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MAXIMAL MEASURE ALGEBRAS IN $\mathcal{P}(\mathbb{N})/\mathcal{Z}_0$

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Let $\mathcal{Z}_0$ denote the ideal of asymptotic density zero subsets of $\mathbb{N}$,

$$\mathcal{Z}_0=\{X\subseteq\mathbb{N}:\lim_{n\to\infty}|X\cap n|/n=0\}$$

and let $\mathcal{P}(\mathbb{N})/\mathcal{Z}_0$ denote the quotient Boolean algebra. It is known that $\mathcal{P}(\mathbb{N})/\mathcal{Z}_0$ includes a measure algebra of Maharam character $2^{\aleph_0}$ as a subalgebra ([3], see also [2] and [1]). In this note I will show that the existence of a maximal measure subalgebra of $\mathcal{P}(\mathbb{N})/\mathcal{Z}_0$ of character strictly smaller than $2^{\aleph_0}$ is relatively consistent with ZFC, answering a question of David Fremlin.

Let $\mathcal{S}_\kappa$ be the forcing for adding κ side-by-side Sacks reals, with countable support. Let $\mathcal{D}$ be the family of all subsets of $\mathbb{N}$ that have density. Let $\mathcal{Z}_0$ be the ideal of sets of asymptotic density zero. Recall that $I_n=[2^n,2^{n+1})$ and $\nu_n(A)=|A\cap I_n|/2^n$, then $d^*(A)=\limsup_{n\to\infty}\nu_n(A)$ and $d(A)=\lim_{n\to\infty}\nu_n(A)$, if it exists. A family of sets $\mathcal{A}$ is *ε-independent with respect to μ* if for every finite $F\subseteq\mathcal{A}$ and every $p\colon F\to\{\pm1\}$ we have $|\mu(\bigcap_{A\in F}A^{p(A)})-2^{-|F|}|\leq\varepsilon$. Here μ can be a measure or a convex mean.

If $\mathcal{A}\subseteq\mathcal{P}(\mathbb{N})$ and $m\in\mathbb{N}$ then we say that $\mathcal{A}$ is *ε-independent at m* if it is ε-independent with respect to ν_m.

A family $\mathcal{A}\subseteq\mathcal{P}(\mathbb{N})$ is a *maximal stochastically independent family with respect to d* if it is included in $\mathcal{D}$, stochastically independent with respect to d, and maximal with respect to these properties.

Lemma 1. *A family $\mathcal{A}\subseteq\mathcal{P}(\mathbb{N})$ is stochastically independent (with respect to d) if and only if for every finite $F\subseteq\mathcal{A}$ and every $\varepsilon>0$ there exists n such that F is ε-independent at every $m\geq n$.* $\square$

Fix an uncountable cardinal κ.

Lemma 2. *Assume CH. Then there is a family $\{A_\alpha:\alpha<\omega_1\}$ that is maximal stochastically independent with respect to d such that in the extension by $\mathcal{S}_\kappa$ it remains maximal.*

Proof. Let $(P_\alpha,\dot r_\alpha)$ $(\alpha<\omega_1)$ enumerate all pairs such that P_α is a condition in $\mathcal{S}_{\omega_1}$ and $\dot r_\alpha$ is a name for a subset of $\mathbb{N}$. We construct

I would like to thank David Fremlin for conversations on this topic, and in particular for providing a direct proof of Theorem 3 from Lemma 2.

A_α ($\alpha < \omega_1$) by recursion. Assume $\mathcal{A}_\delta = \{A_\alpha : \alpha < \delta\}$ has been constructed. Consider $(P_\delta, \dot{r}_\delta)$. If P_δ does not force that $\mathcal{A}_\delta \cup \{\dot{r}_\delta\}$ is stochastically independent, choose any A_δ such that $\mathcal{A}_\delta \cup \{A_\delta\}$ is stochastically independent.

Otherwise, find a fusion sequence Q_t ($t \in T$) of conditions extending P_δ indexed by a perfect tree $T \subseteq 2^{<\mathbb{N}}$ and $\{n_i : i \in \mathbb{N}\}$ as follows. Let T_k be the k-th level of T, and re-enumerate $\mathcal{A}_\alpha$ as $\{A'_i : i \in \mathbb{N}\}$. Write $\mathcal{A}'_k = \{A'_i : i \leq k\}$.

(1) $(i+1)n_i < n_{i+1}$, $n_1 = 1$,
(2) T branches only at the $n_i + 1$-st level for $i \in \mathbb{N}$, and only once. Thus $|T_{n_i}| = i$.
(3) $Q_t \leq Q_s$ if $s \subseteq t$.
(4) the fusion $Q = \bigcup_{f \in [T]} \bigcap_{n=1}^{\infty} Q_{f \restriction n}$ is a condition in $\mathcal{S}_{\omega_1}$.
(5) If $t \in T_{n_i}$, then Q_t forces that $\mathcal{A}'_i \cup \{\dot{r}\}$ is 2^{-i}-independent at every $m \geq n_{i+1}$.
(6) If $t \in T_{n_i}$, then Q_t decides $\dot{r} \cap I_m$ for all $m < n_{i+1}$.

This can be accomplished by using the standard means. That such sequences can be found is the only property of $\mathcal{S}_\kappa$ that we shall need.

Enumerate each T_{n_i} as $t^i_1, \ldots t^i_i$, and write $t^i_j = t^i_i$ for $j > i$. Now pick A_δ so that for all i and $j < n_{i+1} - n_i$ we have

(7) $A_\delta \cap I_{n_i+j} = u^i_j$, where $Q_{t^i_j} \Vdash \dot{r} \cap I_{n_i+j} = u^i_j$.

Then for every i the family $\mathcal{A}'_i \cup \{A_\delta\}$ is 2^{-i}-independent at each $m \geq n_{i+1}$, hence $\mathcal{A}_\delta \cup \{A_\delta\}$ is stochastically independent.

It remains to prove that $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$ is maximal in the extension by $\mathcal{S}_\kappa$. We need to prove that for every name $\dot{r}$ for a subset of $\mathbb{N}$ and every condition P, P does not force that $\mathcal{A} \cup \{\dot{r}\}$ is independent. Assume otherwise. We may assume $\kappa = \omega_1$, by picking an elementary submodel M of a sufficiently large H_λ such that M is closed under ω-sequences, of size $\aleph_1$, and large enough, and intersecting $\mathcal{S}_\kappa$ with M.

Fix δ such that $(P_\delta, \dot{r}_\delta) = (P, \dot{r})$. We claim that Q as in (4) forces that $\{A_\delta, \dot{r}\}$ is not independent. Otherwise some $R \leq Q$ decides i such that $\{A_\delta, \dot{r}\}$ are 1/4-independent at all $m \geq n_i$. But some Q_t, for $t \in T_{n_i}$ is compatible with R, and by (7) it forces that $A_\delta \cap I_m = \dot{r} \cap I_m$ for some $m \geq n_i$, a contradiction. □

Theorem 3. *Assume CH. Then there is a subalgebra $\mathcal{B}$ of $\mathcal{P}(\mathbb{N})/\mathcal{Z}_0$ such that $(\mathcal{B}, d)$ is a measure algebra of Maharam character $\aleph_1$ and in the extension by $\mathcal{S}_\kappa$ it is a maximal subalgebra of $\mathcal{P}(\mathbb{N})/\mathcal{Z}_0$ with this property.*

After preliminary lemmas, we give two proofs of this theorem. The first one is shorter and it uses Lemma 2, while the second one provides a more robust object and involves an extension of the proof of Lemma 2 that may be of an independent interest.

Lemma 4. *If $A \in \mathcal{D}$ and $f\colon A \to \mathbb{N}$ is a strictly increasing surjection, then $d^*(f(B)) = d^*(B)d(A)$.*

Proof. Let $A = \{n_i : i \in \mathbb{N}\}$ be its increasing enumeration, and let $g\colon \mathbb{N} \to \mathbb{N}$ be such that $g(m) = |A \cap m|$. and let $B = \{n_i : i \in C\}$. Then $d^*(B) = \limsup_j |B \cap j|/j = \limsup_j |B \cap j|/g(j) \cdot g(j)/j$. But $\lim_j g(j)/j = d(A)$, and $\limsup_j |B \cap j|/g(j) = d^*(C)$. □

A proof of Theorem 3 using Lemma 2. By Lemma 2, in the extension by $\mathcal{S}_\kappa$ there is a maximal stochastically independent family $\mathcal{A}$ of size $\aleph_1$. By [4, §491], $\mathcal{A}$ generates a subalgebra $\mathcal{B}_0$ of $\mathcal{P}(\mathbb{N})/\mathcal{Z}_0$ that is isomorphic to a measure algebra of character $\aleph_1$, and the measure on $\mathcal{B}$ is given by d. Let $\mathcal{B}'$ be a maximal subalgebra of $\mathcal{P}(\mathbb{N})/\mathcal{Z}_0$ that includes $\mathcal{B}$ and such that $(\mathcal{B}', d)$ is a measure algebra.

Let $\mathcal{B}_A$ denote the factor algebra, $\mathcal{B}_A = \{B \cap A : B \in \mathcal{B}\}$. Assume there is a nonzero $A \in \mathcal{B}_0$ such that $(\mathcal{B}_0)_A = \mathcal{B}'_A$. Let $A = \{n_i : i \in \mathbb{N}\}$ be its increasing enumeration. The map $\Phi\colon \mathcal{P}(A) \to \mathcal{P}(\mathbb{N})$ defined by

$$\Phi(\{n_j : j \in C\}) = C$$

satisfies the formula $d(A)d^*(\Phi(B)) = d^*(B)$, by Lemma 4. Therefore it sends $(\mathcal{B}_0)_A$ to a subalgebra of $\mathcal{P}(\mathbb{N})/\mathcal{Z}_0$ that is its maximal measure subalgebra of Maharam character $\aleph_1$.

We may therefore assume that for every nonzero $A \in \mathcal{B}_0$ the relative Maharam type of $\mathcal{B}_A$ over $(\mathcal{B}_0)_A$ is infinite. By [3, §333], there is a partition of unity A_i ($i \in \mathbb{N}$) such that each $\mathcal{B}_{A_i}$ is relatively Maharam homogeneous and atomless. Therefore by applying Maharam's theorem we may find $A \in \mathcal{B}\setminus\mathcal{B}_0$ such that $\mathcal{A}\cup\{A\}$ is stochastically independent, contradicting the maximality of $\mathcal{A}$. □

Lemma 5. *Assume $A_0, \dots, A_{n-1}$ are stochastically independent in some atomless measure space (X, μ) and B is a measurable set of measure $1/2$ such that for every Boolean combination C of $A_0, \dots, A_{n-1}$ we have $\mu(B\Delta C) \geq \varepsilon$ for some $\varepsilon > 0$. Then there is A_n stochastically independent with $A_0, \dots, A_{n-1}$ and such that $\mu(A_n \cap B) \leq 1/2 - \varepsilon$.*

Proof. Let $C_s = \bigcap_{i=0}^{n-1} A_i^{s(i)}$ for $s\colon n \to \{\pm 1\}$.Choose A_n so that $\mu(A_n \cap C_s) = 1/2$ and $\mu(A_n \cap C_s \cap B)$ is minimal for all s. □

A proof of Theorem 3 using the proof of Lemma 2. Let $(P_\alpha, \dot{r}_\alpha)$ ($\alpha < \omega_1$) enumerate all pairs such that P_α is a condition in $\mathcal{S}_{\omega_1}$ and $\dot{r}_\alpha$ is a name for a subset of $\mathbb{N}$. We construct A_α ($\alpha < \omega_1$) by recursion. Assume $\mathcal{A}_\delta = \{A_\alpha : \alpha < \delta\}$ has been constructed. Consider $(P_\delta, \dot{r}_\delta)$.

If P_δ forces that $\dot{r}$ belongs to the measure algebra generated by $\mathcal{A}_\delta$, or if it does not force that $d(\dot{r}) = 1/2$, then choose any A_δ such that $\mathcal{A}_\delta \cup \{A_\delta\}$ is stochastically independent.

Otherwise, some $P \leq P_\delta$ forces that $\dot{r}$ does not belong to the measure algebra generated by $\mathcal{A}_\delta$. If in the forcing extension for every m there is a Boolean combination C_m of elements of $\mathcal{A}_\delta$ such that $d(C_m \Delta \dot{r}) \leq 2^{-m}$, then $D = \bigcup_m \bigcap_{n=m}^\infty C_m$ satisfies $d(D\Delta\dot{r}) = 0$. Therefore we may extend P further to decide a rational number $\varepsilon > 0$ such that for every finite Boolean combination C of elements of $\mathcal{A}_\delta$ we have $d(C\Delta\dot{r}) \geq \varepsilon$. Find a fusion sequence Q_t $(t \in T)$ of conditions extending P indexed by a perfect tree $T \subseteq 2^{<\mathbb{N}}$ and $\{n_i : i \in \mathbb{N}\}$ as follows. Let T_k be the k-th level of T, and re-enumerate $\mathcal{A}_\alpha$ as $\{A'_i : i \in \mathbb{N}\}$. Write $\mathcal{A}'_k = \{A'_i : i \leq k\}$.

(8) $2(i+1)n_i < n_{i+1}$, $n_1 = 1$,
(9) T branches only at the $n_i + 1$-st level for $i \in \mathbb{N}$, and only once. Thus $|T_{n_i}| = i$.
(10) $Q_t \leq Q_s$ if $s \subseteq t$.
(11) the fusion $Q = \bigcup_{f\in[T]} \bigcap_{n=1}^\infty Q_{f\restriction n}$ is a condition in $\mathcal{S}_{\omega_1}$.
(12) If $t \in T_{n_i}$, then for every Boolean combination C of elements of $\mathcal{A}'_i$, the condition Q_t forces that
$$\max(\nu_m(C\Delta\dot{r}), \nu_m(d(C \setminus \dot{r}))) \geq \varepsilon/2$$
for every $m \geq n_{i+1}$.
(13) $\mathcal{A}'_i$ is 2^{-i-1}-independent at each $m \geq n_{i+1}$.
(14) If $t \in T_{n_i}$, then Q_t decides $\dot{r} \cap I_m$ for all $m < n_{i+1}$.

This can be accomplished by using the standard means. That such sequences can be found is the only property of $\mathcal{S}_\kappa$ that we shall need.

Enumerate each T_{n_i} as $t^i_1, \dots t^i_i$, and write $t^i_j = t^i_i$ for $j > i$. Now pick A_δ so that for all i and $j < n_{i+1} - n_i$ we have (let $m = n_i + j$)

(15) A_δ is 2^{-i}-independent with $\mathcal{A}'_i$ at m, and
(16) If $j < i$, then $Q_{t^i_j} \Vdash \nu_m(A_\delta \cap \dot{r}) \leq 1/2 - \varepsilon/2 + 2^{-i}$,
(17) If $j \geq i$, then $Q_{t^i_{j-i}} \Vdash \nu_m(A_\delta \setminus \dot{r}) \leq 1/2 - \varepsilon/2 + 2^{-i}$.

This can be achieved by using a discrete version of Lemma 5. Then for every i the family $\mathcal{A}'_i \cup \{A_\delta\}$ is 2^{-i}-independent at each $m \geq n_{i+1}$, hence $\mathcal{A}_\delta \cup \{A_\delta\}$ is stochastically independent.

It remains to prove that the algebra $\mathcal{B}$ generated by $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$ is maximal in the extension by $\mathcal{S}_\kappa$. We will prove that for every name $\dot{r}$ for a subset of $\mathbb{N}$ and every condition P, P either forces that $\dot{r}$ belongs to $\mathcal{B}$ or some extension of P forces that $A_\delta \cap \dot{r} \notin \mathcal{D}$.

Assume otherwise, and let $(P,\dot r)$ be a pair such that P forces that $\dot r$ does not belong to B and that $\dot r\cap A_\delta$ is in $\mathcal{D}$ for all δ. We may assume that $P\Vdash d(\dot r)=1/2$ and $\kappa=\omega_1$.

Fix δ such that $(P_\delta,\dot r_\delta)=(P,\dot r)$. Let $\varepsilon>0$ be as in the construction of A_δ. Then Q as in (11) forces that $d(A_\delta\cap\dot r)$ is not defined. Otherwise some $R\leq Q$ forces that for some rational $a\in[0,1]$ we have $|d(A_\delta\cap\dot r)-a|<\varepsilon/8$. By extending R further, we may decide i such that for all $m\geq n_i$

(18) $R\Vdash|\nu_m(A_\delta\cap\dot r)-a|<\varepsilon/8$.

We may assume that i is large enough so that $2^{-i}<\varepsilon/4$. But some Q_t for $t\in T_{n_i}$ is compatible with R, and by (16) and (17) it forces that there are m and m' greater than n_i such that

$$R\Vdash|\nu_m(A_\delta\cap\dot r)-\nu_{m'}(A_\delta\cap\dot r)|\geq 2\varepsilon/2-2^{-i+1}>\varepsilon/2.$$

But this contradicts (18). $\square$

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