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Coanalytic sets with Borel sections

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Fact. (Fujita and Mátrai) *Let $B \subset \boldsymbol{R} \times \boldsymbol{R}$ be a Borel set such that the horizontal section B^y is $\boldsymbol{\Sigma}^0_\alpha$ for every $y \in \boldsymbol{R}$. Then there is a dense G_δ set $D \subset \boldsymbol{R}$ such that $B \cap (\boldsymbol{R} \times D)$ is $\boldsymbol{\Sigma}^0_\alpha \restriction \boldsymbol{R} \times D$.*

This can be proved by a straightforward induction using A.Louveau's solution ([Lo]) of the section problem of Borel sets. This fact has been used in order to solve an old problem by M.Laczkovich about differences of Borel measurable function (See [FM].)

Theorem. *The following statements are equivalent:*

(1) *If $A \subset \boldsymbol{R} \times \boldsymbol{R}$ is $\boldsymbol{\Pi}^1_1$ and all horizontal sections A^y are Borel, then there is a dense G_δ set $D \subset \boldsymbol{R}$ such that $A \cap (\boldsymbol{R} \times D)$ is Borel;*

(2) *similar, but A^y are $\boldsymbol{\Sigma}^0_\alpha$ and $A \cap (\boldsymbol{R} \times D)$ is $\boldsymbol{\Sigma}^0_\alpha \restriction \boldsymbol{R} \times D$;*

(3) *similar, but A^y are closed and $A \cap (\boldsymbol{R} \times D)$ is Borel;*

(4) BP($\boldsymbol{\Sigma}^1_2$), *i.e., every $\boldsymbol{\Sigma}^1_2$ set of reals has the property of Baire.* ◀

Proof. From (1) to (2): use Fact.

From (2) to (3): immediate from the case $\alpha = 1$ of (2).

From (3) to (4): given $\boldsymbol{\Sigma}^1_2$ set $P \subset \boldsymbol{R}$, let $A \subset \boldsymbol{R} \times \boldsymbol{R}$ be $\boldsymbol{\Pi}^1_1$ such that $y \in P \iff \exists x[\langle x, y \rangle \in A]$. Uniformize A by a function $f : P \to \boldsymbol{R}$ with $\boldsymbol{\Pi}^1_1$ graph. Apply (3) to the graph of f. Then $P \cap D$ is $\boldsymbol{\Sigma}^1_1$ and D is co-meager. So P has BP.

From (4) to (1): this is the main part of this note.

Let $\mathbb{C}$ be the Cohen poset. Given a transitive model M of set theory, let $\mathrm{Co}(M)$ be the set of all $\mathbb{C}$-generic reals over M.

Lemma A. (Solovay) $\mathrm{BP}(\mathbf{\Sigma}^1_2)$ *holds if and only if* $\mathrm{Co}(L[r])$ *is co-meager for every* $r \in \boldsymbol{R}$. ◀

Let WO be set the of $w \in {}^{\omega}2$ which codes a wellordering on ω. For each $q \in \mathrm{WO}$ let $\|w\|$ be the order-type (i.e., countable ordinal) that w codes.

Definition. A set $X \subset \boldsymbol{R} \times \omega_1$ is Π^1_2 *in the codes* if the set

$$\left\{ \langle x, w\rangle \in \boldsymbol{R} \times {}^{\omega}2 \,\middle|\, w \in \mathrm{WO},\ \langle x, \|w\| ra \in X \right\}$$

is (lightface) Π^1_2. ◀

Lemma B. *Let* $X \subset \boldsymbol{R} \times \omega_1$ *be* Π^1_2 *in the codes. Suppose that for every* $y \in \boldsymbol{R}$ *there is* $\xi < \omega_1$ *such that* $\langle y, \xi\rangle \in X$. *Then there is a countable ordinal* δ *such that for every* $c \in \mathrm{Co}(L)$ *there is* $\xi < \delta$ *such that* $\langle c, \xi\rangle \in X$. ◀

Proof of (4) $\Longrightarrow$ (1) [taking Lemmas for granted]. We put $\boldsymbol{R} = {}^{\omega}\omega$ and assume A is lightface Π^1_1. Let $f : \boldsymbol{R} \times \boldsymbol{R}$ be a recursive function such that $A = f^{-1}[\mathrm{WO}]$. Since A^y is Borel, the image $f[A^y \times \{y\}]$ si bounded in WO, that is to say,

$$\forall y \in \boldsymbol{R} \exists \xi < \omega_1 \forall x \Big[\langle x, y\rangle \in A \implies \|f(x,y)\| < \xi \Big].$$

For each $\xi < \omega_1$ set

$$\mathrm{WO}_\xi = \left\{ w \in \mathrm{WO} \,\middle|\, \|w\| < \xi \right\}$$

and let

$$X = \left\{ \langle y, \xi\rangle \,\middle|\, f[A^y \times \{y\}] \subset \mathrm{WO}_\xi \right\}.$$

Observe that X is Π^1_2 in the codes. Applying Lemma B we find a countable ordinal δ such that

$$\forall c \in \mathrm{Co}(L) \exists \xi < \delta \Big[\langle c, \xi\rangle \in X \Big].$$

Then we have

$$A \cap \big(\boldsymbol{R} \times \mathrm{Co}(L)\big) = f^{-1}[\mathrm{WO}_\delta] \cap \big(\boldsymbol{R} \times \mathrm{Co}(L)\big).$$

By Lemma A there is a dense G_δ set $D \subset \mathrm{Co}(L)$. ◀

Proof of Lemma B. Let $\varphi(y,w)$ be a Π^1_2 formula such that

$$\begin{aligned}\langle y,\xi\rangle \in X &\iff \exists w \in \mathrm{WO}\Big[\,\xi = \|w\| \wedge \varphi(y,w)\,\Big]\\ &\iff \forall w \in \mathrm{WO}\Big[\,\xi = \|w\| \implies \varphi(y,w)\,\Big].\end{aligned}$$

Then we have, by the assumption of the lemma,

$$(*)\qquad \forall y \exists \xi < \omega_1 \forall w\Big[\,w \in \mathrm{WO} \wedge \|w\| = \xi \implies \varphi(y,w)\,\Big].$$

Let $\varphi^*(y,\xi)$ stand for "$\forall w \cdots$" part of (*). Then $\varphi^*(y,\xi)$ is absolute for every proper class model in which ξ is countable.

Let $c \in \mathrm{Co}(L)$ and suppose that $\langle c,\xi\rangle \in X$. Let $g : \omega \to \xi$ be $\mathrm{Coll}(\xi)$-generic over $L[c]$. Then

$$L[c,q] \models \varphi^*(c,\xi)$$

so that there are forcing conditions $p \in \mathbb{C}$ and $q \in \mathrm{Coll}(\xi)$ such that c meets p, g meets q and

$$\langle p,q\rangle \Vdash_{\mathbb{C}\times\mathrm{Coll}(\xi)} L[\dot{c},\dot{g}] \models \varphi^*(\dot{c},\check{\xi}).$$

Then by absoluteness of forcing relations,

$$L \models \Big[\ \langle p,q\rangle \Vdash_{\mathbb{C}\times\mathrm{Coll}(\xi)} \varphi^*(\dot{c},\check{\xi})\ \Big].$$

By homogeneity of the poset $\mathrm{Coll}(\xi)$,

$$L \models \Big[\ \langle p,\emptyset\rangle \Vdash_{\mathbb{C}\times\mathrm{Coll}(\xi)} \varphi^*(\dot{c},\check{\xi})\ \Big].$$

where $\emptyset$ is the weakest member of $\mathrm{Co}(\xi)$.

For each $\xi < \omega_1$ let

$$Y_\xi = \Big\{p \in \mathbb{C}\ \Big|\ L \models \Big[\ \langle p,\emptyset\rangle \Vdash_{\mathbb{C}\times\mathrm{Coll}(\xi)} \varphi^*(\dot{c},\check{\xi})\ \Big]\Big\}.$$

Then $\bigcup_{\xi<\omega_1} Y_\xi$ is pre-dense in $\mathbb{C}$. By ccc, there is $\delta < \omega_1$ such that $\bigcup_{\xi<\delta} Y_\xi$ is already pre-dense in $\mathbb{C}$. ◀

Daisuke Ikegami observed that $\mathbb{C}$ in Lemma B can be replaced by other alboreal Suslin ccc forcing notions. Daisuke also pointed out that Sacks forcing does not satisfy Lemma B nor clause (3) of Theorem.

By Montgomery's result on the category quantifier, we obtain

Corollary. *Assume* $\mathrm{BP}(\mathbf{\Sigma}^1_2)$*. Let* $A \subseteq \boldsymbol{R} \times \boldsymbol{R}$ *be* $\mathbf{\Pi}^1_1$ *such that* A^y *is* $\mathbf{\Sigma}^0_\alpha$ *for every* $y \in \boldsymbol{R}$*. Then the set*

$$\exists^* A = \left\{ x \in \boldsymbol{R} \;\middle|\; A_x \textit{ is non-meager} \right\}$$

is $\mathbf{\Sigma}^0_\alpha$*.* ◄

Question. Does the last statement imply $\mathrm{BP}(\mathbf{\Sigma}^1_2)$?

References

[FM] H.Fujita and T.Mátrai, *On the difference property of Borel measurable functions*, Fund. Math. to appear.

[Lo] A.Louveau, *A separation theorem for* Σ^1_1, Trans. Amer. Math. Soc. **260** (1980), 363–378.