpha)$, and assume that $\bar{x} \notin \mathrm{lev}^l_r(I)$. Then, there exists $\epsilon > 0$ such that $t_\alpha \le r < r + \epsilon < t_{\bar{x}}$ and so we obtain

$$F(\bar{x}) \not\le^{(j)}_C (r+\epsilon)k + V' \quad \text{and} \quad F(x_\alpha) \le^{(j)}_C (r+\epsilon)k + V'. \tag{3.1}$$

At first, we consider the case of $j = 1$. By (3.1) and the definition of type (1) set-relation, we obtain

$$F(\bar{x}) \not\subset \bigcap_{z \in (r+\epsilon)k+V'} (z - C) \quad \text{and} \quad F(x_\alpha) \subset \bigcap_{z \in (r+\epsilon)k+V'} (z - C).$$

Since $r + \epsilon < t_{\bar{x}}$, there exists $\delta > 0$ such that $r + \epsilon < r + \epsilon + \delta < t_{\bar{x}}$ and so we obtain

$$F(\bar{x}) \not\le^{(1)}_C (r+\epsilon+\delta)k + V' \quad \text{and} \quad F(x_\alpha) \le^{(1)}_C (r+\epsilon+\delta)k + V'. \tag{3.2}$$

Since $\delta > 0$,

$$\bigcap_{z \in (r+\epsilon)k+V'} (z - C) \subset \bigcap_{z \in (r+\epsilon+\delta)k+V'} (z - C),$$

and so we obtain

$$\left\{ \bigcap_{z \in (r+\epsilon+\delta)k+V'} (z - C) \right\}^c \subset \left\{ \bigcap_{z \in (r+\epsilon)k+V'} (z - C) \right\}^c.$$

Since C is a convex cone, we obtain

$$\left\{ \bigcap_{z \in (r+\epsilon+\delta)k+V'} (z - C) \right\}^c + C \subset \left\{ \bigcap_{z \in (r+\epsilon)k+V'} (z - C) \right\}^c + C.$$

Hence, by (3.2), we have

$$F(\bar{x}) \cap \left(\mathrm{int} \left(\left\{ \bigcap_{z \in (r+\epsilon)k+V'} (z - C) \right\}^c + C \right) \right) \ne \emptyset,$$

and

$$F(x_\alpha) \cap \left(\text{int}\left(\left\{\bigcap_{z\in(r+\epsilon)k+V'}(z-C)\right\}^c + C\right)\right) = \emptyset.$$

Since α is an arbitrary and $x_\alpha \to \bar{x}$, for any U which is a neighborhood of x, there exists $\alpha(U) \in J$ such that

$$x_{\alpha(U)} \in U \quad \text{and} \quad F(x_{\alpha(U)}) \cap \left(\text{int}\left(\left\{\bigcap_{z\in(r+\epsilon)k+V'}(z-C)\right\}^c + C\right)\right) = \emptyset.$$

This is a contradiction to the C-lower continuity of F on X.

Next, we consider the case of $j = 4$. By (3.1) and the definition of type (4) set-relation, we obtain

$$\left(\bigcap_{y_1\in F(\bar{x})} (y_1 + C)\right) \cap ((r+\epsilon)k + V') = \emptyset,$$

and

$$\left(\bigcap_{y_2\in F(x_\alpha)} (y_2 + C)\right) \cap ((r+\epsilon)k + V') \neq \emptyset.$$

Let $T := \bigcup_{\alpha\in J} F(x_\alpha)$. Then we obtain

$$F(\bar{x}) \not\subset \text{cl}\,(\sup_C(T) - C) \quad \text{and} \quad F(x_\alpha) \subset \text{cl}\,(\sup_C(T) - C).$$

Hence we have

$$F(\bar{x}) \cap \{\text{cl}\,(\sup_C(T) - C)\}^c \neq \emptyset \quad \text{and} \quad F(x_\alpha) \cap \{\text{cl}\,(\sup_C(T) - C)\}^c = \emptyset.$$

So we obtain

$$F(\bar{x}) \cap (\{\text{cl}\,(\sup_C(T) - C)\}^c + C) \neq \emptyset \quad \text{and} \quad F(x_\alpha) \cap (\{\text{cl}\,(\sup_C(T) - C)\}^c + C) = \emptyset.$$

This is a contradiction to the C-lower continuity of F on X.

Next, we consider the case of $j = 5$. By (3.1) and the definition of type (5) set-relation, we obtain

$$F(\bar{x}) \not\subset (r+\epsilon)k + V' - C \quad \text{and} \quad F(x_\alpha) \subset (r+\epsilon)k + V' - C.$$

Since $r + \epsilon < t_{\bar{x}}$, there exists $\delta > 0$ such that $r + \epsilon < r + \epsilon + \delta < t_{\bar{x}}$ and so we obtain

$$F(\bar{x}) \not\leq_C^{(5)} (r+\epsilon+\delta)k + V' \quad \text{and} \quad F(x_\alpha) \leq_C^{(5)} (r+\epsilon+\delta)k + V'. \tag{3.3}$$

Since $\delta > 0$,

$$\text{cl}\,((r+\epsilon)k + V' - C) \subset (r+\epsilon+\delta)k + V' - C,$$

and so we obtain

$$\{(r+\epsilon+\delta)k + V' - C\}^c \subset \{\text{cl}\,((r+\epsilon)k + V' - C)\}^c.$$

Since C is a convex cone, we obtain

$$\{(r+\epsilon+\delta)k + V' - C\}^c + C \subset \{\text{cl}\,((r+\epsilon)k + V' - C)\}^c + C.$$

Hence, by (3.3), we have

$$F(\bar{x}) \cap (\{\text{cl}\,((r+\epsilon)k + V' - C)\}^c + C) \neq \emptyset,$$

and

$$F(x_\alpha) \cap (\{\text{cl}\,((r+\epsilon)k + V' - C)\}^c + C) = \emptyset.$$

This is a contradiction to the C-lower continuity of F on X. Consequently, for each $j = 1, 4, 5$, $I_{k,V'}^{(j)} \circ F$ is lower semicontinuous on X.

Next, we consider the case of $j = 2$. By (3.1) and the definition of type (2) set-relation, we obtain

$$F(\bar{x}) \cap \left(\bigcap_{z \in (r+\epsilon)k+V'} (z - C) \right) = \emptyset \quad \text{and} \quad F(x_\alpha) \cap \left(\bigcap_{z \in (r+\epsilon)k+V'} (z - C) \right) \neq \emptyset.$$

Since $r < r + \epsilon < t_{\bar{x}}$, there exists $\delta > 0$ such that $r < r + \epsilon - \delta < r + \epsilon < t_{\bar{x}}$, and so we obtain

$$F(\bar{x}) \not\leq_C^{(2)} (r + \epsilon - \delta)k + V' \quad \text{and} \quad F(x_\alpha) \leq_C^{(2)} (r + \epsilon - \delta)k + V'. \tag{3.4}$$

So,

$$\bigcap_{z' \in (r+\epsilon-\delta)k+V'} (z' - C) \subset \operatorname{cl}\left(\bigcap\nolimits_{z \in (r+\epsilon)k+V'} (z' - C) \right),$$

and so, we obtain

$$\left\{ \operatorname{cl}\left(\bigcap\nolimits_{z \in (r+\epsilon)k+V'} (z' - C) \right) \right\}^c \subset \left\{ \bigcap_{z' \in (r+\epsilon-\delta)k+V'} (z' - C) \right\}^c.$$

Since C is a convex cone, we obtain

$$\left\{ \operatorname{cl}\left(\bigcap\nolimits_{z \in (r+\epsilon)k+V'} (z' - C) \right) \right\}^c + C \subset \left\{ \bigcap_{z' \in (r+\epsilon-\delta)k+V'} (z' - C) \right\}^c + C.$$

Hence, by (3.4), we have

$$F(\bar{x}) \subset \left\{ \operatorname{cl}\left(\bigcap\nolimits_{z \in (r+\epsilon)k+V'} (z' - C) \right) \right\}^c + C,$$

and

$$F(x_\alpha) \not\subset \left\{ \operatorname{cl}\left(\bigcap\nolimits_{z \in (r+\epsilon)k+V'} (z' - C) \right) \right\}^c + C.$$

Since α is an arbitrary and $x_\alpha \to \bar{x}$, for any U which is a neighborhood of x, there exists $\alpha(U) \in J$ such that

$$x_{\alpha(U)} \in U \quad \text{and} \quad F(x_{\alpha(U)}) \not\subset \left\{ \operatorname{cl}\left(\bigcap\nolimits_{z \in (r+\epsilon)k+V'} (z' - C) \right) \right\}^c + C.$$

This is a contradiction to the C-upper continuity of F on X.

Next, we consider the case of $j = 3$. By (3.1) and the definition of type (3) set-relation, we obtain

$$(r + \epsilon)k + V' \not\subset F(\bar{x}) + C \quad \text{and} \quad (r + \epsilon)k + V' \subset F(x_\alpha) + C.$$

Since $r < r + \epsilon < t_{\bar{x}}$, there exists $\delta > 0$ such that $r \leq r + \epsilon - \delta < r + \epsilon < t_{\bar{x}}$, and so we obtain

$$(r + \epsilon - \delta)k + V' \not\subset F(\bar{x}) + C \quad \text{and} \quad (r + \epsilon - \delta)k + V' \subset F(x_\alpha) + C.$$

Hence,

$$(r + \epsilon)k + V' \not\subset F(\bar{x}) + C + \delta k \quad \text{and} \quad (r + \epsilon)k + V' \subset F(x_\alpha) + C + \delta k,$$

and so, by $\delta > 0$, we have

$$F(\bar{x}) \subset \operatorname{int}(F(\bar{x}) + C + \delta k) \quad \text{and} \quad F(x_\alpha) \not\subset \operatorname{int}(F(\bar{x}) + C + \delta k).$$

Since C is a convex cone, we obtain

$$F(\bar{x}) \subset \operatorname{int}(F(\bar{x}) + C + \delta k) + C \quad \text{and} \quad F(x_\alpha) \not\subset \operatorname{int}(F(\bar{x}) + C + \delta k) + C.$$

This is a contradiction to the C-upper continuity of F on X.

Finally, we consider the case of $j=6$. By (3.1) and the definition of type (6) set-relation, we obtain

$$F(\bar{x})\cap\{(r+\epsilon)k+V'-C\}=\emptyset \quad\text{and}\quad F(x_\alpha)\cap\{(r+\epsilon)k+V'-C\}\neq\emptyset.$$

Since $r<r+\epsilon<t_{\bar{x}}$, there exists $\delta>0$ such that $r<r+\epsilon<r+\epsilon+\delta<t_{\bar{x}}$, and so we obtain

$$F(\bar{x})\not\leq_C^{(6)}(r+\epsilon+\delta)k+V' \quad\text{and}\quad F(x_\alpha)\leq_C^{(6)}(r+\epsilon+\delta)k+V'. \tag{3.5}$$

Since $\delta>0$,

$$\mathrm{cl}\,((r+\epsilon)k+V'-C)\subset(r+\epsilon+\delta)k+V'-C,$$

and so we obtain

$$\{(r+\epsilon)k+V'-C\}^c\subset\{\mathrm{cl}\,((r+\epsilon)k+V'-C)\}^c.$$

Since C is a convex cone, we obtain and so we obtain

$$\{(r+\epsilon)k+V'-C\}^c+C\subset\{\mathrm{cl}\,((r+\epsilon)k+V'-C)\}^c+C.$$

Hence, by (3.5), we have

$$F(\bar{x})\subset\{\mathrm{cl}\,((r+\epsilon)k+V'-C)\}^c+C \quad\text{and}\quad F(x_\alpha)\not\subset\{\mathrm{cl}\,((r+\epsilon)k+V'-C)\}^c+C.$$

This is a contradiction to the C-upper continuity of F on X. Consequently, for each $j=2,3,6$, $I_{k,V'}^{(j)}\circ F$ is lower semicontinuous on X.

Next, we prove (i)-(b) and (ii)-(a). For each $j=1,\dots,6$, we show that

$$\mathrm{lev}_r^u(I):=\{x\in X|r\leq(I_{k,V'}^{(j)}\circ F)(x)\}$$

is closed for any $r\in\mathbb{R}$, that is, for any $\{x_\alpha\}_{\alpha\in J}\subset\mathrm{lev}_r^u(I)$,

$$x_\alpha\to\bar{x}\Rightarrow\bar{x}\in\mathrm{lev}_r^u(I),$$

where J is a directed set. Let $t_{\bar{x}}:=I_{k,V'}^{(j)}\circ F(\bar{x})$, $t_\alpha:=I_{k,V'}^{(j)}\circ F(x_\alpha)$, and assume that $\bar{x}\notin\mathrm{lev}_r^u(I)$. Then, there exists $\epsilon>0$ such that $t_{\bar{x}}<r-\epsilon<r\leq t_\alpha$ and so we obtain

$$F(\bar{x})\leq_C^{(j)}(r-\epsilon)k+V' \quad\text{and}\quad F(x_\alpha)\not\leq_C^{(j)}(r-\epsilon)k+V'. \tag{3.6}$$

At first, we consider the case of $j=1$. By (3.6) and the definition of type (1) set-relation, we obtain

$$F(\bar{x})\subset\bigcap_{z\in(r-\epsilon)k+V'}(z-C) \quad\text{and}\quad F(x_\alpha)\not\subset\bigcap_{z\in(r-\epsilon)k+V'}(z-C).$$

Since $t_{\bar{x}}<r-\epsilon$, there exists $\delta>0$ such that $t_{\bar{x}}<r-\epsilon-\delta<r-\epsilon$ and so we obtain

$$F(\bar{x})\leq_C^{(1)}(r-\epsilon-\delta)k+V' \quad\text{and}\quad F(x_\alpha)\not\leq_C^{(1)}(r-\epsilon-\delta)k+V'. \tag{3.7}$$

Since $\delta>0$, we obtain

$$\bigcap_{z\in(r-\epsilon-\delta)k+V'}(z-C)\subset\mathrm{int}\left(\bigcap\nolimits_{z\in(r-\epsilon)k+V'}(z-C)\right).$$

Since C is a convex cone, we obtain

$$\bigcap_{z\in(r-\epsilon-\delta)k+V'}(z-C)-C\subset\mathrm{int}\left(\bigcap\nolimits_{z\in(r-\epsilon)k+V'}(z-C)-C\right).$$

Hence, by (3.7) we obtain

$$F(\bar{x}) \subset \operatorname{int}\left(\bigcap_{z\in(r-\epsilon)k+V'}(z-C)-C\right) \quad \text{and} \quad F(x_\alpha) \not\subset \operatorname{int}\left(\bigcap_{z\in(r-\epsilon)k+V'}(z-C)-C\right).$$

Since α is an arbitrary and $x_\alpha \to \bar{x}$, for any U which is a neighborhood of x, there exists $\alpha(U) \in J$ such that

$$x_{\alpha(U)} \in U \quad \text{and} \quad F(x_{\alpha(U)}) \not\subset \operatorname{int}\left(\bigcap_{z\in(r-\epsilon)k+V'}(z-C)-C\right).$$

This is a contradiction to the $-C$-upper continuity of F on X.

Next, we consider the case of $j = 4$. By (3.6) and the definition of type (4) set-relation, we obtain

$$\left(\bigcap_{y_1\in F(\bar{x})}(y_1+C)\right) \cap ((r-\epsilon)k+V') \neq \emptyset,$$

and

$$\left(\bigcap_{y_2\in F(x_\alpha)}(y_2+C)\right) \cap ((r-\epsilon)k+V') = \emptyset.$$

Hence we have

$$F(\bar{x}) \subset \sup_C(F(\bar{x})) - C \quad \text{and} \quad F(x_\alpha) \not\subset \sup_C(F(\bar{x})) - C. \tag{3.8}$$

Since $t_{\bar{x}} < r-\epsilon$, there exists $\delta > 0$ such that $t_{\bar{x}} < r-\epsilon-\delta < r-\epsilon$, and so we obtain

$$F(\bar{x}) \leq_C^{(4)} (r-\epsilon-\delta)k+V' \quad \text{and} \quad F(x_\alpha) \not\leq_C^{(4)} (r-\epsilon-\delta)k+V'.$$

Since C is a convex cone, $\delta > 0$ and by (3.8), we obtain

$$F(\bar{x}) \subset \sup_C(F(\bar{x})) + \delta k - C \quad \text{and} \quad F(x_\alpha) \not\subset \sup_C(F(\bar{x})) + \delta k - C. \tag{3.9}$$

Moreover

$$\sup_C(F(\bar{x})) - C \subset \operatorname{int}(\sup_C(F(\bar{x})) + \delta k - C). \tag{3.10}$$

Hence, by (3.8), (3.9) and (3.10) we have

$$F(\bar{x}) \subset \operatorname{int}(\sup_C(F(\bar{x})) + \delta k - C) \quad \text{and} \quad F(x_\alpha) \not\subset \operatorname{int}(\sup_C(F(\bar{x})) + \delta k - C).$$

Thus, we obtain

$$F(\bar{x}) \subset \operatorname{int}(\sup_C(F(\bar{x})) + \delta k - C) - C \quad \text{and} \quad F(x_\alpha) \not\subset \operatorname{int}(\sup_C(F(\bar{x})) + \delta k - C) - C.$$

This is a contradiction to the $(-C)$-upper continuity of F on X.

Next, we consider the case of $j = 5$. By (3.6) and the definition of type (5) set-relation, we obtain

$$F(\bar{x}) \subset (r-\epsilon)k+V'-C \quad \text{and} \quad F(x_\alpha) \not\subset (r-\epsilon)k+V'-C.$$

Since $t_{\bar{x}} < r-\epsilon$, there exists $\delta > 0$ such that $t_{\bar{x}} < r-\epsilon-\delta < r-\epsilon$ and so we obtain

$$F(\bar{x}) \leq_C^{(5)} (r-\epsilon-\delta)k+V' \quad \text{and} \quad F(x_\alpha) \not\leq_C^{(5)} (r-\epsilon-\delta)k+V'. \tag{3.11}$$

Since $\delta > 0$, we obtain

$$(r-\epsilon-\delta)k+V'-C \subset \operatorname{int}((r-\epsilon)k+V'-C).$$

Since C is a convex cone, we obtain

$$(r-\epsilon-\delta)k+V'-C \subset \operatorname{int}((r-\epsilon)k+V'-C)-C.$$

Hence, by (3.11) we have

$$F(\bar{x}) \subset \operatorname{int}((r-\epsilon)k + V' - C) - C \quad \text{and} \quad F(x_\alpha) \not\subset \operatorname{int}((r-\epsilon)k + V' - C) - C.$$

This is a contradiction to the $(-C)$-upper continuity of F on X. Consequently, for each $j = 1, 4, 5$, $I^{(j)}_{k,V'} \circ F$ is upper semicontinuous on X.

Next, we consider the case of $j = 2$. By (3.6) and the definition of type (2) set-relation, we obtain

$$F(\bar{x}) \cap \left(\bigcap_{z \in (r-\epsilon)k + V'} (z - C) \right) \neq \emptyset \quad \text{and} \quad F(x_\alpha) \cap \left(\bigcap_{z \in (r-\epsilon)k + V'} (z - C) \right) = \emptyset.$$

Since $t_{\bar{x}} < r - \epsilon$, there exists $\delta > 0$ such that $t_{\bar{x}} < r - \epsilon - \delta < r - \epsilon$ and so we obtain

$$F(\bar{x}) \leq^{(2)}_C (r - \epsilon - \delta)k + V' \quad \text{and} \quad F(x_\alpha) \not\leq^{(2)}_C (r - \epsilon - \delta)k + V'. \tag{3.12}$$

Since $\delta > 0$, we obtain

$$\bigcap_{z' \in (r-\epsilon-\delta)k + V'} (z' - C) \subset \operatorname{int}\left(\bigcap\nolimits_{z \in (r-\epsilon)k + V'} (z - C) \right).$$

Since C is a convex cone, we obtain

$$\left(\bigcap_{z' \in (r-\epsilon-\delta)k + V'} (z' - C) \right) - C \subset \operatorname{int}\left\{ \left(\bigcap\nolimits_{z \in (r-\epsilon)k + V'} (z - C) \right) - C \right\}.$$

Hence, by (3.12), we have

$$F(\bar{x}) \cap \left(\operatorname{int}\left(\bigcap\nolimits_{z \in (r-\epsilon)k + V'} (z - C) \right) - C \right) \neq \emptyset,$$

and

$$F(x_\alpha) \cap \left(\operatorname{int}\left(\bigcap\nolimits_{z \in (r-\epsilon)k + V'} (z - C) \right) - C \right) = \emptyset.$$

Since α is an arbitrary and $x_\alpha \to \bar{x}$, for any U which is a neighborhood of x, there exists $\alpha(U) \in J$ such that

$$x_{\alpha(U)} \in U \quad \text{and} \quad F(x_{\alpha(U)}) \cap \left(\operatorname{int}\left(\bigcap\nolimits_{z \in (r-\epsilon)k + V'} (z - C) \right) - C \right) = \emptyset.$$

This is a contradiction to the $(-C)$-lower continuity of F on X.

Next, we consider the case of $j = 3$. By (3.6) and the definition of type (3) set-relation, we obtain

$$(r-\epsilon)k + V' \subset F(\bar{x}) + C \quad \text{and} \quad (r-\epsilon)k + V' \not\subset F(x_\alpha) + C.$$

By the transitivity of type (3) set-relation, we obtain $F(\bar{x}) \not\subset F(x_\alpha) + C$ and so,

$$F(\bar{x}) \cap (\operatorname{int}(\{(F(x_\alpha) + C\}^c)) \neq \emptyset \quad \text{and} \quad F(x_\alpha) \cap (\operatorname{int}(\{(F(x_\alpha) + C\}^c)) = \emptyset.$$

Since C is a convex cone, we obtain

$$F(\bar{x}) \cap (\operatorname{int}(\{(F(x_\alpha) + C\}^c - C)) \neq \emptyset \quad \text{and} \quad F(x_\alpha) \cap (\operatorname{int}(\{(F(x_\alpha) + C\}^c - C)) = \emptyset.$$

This is a contradiction to the $(-C)$-lower continuity of F on X.

Finally, we consider the case of $j = 6$. By (3.6) and the definition of type (6) set-relation, we obtain

$$F(\bar{x}) \cap ((r-\epsilon)k + V' - C) \neq \emptyset \quad \text{and} \quad F(x_\alpha) \cap ((r-\epsilon)k + V' - C) = \emptyset.$$

Since $t_{\bar{x}} < r - \epsilon$, there exists $\delta > 0$ such that $t_{\bar{x}} < r - \epsilon - \delta < r - \epsilon$ and so we obtain

$$F(\bar{x}) \leq_C^{(6)} (r - \epsilon - \delta)k + V' \quad \text{and} \quad F(x_\alpha) \not\leq_C^{(6)} (r - \epsilon - \delta)k + V'. \tag{3.13}$$

Since $\delta > 0$, we obtain

$$(r - \epsilon - \delta)k + V' - C \subset \operatorname{int}((r - \epsilon)k + V' - C).$$

Since C is a convex cone, we obtain

$$(r - \epsilon - \delta)k + V' - C \subset \operatorname{int}((r - \epsilon)k + V' - C) - C.$$

Hence, by (3.13) we have

$$F(\bar{x}) \cap (\operatorname{int}((r - \epsilon)k + V') - C) \neq \emptyset \quad \text{and} \quad F(x_\alpha) \cap (\operatorname{int}((r - \epsilon)k + V') - C) = \emptyset.$$

This is a contradiction to the $(-C)$-lower continuity of F on X. Consequently, for each $j = 2, 3, 6$, $I_{k,V'}^{(j)} \circ F$ is upper semicontinuous on X.

□

Corollary 3.2. *Let F be a set-valued map and C be a convex cone in Y. Then, the following statements are hold.*

i. For each $j = 1, 2, 3$,
- *(a) If $F$is $(-C)$-lower continuous, then $S_{k,V'}^{(j)} \circ F$ is upper semicontinuous on X,*
- *(b) If $F$is C-upper continuous, then $S_{k,V'}^{(j)} \circ F$ is lower semicontinuous on X.*

ii. For each $j = 4, 5, 6$,
- *(a) If $F$is C-lower continuous, then $S_{k,V'}^{(j)} \circ F$ is lower semicontinuous on X,*
- *(b) If $F$is $(-C)$-upper continuous, then $S_{k,V'}^{(j)} \circ F$ is upper semicontinuous on X.*

Proof. By the same way in Theorem 3.2, the statements are proved. □

Finally, for each $j = 3, 5$, we explain the inherited properties of cone-continuities for parent set-valued maps with the following examples.

Example 3.1. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$ and $C = \mathbb{R}^2_+$. We consider a set-valued map $F : X \to 2^Y$

$$F(x) := \begin{cases} \bigcup\limits_{0 \leq z \leq x} [1, z + 1] & (x > 0) \\ \bigcup\limits_{x \leq z \leq 0} [z - 1, 0] & (x \leq 0) \end{cases}$$

Then, F is C-upper continuous on X.

Let $k = \binom{1}{1}$ and $V' = [0, 1]$. Then,

$$(I_{k,V'}^{(3)} \circ F)(x) := \begin{cases} 1 & (x > 0) \\ x & (x \leq 0) \end{cases}$$

Hence we obtain the same result in Theorem 3.2.

Example 3.2. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$ and $C = \mathbb{R}^2_+$. We consider a set-valued map $F : X \to 2^Y$

$$F(x) := \begin{cases} \bigcup\limits_{0 \leq z \leq x} [1, z + 1] & (x > 0) \\ \bigcup\limits_{x \leq z \leq 0} (-\infty, z - 1] & (x \leq 0) \end{cases}$$

Then, F is C-lower continuous on X.

Let $k = \binom{1}{1}$ and $V' = [0,1]$. Then,

$$(I^{(3)}_{k,V'} \circ F)(x) := \begin{cases} x+1 & (x > 0) \\ 0 & (x \leq 0) \end{cases}$$

Hence we obtain the same result in Theorem 3.2.

References

[1] A. Göpfert, H. Riahi, C. Tammer and C. Zălinescu, *Variational methods in partially ordered spaces*, Springer-Verlag, New York, 2003.

[2] Y. Kimura, K. Tanaka, and T. Tanaka, *On semicontinuity of set-valued maps and marginal functions*, in Nonlinear Anlysis and Convex Analysis, 181-188(1999).

[3] D. Kuroiwa, T. Tanaka, and T. X. D. Ha, *On cone convexity of set-valued maps*, Nonlinear Anal. 30 (1997), 1487–1496.

[4] I. Kuwano, T. Tanaka, and S. Yamada, *Characterization of nonlinear scalarizing functions for sets*, to appear in the proceedings of Nonlinear Analysis and Optimization, Yokohama Publishers, Yokohama, 2008.

[5] I. Kuwano, T. Tanaka, and S. Yamada, *Inherited properties of unified types of scalarizing functions for sets*, submited to the proceedings of Nonlinear Analysis and Convex Analysis (NACA-2009).

[6] S. Nishizawa, T. Tanaka, and P. Gr. Georgiev, *On inherited properties of set-valued maps*, Nonlinear Analysis and Convex Analysis, W. Takahashi and T. Tanaka (eds.), Yokohama Publishers, Yokohama, 2003, 341–350.

[7] A. Shimizu, T. Tanaka, and S. Yamada, *A unified approach for scalarization on sets*, in the Asian Conference on Nonlinear Analysis and Optimization (NAO-Asia2008) held at Kunibiki Messe, Matsue, Japan, September in 2008.