Coanalytic sets with Borel sections

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**Fact.** (Fujita and Mátrai) Let $B \subset \mathbb{R} \times \mathbb{R}$ be a Borel set such that the horizontal section $B^y$ is $\Sigma^0_\alpha$ for every $y \in \mathbb{R}$. Then there is a dense $G_\delta$ set $D \subset \mathbb{R}$ such that $B \cap (\mathbb{R} \times D)$ is $\Sigma^0_\alpha | \mathbb{R} \times D$.

This can be proved by a straightforward induction using A.Louveau's solution ([Lo]) of the section problem of Borel sets. This fact has been used in order to solve an old problem by M.Laczkovich about differences of Borel measurable function (See [FM].)

**Theorem.** The following statements are equivalent:

1. If $A \subset \mathbb{R} \times \mathbb{R}$ is $\Pi^1_1$ and all horizontal sections $A^y$ are Borel, then there is a dense $G_\delta$ set $D \subset \mathbb{R}$ such that $A \cap (\mathbb{R} \times D)$ is Borel;

2. similar, but $A^y$ are $\Sigma^0_\alpha$ and $A \cap (\mathbb{R} \times D)$ is $\Sigma^0_\alpha | \mathbb{R} \times D$;

3. similar, but $A^y$ are closed and $A \cap (\mathbb{R} \times D)$ is Borel;

4. BP($\Sigma^1_2$), i.e., every $\Sigma^1_2$ set of reals has the property of Baire. □

**Proof.** From (1) to (2): use Fact.

From (2) to (3): immediate from the case $\alpha = 1$ of (2).

From (3) to (4): given $\Sigma^1_2$ set $P \subset \mathbb{R}$, let $A \subset \mathbb{R} \times \mathbb{R}$ be $\Pi^1_1$ such that $y \in P \iff \exists x [\langle x, y \rangle \in A]$. Uniformize $A$ by a function $f : P \to \mathbb{R}$ with $\Pi^1_1$ graph. Apply (3) to the graph of $f$. Then $P \cap D$ is $\Sigma^1_1$ and $D$ is co-meager. So $P$ has BP.

From (4) to (1): this is the main part of this note.

Let $\mathbb{C}$ be the Cohen poset. Given a transitive model $M$ of set theory, let $\text{Co}(M)$ be the set of all $\mathbb{C}$-generic reals over $M$. 

Lemma A. (Solovay) BP($\Sigma^1_2$) holds if and only if Co($L[r]$) is co-meager for every $r \in R$.

Let WO be set the of $w \in \omega^2$ which codes a wellordering on $\omega$. For each $q \in \text{WO}$ let $\|w\|$ be the order-type (i.e., countable ordinal) that $w$ codes.

Definition. A set $X \subset R \times \omega_1$ is $\Pi^1_2$ in the codes if the set

$$\{ (x, w) \in R \times \omega^2 \mid w \in \text{WO}, (x, \|w\|) \in X \}$$

is (lightface) $\Pi^1_2$.

Lemma B. Let $X \subset R \times \omega_1$ be $\Pi^1_2$ in the codes. Suppose that for every $y \in R$ there is $\xi < \omega_1$ such that $\langle y, \xi \rangle \in X$. Then there is a countable ordinal $\delta$ such that for every $c \in \text{Co}(L)$ there is $\xi < \delta$ such that $\langle c, \xi \rangle \in X$.

Proof of (4) $\Rightarrow$ (1) [taking Lemmas for granted]. We put $R = \omega^\omega$ and assume $A$ is lightface $\Pi^1_1$. Let $f : R \times R$ be a recursive function such that $A = f^{-1}[\text{WO}]$. Since $A^y$ is Borel, the image $f[A^y \times \{y\}]$ is bounded in WO, that is to say,

$$\forall y \in R \exists \xi < \omega_1 \forall x \left[ \langle x, y \rangle \in A \implies \|f(x, y)\| < \xi \right].$$

For each $\xi < \omega_1$ set

$$\text{WO}_\xi = \left\{ w \in \text{WO} \mid \|w\| < \xi \right\}$$

and let

$$X = \left\{ \langle y, \xi \rangle \mid f[A^y \times \{y\}] \subset \text{WO}_\xi \right\}.$$

Observe that $X$ is $\Pi^1_2$ in the codes. Applying Lemma B we find a countable ordinal $\delta$ such that

$$\forall c \in \text{Co}(L) \exists \xi < \delta \left[ \langle c, \xi \rangle \in X \right].$$

Then we have

$$A \cap (R \times \text{Co}(L)) = f^{-1}[\text{WO}_\delta] \cap (R \times \text{Co}(L)).$$

By Lemma A there is a dense $G_\delta$ set $D \subset \text{Co}(L).$
Proof of Lemma B. Let $\varphi(y, w)$ be a $\Pi^1_2$ formula such that
\[
\langle y, \xi \rangle \in X \iff \exists w \in \text{WO} \left[ \xi = \|w\| \land \varphi(y, w) \right] \\
\iff \forall w \in \text{WO} \left[ \xi = \|w\| \implies \varphi(y, w) \right].
\]
Then we have, by the assumption of the lemma,
\[\forall y \exists \xi < \omega_1 \forall w \left[ w \in \text{WO} \land \|w\| = \xi \implies \varphi(y, w) \right]. \tag{\*}\]
Let $\varphi^*(y, \xi)$ stand for "$\forall w \cdots$" part of (\*). Then $\varphi^*(y, \xi)$ is absolute for every proper class model in which $\xi$ is countable.

Let $c \in \text{Co}(L)$ and suppose that $\langle c, \xi \rangle \in X$. Let $g : \omega \to \xi$ be $\text{Coll}(\xi)$-generic over $L[c]$. Then
\[L[c, q] \models \varphi^*(c, \xi)\]
so that there are forcing conditions $p \in \mathbb{C}$ and $q \in \text{Coll}(\xi)$ such that $c$ meets $p$, $g$ meets $q$ and
\[\langle p, q \rangle \Vdash_{\mathbb{C} \times \text{Coll}(\xi)} L[\dot{c}, \dot{g}] \models \varphi^*(\dot{c}, \dot{\xi}).\]
Then by absoluteness of forcing relations,
\[L \models \left[ \langle p, q \rangle \Vdash_{\mathbb{C} \times \text{Coll}(\xi)} \varphi^*(\dot{c}, \dot{\xi}) \right].\]
By homogeneity of the poset $\text{Coll}(\xi)$,
\[L \models \left[ \langle p, \emptyset \rangle \Vdash_{\mathbb{C} \times \text{Coll}(\xi)} \varphi^*(\dot{c}, \dot{\xi}) \right].\]
where $\emptyset$ is the weakest member of $\text{Co}(\xi)$.
For each $\xi < \omega_1$ let
\[Y_\xi = \left\{ p \in \mathbb{C} \mid L \models \left[ \langle p, \emptyset \rangle \Vdash_{\mathbb{C} \times \text{Coll}(\xi)} \varphi^*(\dot{c}, \dot{\xi}) \right] \right\}.
\]
Then $\bigcup_{\xi < \omega_1} Y_\xi$ is pre-dense in $\mathbb{C}$. By ccc, there is $\delta < \omega_1$ such that $\bigcup_{\xi < \delta} Y_\xi$ is already pre-dense in $\mathbb{C}$. ▷

Daisuke Ikegami observed that $\mathbb{C}$ in Lemma B can be replaced by other alboreal Suslin ccc forcing notions. Daisuke also pointed out that Sacks forcing does not satisfy Lemma B nor clause (3) of Theorem.
By Montgomery's result on the category quantifier, we obtain

**Corollary.** Assume BP($\Sigma^1_2$). Let $A \subseteq R \times R$ be $\Pi^1_1$ such that $A^y$ is $\Sigma^0_\alpha$ for every $y \in R$. Then the set

$$\exists^* A = \{ x \in R \mid A_x \text{ is non-meager} \}$$

is $\Sigma^0_\alpha$. ◀

**Question.** Does the last statement imply BP($\Sigma^1_2$)?

**References**
