

Notes on Skinny Stationary Subsets of $\mathcal{P}_\kappa\lambda$

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§ 1 Introduction

In this article we will introduce the notion of skinniness for subsets of $\mathcal{P}_\kappa\lambda$ where κ denotes a regular uncountable cardinal and λ denotes a cardinal $> \kappa$. We will state a theorem showing the close connection between the existence of a skinny stationary subset of $\mathcal{P}_\kappa\lambda$ and \diamond_λ , the Diamond Principle for λ . We will also discuss the existence of a skinny stationary subset of $\mathcal{P}_\kappa\lambda$ as a consequence of the precipitousness of $\text{NS}_{\kappa\lambda}$, the non-stationary ideal over $\mathcal{P}_\kappa\lambda$, for some λ s. Due to the nature and the length of this article, we will omit the proofs. They will appear in our forthcoming papers [3] and [4]. The aim of this article is to show the validity of the study of skinny subsets of $\mathcal{P}_\kappa\lambda$ for furthering the investigation of the combinatorics of $\mathcal{P}_\kappa\lambda$. Throughout this article, we let κ denote a regular uncountable cardinal and λ a cardinal $> \kappa$.

§ 2 Skinny subsets of $\mathcal{P}_\kappa\lambda$

We denote the set $\{s \subset \lambda : |s| < \kappa\}$ by $\mathcal{P}_\kappa\lambda$. We also denote the non-stationary ideal over $\mathcal{P}_\kappa\lambda$ by $\text{NS}_{\kappa\lambda}$. For the basic facts on $\mathcal{P}_\kappa\lambda$ and $\text{NS}_{\kappa\lambda}$, we refer the reader to [2].

In order to introduce the notion of skinniness, we need the next notation:

Notation. For an ordinal $\delta \leq \lambda$ and a set $X \subseteq \mathcal{P}_\kappa\lambda$, let

$$X^\delta = \{s \in X \cap \mathcal{P}_\kappa\delta : \sup(s) = \delta\}.$$

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It is clear that $X^\delta = \emptyset$ if $\text{cf}(\delta) \geq \kappa$ for every $X \subseteq \mathcal{P}_\kappa \lambda$.

Now we present the definition of skinniness:

Definition. Let X be a subset of $\mathcal{P}_\kappa \lambda$.

- (i) X is said to be *skinny* if $|X^\delta| < |\mathcal{P}_\kappa \delta|$ for every $\delta \leq \lambda$.
- (ii) X is said to be *skinnier* if $|X^\delta| \leq |\delta|$ for every $\delta \leq \lambda$.
- (iii) X is said to be *skinniest* if $|X^\delta| \leq 1$ for every $\delta \leq \lambda$.

We note that unless λ is the successor cardinal of some cardinal δ with $\delta^{<\kappa} = \delta$, we can show that there exists a club subset C of $\mathcal{P}_\kappa \lambda$ such that, for every subset X of $\mathcal{P}_\kappa \lambda$, if X is skinnier, then $X \cap C$ is skinny. For some λ s, $\mathcal{P}_\kappa \lambda$ does not have any skinny stationary subsets.

Proposition 1 ([5]). *If λ is a strong limit singular cardinal, then there is no skinny stationary subset of $\mathcal{P}_\kappa \lambda$.*

As the next result indicates, the situation for a regular λ is quite different.

Proposition 2 (Matsubara-Sakai [4]). *In L , the class of constructible sets, if κ and λ are regular cardinals with $\aleph_1 \leq \kappa < \lambda$, then every stationary subset of $\mathcal{P}_\kappa \lambda$ has a skinniest stationary subset.*

The next result is an immediate consequence of the last theorem and Jensen's Covering Theorem.

Corollary 3. *Assume $0^\#$ does not exist. If κ and λ are regular cardinals with $\aleph_1 < \kappa < \lambda$, then there is a skinniest unbounded subset of $\mathcal{P}_\kappa \lambda$.*

One might wonder whether the existence of a skinniest stationary subset of $\mathcal{P}_\kappa \lambda$ where λ is a regular cardinal is a consequence of ZFC. In the following section, we will indicate that this is not the case.

Various researchers have been investigating large cardinal properties of ideals over $\mathcal{P}_\kappa \lambda$. Among these properties, precipitousness and saturation are the properties which have been studied most extensively. For an ideal I over $\mathcal{P}_\kappa \lambda$ and a subset X of $\mathcal{P}_\kappa \lambda$, we say that X is *I -positive* if $X \notin I$ and X is *I -measure one* if $\mathcal{P}_\kappa \lambda - X \in I$. The next theorem shows that if an ideal I is sufficiently saturated, then there is a skinniest I -measure one set.

Proposition 4 (Matsubara-Sakai [4]). *Let κ and λ be uncountable regular cardinals with $\kappa < \lambda$, and let δ be a cardinal with $\delta < \kappa^{+\kappa}$ and $\delta \leq \lambda$. If I is a normal fine κ -complete δ -saturated ideal over $\mathcal{P}_\kappa \lambda$, then there exists a skinniest I -measure one subset of $\mathcal{P}_\kappa \lambda$.*

In the study of $\mathcal{P}_\kappa\lambda$ combinatorics, we are particularly interested in the properties of $\text{NS}_{\kappa\lambda}$, the non-stationary ideal over $\mathcal{P}_\kappa\lambda$. For each stationary $X \subseteq \mathcal{P}_\kappa\lambda$, we let $\text{NS}_{\kappa\lambda}|X$ denote the set $\{Y \subseteq \mathcal{P}_\kappa\lambda : X \cap Y \in \text{NS}_{\kappa\lambda}\}$. $\text{NS}_{\kappa\lambda}|X$ is the smallest κ -complete normal fine ideal extending $\text{NS}_{\kappa\lambda} \cup \{\mathcal{P}_\kappa\lambda - X\}$. In the study of ideals over $\mathcal{P}_\kappa\lambda$, $\text{NS}_{\kappa\lambda}$ and $\text{NS}_{\kappa\lambda}|X$, for various $X \subseteq \mathcal{P}_\kappa\lambda$, play a central role. To obtain a skinny stationary subset of $\mathcal{P}_\kappa\lambda$, the somewhat stringent condition on saturation of the last theorem can be weakened if we assume that I is $\text{NS}_{\kappa\lambda}$ or $\text{NS}_{\kappa\lambda}|X$ for some λ s.

Proposition 5 ([3]). *Assume that λ is either a strong limit cardinal or the successor of a singular cardinal δ with $\delta^{<\kappa} = 2^\delta > 2^{<\kappa}$ and $\text{cf}(\delta) < \kappa$. Let X be a stationary subset of $\mathcal{P}_\kappa\lambda$.*

- (i) *If $\text{NS}_{\kappa\lambda}|X$ is precipitous, then X has a skinny stationary subset.*
- (ii) *If $\text{NS}_{\kappa\lambda}|X$ is 2^λ -saturated, then, for some club $C \subseteq \mathcal{P}_\kappa\lambda$, $X \cap C$ is skinny.*

By combining Proposition 1 and Proposition 5, we get the following result:

Proposition 6. *Let λ be a strong limit singular cardinal $> \kappa$. Then*

- (i) *$\text{NS}_{\kappa\lambda}$ is nowhere 2^λ -saturated, i.e. $\text{NS}_{\kappa\lambda}|X$ is not 2^λ -saturated for every stationary $X \subseteq \mathcal{P}_\kappa\lambda$.*
- (ii) *$\text{NS}_{\kappa\lambda}$ is nowhere precipitous.*

Conclusion (ii) of the above theorem was previously demonstrated in [5]. In the next section, we will state some applications of skinny stationary subsets of $\mathcal{P}_\kappa\lambda$ where λ is a regular cardinal.

§ 3 Diamonds

In this section we will discuss the relationship between the existence of a skinny stationary subset of $\mathcal{P}_\kappa\lambda$ where λ is regular and $\diamond_\lambda(E_{<\kappa}^\lambda)$ where $E_{<\kappa}^\lambda = \{\alpha < \lambda : \text{cf}(\alpha) < \kappa\}$. For the definition of $\diamond_\lambda(E_{<\kappa}^\lambda)$, we refer the reader to [1].

Before we start discussing the Diamond Principle, we want to go back to the notion of skinniness once again. In the first section, we introduced three notions of skinniness, namely “skinny”, “skinnier”, and “skinniest”. The next proposition shows that for the cases of λ which we are interested in, if we assume GCH, then the existence of a skinny stationary subset of $\mathcal{P}_\kappa\lambda$ and the existence of a skinnier one are equivalent. Instead of GCH, we will use a weaker hypothesis, namely the Singular Cardinal Hypothesis.

Proposition 7. *Assume that λ is either a strong limit cardinal or the successor of a strong limit cardinal whose cofinality is less than κ . Suppose the Singular Cardinal Hypothesis holds for cardinals $\leq \lambda$. Then there exists a club $C \subseteq \mathcal{P}_\kappa \lambda$ such that, for every $X \subseteq \mathcal{P}_\kappa \lambda$, $X \cap C$ is skinny if and only if $X \cap C$ is skinnier.*

If λ is a strong limit cardinal with $\text{cf}(\lambda) < \kappa$, then, from the Singular Cardinal Hypothesis, we conclude that $\lambda^{<\kappa} = \lambda^+$. Hence in this case, we just let $C = \{s \in \mathcal{P}_\kappa \lambda : \text{sup}(s) = \lambda\}$. For the other cases covered by the hypotheses of the last proposition, note that $\{\delta < \lambda : |\mathcal{P}_\kappa \delta| = |\delta|^+ \wedge \text{cf}(\delta) < \kappa\}$ contains a $< \kappa$ -club subset C^* of λ . For these λ s, we let $C = \{s \in \mathcal{P}_\kappa \lambda : \text{sup}(s) \in C^* \wedge \text{sup}(s) \notin s\}$. As noted in § 2, unless λ is the successor cardinal of some cardinal δ with $\delta^{<\kappa} = \delta$, we can conclude that every skinnier subset of $\mathcal{P}_\kappa \lambda$ is essentially skinny. Our next result will relate the existence of a skinny stationary set and the Diamond Principle.

Proposition 8. *Let λ be a regular cardinal such that $\lambda > 2^{<\kappa}$ holds. Suppose E is a stationary subset of $\{\delta < \lambda : \text{cf}(\delta) < \kappa \wedge \delta \geq 2^{<\kappa}\}$. The following are equivalent.*

- (i) $\diamond_\lambda(E)$.
- (ii) *There exists a skinniest stationary set $X \subseteq \mathcal{P}_\kappa \lambda$ such that $\{\text{sup}(t) : t \in X\} \subseteq E$, and $\lambda^{<\lambda} = \lambda$ holds.*
- (iii) *There exists a skinnier stationary set $Y \subseteq \mathcal{P}_\kappa \lambda$ such that $\{\text{sup}(t) : t \in Y\} \subseteq E$, and $\lambda^{<\lambda} = \lambda$ holds.*

For $X \subseteq \mathcal{P}_\kappa \lambda$, we let E_X denote the set $\{\text{sup}(t) : t \in X \wedge \text{sup}(t) \notin t\}$. If X is a stationary subset of $\mathcal{P}_\kappa \lambda$ where λ is a cardinal with $\text{cf}(\lambda) \geq \kappa$, then E_X is a stationary subset of $E_{<\kappa}^\lambda$.

Proposition 5, Proposition 7, and Proposition 8 imply the following result:

Proposition 9. *Suppose λ is either an inaccessible cardinal or the successor of a singular δ with $\text{cf}(\delta) < \kappa$. Assume GCH holds below λ . If $\text{NS}_{\kappa\lambda}|X$ is precipitous, then $\diamond_\lambda(E_X)$ holds. Therefore if $\text{NS}_{\kappa\lambda}$ is precipitous, then $\diamond_\lambda(E)$ holds for every stationary subset E of $E_{<\kappa}^\lambda$.*

The last proposition together with a recent theorem of Shelah on \diamond implies the next result:

Proposition 10. *Let λ be a regular cardinal $> \kappa$. Assume that GCH holds below λ . If $\text{NS}_{\kappa\lambda}$ is precipitous, then $\diamond_\lambda(E)$ holds for every stationary subset E of $E_{<\kappa}^\lambda$.*

Theorem (Shelah's Theorem on \diamond). *Let δ be a regular uncountable cardinal. Suppose $\lambda = \delta^+$. If $2^\delta = \delta^+$, then $\diamond_\lambda(E)$ holds for every stationary subset E of $\{\alpha < \lambda : \text{cf}(\alpha) \neq \text{cf}(\delta)\}$.*

The last result shows that if $\text{NS}_{\kappa\lambda}$ is precipitous, where λ is an inaccessible cardinal, then the failure of $\diamond_{\lambda}(E_{<\kappa}^{\lambda})$ implies that the Singular Cardinal Hypothesis fails badly.

Proposition 11. *Suppose E is a stationary subset of $E_{<\kappa}^{\lambda}$ where λ is an inaccessible cardinal. If $\text{NS}_{\kappa\lambda}|\{t \in \mathcal{P}_{\kappa}\lambda : \text{sup}(t) \in E\}$ is precipitous and $\diamond_{\lambda}(E)$ fails, then $\{\delta \in E : \text{the Singular Cardinal Hypothesis holds at } \delta\}$ is non-stationary.*

Note that if E is a stationary subset of $E_{<\kappa}^{\lambda}$ where λ is an inaccessible cardinal, then the set $\{s \in \mathcal{P}_{\kappa}\lambda : \text{sup}(s) \in E\}$ is a stationary subset of $\mathcal{P}_{\kappa}\lambda$.

References

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