On the consistency strength of the FRP for the second uncountable cardinal

Tadatoshi MIYAMOTO

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Abstract

We show that the consistency strength of the Fodor-type Reflection Principle for the second uncountable cardinal is exactly that of a Mahlo cardinal.

Introduction

The Fodor-type Reflection Principles for various uncountable cardinals $\lambda$, denoted by FRP$(\lambda)$, are introduced in [F]. We are interested in the consistency strength of FRP$(\omega_2)$ in this note. Let us recall the following two reflection principles, where $S^2_0 = \{ \alpha < \omega_2 \mid \text{cf} (\alpha) = \omega \}$ and $S^2_1 = \{ \alpha < \omega_2 \mid \text{cf} (\alpha) = \omega_1 \}$.

1. For all stationary $S \subseteq [\omega_2]^\omega$, there exists $\gamma \in S^2_1$ such that $S \cap [\gamma]^{\omega}$ is stationary in $[\gamma]^{\omega}$.
2. For all stationary $S \subseteq S^2_0$, there exists $\gamma \in S^2_1$ such that $S \cap \gamma$ is stationary in $\gamma$.

It is known that FRP$(\omega_2)$ fits in between these two by [F]. Namely, (1) implies FRP$(\omega_2)$. And FRP$(\omega_2)$ implies (2). The consistency strength of (1) is that of a weakly compact cardinal by [V]. And the consistency strength of (2) is that of a Mahlo cardinal by [H-S]. We follow [S] (pp.576-581) to show that the consistency strength of FRP$(\omega_2)$ is that of a Mahlo cardinal.

§1. Main Theorem

Definition. A map $\langle C_\delta \mid \delta \in E \rangle$ is a ladder system, if $E \subseteq S^2_0$ is stationary in $\omega_2$ and each $C_\delta$ is a cofinal subset of $\delta$ such that the order-type of $C_\delta$ is $\omega$. Let $\gamma \in S^2_1$. We say a sequence $\langle X_i \mid i < \omega_1 \rangle$ is a filtration on $\gamma$, if it is continuously $\subseteq$-increasing countable subsets of $\gamma$ with $\bigcup \{ X_i \mid i < \omega_1 \} = \gamma$.

The following is equivalent to the FRP$(\omega_2)$ of [F] and we take this as our definition of FRP$(\omega_2)$.

Definition. The Fodor-type Reflection Principle for the second uncountable cardinal, denoted by FRP$(\omega_2)$, holds, if for any ladder system $\langle C_\delta \mid \delta \in E \rangle$, there exists $\gamma \in S^2_1$ and a filtration $\langle X_i \mid i < \omega_1 \rangle$ on $\gamma$ such that $T = \{ i < \omega_1 \mid \text{sup}(X_i) \in E \text{ and } C_{\text{sup}(X_i)} \subseteq X_i \}$ is stationary in $\omega_1$.

Definition. Let $\kappa$ be a strongly inaccessible cardinal. The Levy collapse which makes $\kappa = \omega_2$ by the countable conditions is denoted by $\text{Lv}(\kappa, \omega_1)$. Hence $p \in \text{Lv}(\kappa, \omega_1)$, if $p$ is a function whose domain is a countable subset of $[\omega_2, \kappa) \times \omega_1$ such that for all $(\xi, i)$ in the domain of $p$, we demand $p(\xi, i) < \xi$. For $p, q \in \text{Lv}(\kappa, \omega_1)$, we define $q \leq p$, if $q \supseteq p$.

Theorem. Let $\kappa$ be a Mahlo cardinal and assume GCH in the ground model $V$. Let $G_\kappa$ be any $\text{Lv}(\kappa, \omega_1)$-generic filter over $V$. Then we have $\kappa = \omega_2$ and $(\kappa^+)^V = \omega_3$ in the generic extension $V[G_\kappa]$. Now in $V[G_\kappa]$, we may construct a $< \omega_2$-support $\omega_3$-stage iterated forcing $(P_{a^*}^\kappa, a^* \leq \omega_2)$ such that for each $\alpha^* < \omega_3, P_{a^*}^\kappa$ is $\omega_2$-Baire and has a dense subset of size $\omega_2$ and that FRP$(\omega_2)$ holds in the generic extensions $V[G_\kappa]^\langle P_{a^*}^\kappa \rangle_\omega$.

§2. An Idea of Proof

Let $\kappa$ be a Mahlo cardinal and assume GCH in the ground model $V$. Let $G_\kappa$ be a fixed $\text{Lv}(\kappa, \omega_1)$-generic filter over $V$. We work in the generic extension $V[G_\kappa]$ where $\kappa = \omega_2$ and GCH holds.

Definition. A ladder system $\langle C_\delta \mid \delta \in E \rangle$ is reflected, if there exist $\gamma \in S^2_0$ and a filtration $\langle X_i \mid i < \omega_1 \rangle$ on $\gamma$ such that $T = \{ i < \omega_1 \mid \text{sup}(X_i) \in E \text{ and } C_{\text{sup}(X_i)} \subseteq X_i \}$ is stationary. We also say that a ladder system is non-reflecting, if it is not reflected. Let $\langle C_\delta \mid \delta \in E \rangle$ be non-reflecting. Then we may associate
a p.o.set $Q$ which shoots a club off $E$. By this we mean that $Q$ forces a club $C$ in $\kappa$ such that for any accumulation point $\alpha$ of $C$, namely $\alpha$ is a limit ordinal and $C \cap \alpha$ is cofinal in $\alpha \in C$, we have $\alpha \not\in E$. The conditions in $Q$ are the possible initial segments of $C$.

We argue in $V[G_\kappa]$. Let $(C_\delta \mid \delta \in E)$ be a non-reflecting ladder system and $Q$ be the associated p.o.set. Since there is no restrictions to put any new point above any condition in $Q$, it is clear that $Q$ adds a cofinal and closed subset of $\kappa$. It is also clear that $Q$ is of size $(2^{<\kappa})^{V[G_\kappa]} = (2^{\omega_1})^{V[G_\kappa]} = \omega_2^{V[G_\kappa]}$. However it is not at all clear that $Q$ is $\kappa$-Baire. Namely, $Q$ does not add any new sequences of ordinals of length $< \kappa$. Before we start iterating, we present the following.

**Observation.** Let $(C_\delta \mid \delta \in E)$ be non-reflecting in $V[G_\kappa]$ and let $Q$ be the associated p.o.set in $V[G_\kappa]$ which shoots a club off $E$ over $V[G_\kappa]$. Now we go back in $V$ for a while. Let $\theta$ be a sufficiently large regular cardinal in $V$ and $N$ be an elementary substructure in $V$ of $(H_\theta)^V$ such that $\kappa \in N$, $N \cap \kappa = \lambda$ is a strongly inaccessible cardinal in $V$, $\kappa < \lambda \in N$ in $V$ and $|N| = \lambda$ in $V$. We further assume that $(C_\delta \mid \delta \in E), Q \in N[G_\kappa]$ in $V[G_\kappa]$. Let $M$ be the transitive collapse of $N$ by the collapse $\pi$ in $V$. Since $L(V(\kappa, \omega_1))$ has the $\kappa$-c.c., every condition in $L(V(\kappa, \omega_1))$ is $(L(V(\kappa, \omega_1)), N)$-generic. Hence $\pi$ gets extended to $\pi$ (same notation in use) collapsing $N[G_\kappa]$ onto $M[G_\lambda]$, where $G_\lambda = G_\kappa \cap \lambda \in \lambda$-generic over $V$.

Notice that we may view $M[G_\lambda]$ as a generic extension of $M$ via $L(V(\lambda, \omega_1))$ over the transitive set model $M$. We also have that $V \cap <\lambda M \subset M$ and $V[G_\lambda] \cap <\lambda M[G_\lambda] \subset M[G_\lambda]$. Since $(C_\delta \mid \delta \in E) \in N[G_\kappa]$, it gets collapsed to $(C_\delta \mid \delta \in E \cap \lambda) \in M[G_\lambda]$. We claim that $E \cap \lambda$ is a non-stationary subset of $\omega_2^{V[G_\kappa]}$ in $V[G_\lambda]$. This is because, if $E \cap \lambda$ were stationary in $V[G_\lambda]$, then it is easy to see by genericity of $\pi$ that for a (any) filtration $(X_\delta \mid \delta \in \omega) \in \lambda$ in $V[G_{\lambda+1}] = V[G_{\lambda}]^{\langle f \rangle}$, where $f : \omega_1 \longrightarrow \lambda$ onto, we have $T\equiv \{\delta \in \omega_1 \mid sup(X_\delta) \in E \cap \lambda \cap \lambda \}$ is stationary in $V[G_{\lambda+1}]$. This $T$ remains stationary in $V[G_{\lambda+1}] = V[G_{\lambda+1}]^{\langle f \rangle}$, where $G_{\lambda+1} = G_{\lambda}^{\langle \lambda+1, \kappa \rangle}$-generic over $V[G_{\lambda+1}]$. Hence the ladder system $(C_\delta \mid \delta \in E)$ gets reflected. Hence there is a club $C$ of $\lambda$ in $V[G_\lambda]$ such that $C \cap (E \cap \lambda) = \emptyset$. Now by making use of this $C$ and the fact $V[G_\lambda] \cap <\lambda M[G_\lambda] \subset M[G_\lambda]$, we may construct a $\pi(Q), M[G_\lambda]$-generic sequence $(q_\kappa \mid \kappa < \lambda)$ in $V[G_\lambda]$. Now take point-wise preimages of the $q_\kappa$. Namely let $p_\kappa \in Q \cap N[G_\kappa]$ such that $\pi(p_\kappa) = q_\kappa$. Then it is routine to show that $(p_\kappa \mid \kappa < \lambda)$ is a $(Q, N[G_\kappa])$-generic sequence in $V[G_\kappa]$. Hence $sup(\{p_\kappa \mid \kappa < \lambda\}) = N[G_\kappa] \cap \kappa = N \cap \kappa = \lambda \not\in E \subset S_\theta$. Hence $q = (\{p_\kappa \mid \kappa < \lambda\}) \cup \{\lambda\} \in Q$ decides $O \cap N[G_\kappa][O] = O \cap N[G_\kappa] = \{p \in Q \cap N[G_\kappa] \mid \mu \geq p_\kappa \}$ for some $k < \lambda \in V[G_\kappa]$, where $O$ are the $Q$-generic filters over $V[G_\kappa]$ with $q \in O$. Hence $Q$ is $\kappa$-Baire.

With this in mind, we are interested in the following class of preorders $P$ in $V[G_\kappa]$.

**Definition.** A preorder $P$ is reasonable, if $P$ has a dense subset of size $\kappa$ and is $\kappa$-Baire.

**Proposition.** Let $P$ be reasonable in $V[G_\kappa]$. Then $P$ preserves every cofinality, every cardinality and GCH.

Typically we will consider a $< \kappa$-support iterated forcing $P = P_\alpha^* \in (H_{\alpha+1})^{V[G_\kappa]}$ with $\alpha^* < (\kappa^*)^V = (\kappa^*)^{V[G_\kappa]} = \omega_3^{V[G_\kappa]}$. We intend to denote some of the objects in $V[G_\kappa]$ with $^*$ in this note.

**Definition.** A sequence $\langle P_\alpha^* \mid \beta^* \leq \alpha^* \rangle$ (together with $\langle Q_\alpha^* \mid \beta^* < \alpha^* \rangle$, $\langle C_\delta^* \mid \delta \in E_\beta^* \rangle \mid \beta^* < \alpha^*$) and enumerations of names of the ladder systems from the intermediate stages $\langle (\alpha_1, \alpha_2) \mapsto \langle C_\delta^* \mid \delta \in E_{\alpha_1, \alpha_2} \rangle \mid \alpha_1 < \alpha^*, \alpha_2 < \alpha^* \rangle \subset V[G_\kappa]$ is our iteration, if

* $\langle P_\alpha^* \mid \beta^* \leq \alpha^* \rangle$ is a $< \kappa$-support iterated forcing with $\alpha^* < (\kappa^*)^V$ such that $P_{\alpha^*+1} = P_{\alpha^*} \ast Q_{\alpha^*}$ for each $\beta^* < \alpha^*$.

* The support of $p^* \in P_\alpha^*$ is defined by $supp(p^*) = \{\xi < \beta^* \mid \pi^*(\xi) \neq \emptyset \}$. And so $supp(p^*)$ is of size $< \kappa$.

* For each $\beta^* < \alpha^*$, $P_\beta^*$ is reasonable and $\|V[G_\kappa]\|^\langle C_\delta^* \mid \delta \in E_{\beta^*}\rangle$ is a non-reflecting ladder system and $\|V[G_\kappa]\|$ the associated $Q_{\beta^*}$ shoots a club off $E_{\beta^*}^*$. 
We would like to consider that the last preorder $P_{\kappa^{++}}^\ast$ has just finished its construction and waits to be explored its reasonability and more. Hence our iteration is known to be reasonable possibly except the last preorder.

We are interested in reasonable preorders and iterations in $(H_{\kappa^{++}})(G_{\kappa})$.

**Proposition.** (Successor) Let $I = (P_{\kappa^{++}}^\ast \mid \gamma^\ast \leq \beta^\ast + 1)$ be our iteration. If $(P_{\gamma^\ast}^\ast \mid \gamma^\ast \leq \beta^\ast) \in (H_{\kappa^{++}})(G_{\kappa})$, then $P_{\beta^\ast+1}^\ast \in (H_{\kappa^{++}})(G_{\kappa})$.

**Proof.** Since $P_{\beta^\ast}^\ast$ is reasonable, $P_{\beta^\ast}$ has a dense subset $D$ of size $\kappa$ and $P_{\beta^\ast}^\ast$ is $\kappa$-Baire. Since $1^{\ast} \models \forall^{(H_{\kappa^{++}})^{V pensions \in (\kappa^{++})^{V}}}$, we may represent each $p \in P_{\beta^\ast+1}^\ast$ as $p[\beta^\ast \in P_{\beta^\ast}$ and $p(\beta^\ast) : [\kappa]^{<\kappa} \rightarrow \mathcal{P}(D)$. Hence $|p(\beta^\ast)| \leq \kappa$ and $p(\beta^\ast) \subset [\kappa]^{<\kappa} \times \mathcal{P}(D) \subset (H_{\kappa^{++}})(G_{\kappa})$. Hence $p(\beta^\ast) \subset (H_{\kappa^{++}})(G_{\kappa})$ and so $p \in (H_{\kappa^{++}})(G_{\kappa})$. Hence $P_{\beta^\ast+1}^\ast \subset (H_{\kappa^{++}})(G_{\kappa})$ and $|P_{\beta^\ast+1}^\ast| \leq |P_{\beta^\ast}| \times |[\kappa]^{<\kappa} \times \mathcal{P}(D)| \leq \kappa^\ast$. Hence $P_{\beta^\ast+1}^\ast \in (H_{\kappa^{++}})(G_{\kappa})$.

\[\square \]

**Proposition.** (Limit) Let $I = (P_{\kappa^{++}}^\ast \mid \gamma^\ast \leq \beta^\ast)$ be our iteration with limit $\beta^\ast$. If for all $\gamma^\ast < \beta^\ast$, we have $P_{\gamma^\ast}^\ast \in (H_{\kappa^{++}})(G_{\kappa})$, then $P_{\beta^\ast}^\ast \in (H_{\kappa^{++}})(G_{\kappa})$.

**Proof.** For $p \in P_{\beta^\ast}$ and $\gamma^\ast < \beta^\ast$, we have $p[\gamma^\ast \in (H_{\kappa^{++}})(G_{\kappa})$. Hence $p \subset (H_{\kappa^{++}})(G_{\kappa})$. But $|p| \leq \kappa$. Hence $P_{\beta^\ast}^\ast \subset (H_{\kappa^{++}})(G_{\kappa})$.

Now if $cf(\beta^\ast) < \kappa$, then $|P_{\beta^\ast}| \leq |(\kappa^{+})^{<\kappa}| \leq \kappa^\ast$. Hence $P_{\beta^\ast}^\ast \subset (H_{\kappa^{++}})(G_{\kappa})$.

Next if $cf(\beta^\ast) = \kappa$, then $|P_{\beta^\ast}| \leq \kappa \times \kappa^\ast = \kappa^\ast$. Hence $P_{\beta^\ast}^\ast \in (H_{\kappa^{++}})(G_{\kappa})$.

\[\square \]

**Corollary.** For every our iteration $(P_{\beta^\ast}^\ast \mid \beta^\ast \leq \alpha^\ast)$, we have $(P_{\beta^\ast}^\ast \mid \beta^\ast \leq \alpha^\ast) \in (H_{\kappa^{++}})(G_{\kappa})$.

**Definition.** For any reasonable $P \in (H_{\kappa^{++}})(G_{\kappa})$, if $1^{\ast} \models \exists P^{G_{\kappa}} \in \dot{P} \mid \delta \in \dot{E}^{P}$ is a non-reflecting ladder system$^\ast$ for some $\langle \dot{C}^{P}_{\delta} \mid \delta \in \dot{E}^{P} \rangle \in (H_{\kappa^{++}})(G_{\kappa})$, then we associate one of them to $P$. Let $\Phi$ denote this association. We see that $\Phi \subset (H_{\kappa^{++}})(G_{\kappa}) \subset (H_{\kappa^{++}})(G_{\kappa})$. Hence $\Phi \in (H_{\kappa^{++}})(G_{\kappa}) = (H_{\kappa^{++}})(G_{\kappa})$. Therefore we may fix a name $\Phi \subset (H_{\kappa^{++}})(G_{\kappa})$.

We think of $\Phi$ as a name of a specific choice function. We may need to fix other names of choice functions $\cdots \in (H_{\kappa^{++}})$ as we go along.

**Definition.** In $V$, let us fix $\kappa^{+} \rightarrow (\kappa^{+}) \times (\kappa^{+})$ for book-keeping. Let $\mathcal{N}$ consists of $N$ such that

- $N$ is an elementary substructure of $(H_{\kappa^{++}})$.
- $\kappa, h, \Phi, \cdots \in N$.
- $N \cap \kappa = \lambda < \kappa$ and $\lambda$ is a strongly inaccessible cardinal.
- $\kappa^{+} \subset N$.
- $|N| = \lambda$.

Since $\kappa$ is Mahlo, there are many elements in $\mathcal{N}$. We aim at the following.

**Target.** Let $N \in \mathcal{N}$ with $P_{\alpha^{\ast}}^{\ast} \in N(G_{\kappa})$. Then for any $p \in P_{\alpha^{\ast}}^{\ast} \cap N(G_{\kappa})$, there exists a $(P_{\alpha^{\ast}}^{\ast}, N(G_{\kappa}))$-generic sequence $(p_{k}^{\ast} \mid k < \lambda)$ such that $(\pi(p_{k}^{\ast}) \mid k < \lambda) \in V(G_{\kappa})$, where $\pi$ is the transitive collapse of $N(G_{\kappa})$ onto $M[G_{\lambda}]$.

**Definition.** Our iteration $I = (P_{\beta^\ast}^\ast \mid \beta^\ast \leq \alpha^\ast)$ is wonderful, if for any $N \in \mathcal{N}$ with $I \in N(G_{\kappa})$ (by this we mean that the other associated sequent sequences of objects with our iteration are also assumed to be in $N(G_{\kappa})$ and we may simply denote this as $P_{\alpha^{\ast}}^{\ast} \in N(G_{\kappa})$), any $p^{\ast} \in P_{\alpha^{\ast}}^{\ast} \cap N(G_{\kappa})$, there exists a $(P_{\alpha^{\ast}}^{\ast}, N(G_{\kappa}))$-generic
sequence \( p^*_k \mid k < \lambda \) below \( p^* \) such that \( \langle \pi(p^*_k) \mid k < \lambda \rangle \in V[G\lambda] \), where \( \lambda = N \cap \kappa \) and \( \pi \) is the transitive collapse of \( N[G\kappa] \) onto \( M[G\lambda] \).

**Proposition.** If \( \langle P^*_\beta \mid \beta^* \leq \alpha^* \rangle \) is wonderful, then the last preorder \( P^*_\alpha \) is reasonable.

**Proof.** Fix any \( p^* \in P^*_\alpha \). Since \( \kappa \) is Mahlo, we may pick \( N \in \mathcal{N} \) such that \( P^*_\beta \in N[G\kappa] \) and \( p^* \in N[G\kappa] \). Let \( \lambda = N \cap \kappa \). By assumption, we may pick a \( (P^*_\alpha, N[G\kappa]) \)-generic sequence \( \langle p^*_k \mid k < \lambda \rangle \in V[G\kappa] \) below \( p^* \).

**Claim.** There exist \( q^* \in P^*_\alpha \) and \( \langle s^*_\beta \mid \beta^* \in N[G\kappa] \cap \alpha^* = N \cap \alpha^* \rangle \) such that

- For all \( k \leq \lambda \), we have \( q^* \models p^*_k \).
- \( \supp(q^*) = \lambda \cap N \).
- Each \( s^*_\beta \) is a cofinal and closed subset of \( \lambda \) with \( \supp(s^*_\beta) = \lambda \).
- If \( \beta^* \in N \cap \alpha^* \), then \( q^*[\beta^*] \models \langle p^*_k \rangle_{k < \lambda} \).

Since \( q^* \) gets classified by the \( \langle s^*_\beta \mid \beta^* \in N \cap \alpha^* \rangle \) and are at most \( \kappa \)-many such sequences, we conclude that \( P^*_\alpha \) is reasonable.

**Proof.** For each \( \beta^* \in N \cap \alpha^* \), let \( s^*_\beta = \bigcup \{ s \mid \exists k < \lambda \exists l \geq k \ p^*_k[\beta^*] \models [V[G\kappa]]\langle q^*_k(\beta^*) = s \rangle \} \).

We construct \( q^*[\beta^*] \) by recursion on \( \beta^* \leq \alpha^* \) in \( V[G\kappa] \). Suppose \( \beta^* < \alpha^* \) and for all \( k < \lambda \), we have \( q^*[\beta^*] \models p^*_k \). We want to specify \( q^*(\beta^*) \).

We first assume \( \beta^* \notin N \). Then let \( q^*(\beta^*) = \emptyset \). Since each \( p^*_k \in N[G\kappa] \), we have \( \supp(p^*_k) \subseteq N[G\kappa] \cap \alpha^* = N \cap \alpha^* \). Hence for all \( k < \lambda \), we have \( p^*_k(\beta^*) = \emptyset \) and so \( q^*[\beta^* + 1] = p^*_\alpha[\beta^* + 1] \).

We next assume \( \beta^* \in N \). By assumption \( P^*_\alpha \) is \( \kappa \)-Baire and \( \langle p^*_k[\beta^*] \mid k < \lambda \rangle \) is an induced \( (P^*_\alpha, N[G\kappa]) \)-generic sequence. Hence for any \( k < \lambda \), there exists \( l \) such that \( k < l < \lambda \) and \( p^*_l[\beta^*] \) decides the value of \( p^*_k(\beta^*) \) to be some \( s \).

Let \( O_{\beta^*} \) be any \( P^*_\alpha \)-generic filter over \( V[G\kappa] \) with \( \langle s^* \mid \beta^* \in O_{\beta^*} \rangle \). Since \( q^*[\beta^*] \) is below every \( p^*_k \), we have in \( V[G\kappa][O_{\beta^*}] \) that \( \langle p^*_k[\beta^*] \rangle_{k < \lambda} \) is a \( (O_{\beta^*}, N[G\kappa][O_{\beta^*}]) \)-generic sequence. Hence we conclude \( s^*_\beta \) is a cofinal and closed subset of \( N[G\kappa][O_{\beta^*}] \cap \alpha^* = N \cap \alpha^* \). Hence \( \lambda \in (S^2)^{V[G\kappa]} \), we have \( q^*[\beta^*] \models [V[G\kappa]]\langle p^*_k(\beta^*) \mid k < \lambda \rangle \cup \lambda] \models s^*_\beta \cup \lambda] \in Q_{\beta^*} \).

Then for all \( k < \lambda \), we have \( q^*[\beta^* + 1] = p^*_k[\beta^* + 1] \). Since \( p^*_k \) is below \( \alpha^* \), we have that for any \( \beta^* \in N[G\kappa] \cap \alpha^* = N \cap \alpha^* \), there exists \( k < \lambda \) such that \( p^*_k[\beta^*] \models [V[G\kappa]]u p^*_k(\beta^*) \neq \emptyset \).

Hence \( \supp(q^*) = N[G\kappa] \cap \alpha^* = N \cap \alpha^* \) holds.

Notice that we did not make use of \( \langle \pi(p^*_k) \mid k < \lambda \rangle \in V[G\lambda] \) in the above.

**Notation.** Let \( N \in \mathcal{N} \). Let \( \pi : N[G\kappa] \rightarrow M[G\lambda] \) be the transitive collapse. The images of ordinals \( \alpha^* \), preorders \( P^* \) and \( P^* \)-names \( \dot{Q}^* \) etc under \( \pi \) will be denoted as \( \alpha = \pi(\alpha^*), P = \pi(P^*) \) and \( \dot{Q} = \pi(\dot{Q}^*) \).

We prove the following two lemmas later in this section. We assume these two for the rest of this section to finish our proof of theorem.

**Lemma.** (Successor) Let \( \mathcal{I} = \langle P^*_\beta \mid \beta^* \leq \alpha^* + 1 \rangle \) be our iteration. If \( \langle P^*_\beta \mid \beta^* \leq \alpha^* \rangle \) is wonderful, then so is \( \mathcal{I} \).

**Lemma.** (Limit) Let \( \mathcal{I} = \langle P^*_\gamma \mid \gamma^* \leq \alpha^* \rangle \) be our iteration with limit \( \alpha^* \). If for all \( \gamma^* < \alpha^* \), \( \langle P^*_\gamma \mid \beta^* \leq \gamma^* \rangle \) are wonderful, then so is \( \mathcal{I} \).

**Corollary.** Every our iteration \( \mathcal{I} \) is wonderful.

**Proof.** Since \( P^*_0 = \{ \emptyset \} \), it is trivial that \( \langle P^*_0 \rangle \) is wonderful. Hence by recursion we may conclude \( \mathcal{I} \) is wonderful.
Assuming that we have done with these two, we may finish our proof.

Proof of theorem. We argue in two cases.

Case 1. There exist our iteration $\mathcal{I} = \langle P^*_\beta, \beta^* \leq \alpha^* \rangle$ and $p \in P^*_\alpha$ such that $p \not\Vdash_{\mathcal{I}} \text{"FRP}(\omega_2)$ holds. Now think of doing trivial iteration to satisfy the statement of the theorem. Hence we are done.

Case 2. For any our iteration $\mathcal{I} = \langle P^*_\beta, \beta^* \leq \alpha^* \rangle$, we have $1 \not\Vdash_{\mathcal{I}} \text{"FRP}(\omega_2)$ fails:

In this case, recall we have a fixed map $\Phi = \langle P \mapsto \langle C^\delta_p \mid \delta \in \dot{E}^P \rangle \mid P \in \{H_{\kappa+}^+\}^{\mathbb{G}_{\kappa}} \rangle$ is a relevant reasonable preorder, where $1 \not\Vdash_{\mathcal{I}} \text{"the ladder system} \langle C^\delta_p \mid \delta \in \dot{E}^P \rangle$ is non-reflecting.

Now we begin to construct a $< \kappa$-support iterated forcing $\langle P^*_\alpha, \alpha^* \leq (\kappa^+) \rangle^{\mathbb{G}_{\kappa}}$ by recursion on $\alpha^*$. Suppose $\alpha^* < \kappa^+$ and that we have constructed $\mathcal{I} = \langle P^*_\beta, \beta^* \leq \alpha^* \rangle$ which is our iteration. Since the last preorder $P^*_\alpha$ is reasonable, we may fix an enumeration of names $\langle (C^\alpha_{\alpha^*} \mid \delta \in \dot{E}^{\alpha_{\alpha^*}}) \mid \alpha^* < \kappa^+ \rangle$ of the ladder systems in $V[\mathbb{G}_{\kappa}]^{P^*_\alpha}$ in addition to the fixed enumeration of suitable names of the ladder systems $\langle (C^\alpha_{\alpha_1}, \alpha_2) \mapsto \langle \dot{C}_{\delta}^{\alpha_1\alpha_2} \mid \delta \in \dot{E}^{\alpha_{\alpha^*}} \rangle \mid \alpha_1 < \alpha^* < \alpha^* \rangle$ in $V[\mathbb{G}_{\kappa}]^{P^*_\alpha}$ with $\alpha_1 < \alpha^*$.

It suffices to specify a non-reflecting ladder system $\langle C^\ast_{\alpha, \delta} \mid \delta \in \dot{E}^{\ast}_{\alpha^*} \rangle$ in $V[\mathbb{G}_{\kappa}]^{P^*_\alpha} = V[\mathbb{G}_{\kappa}][O_{\alpha \ast}]$ as follows:

Let $h(\alpha^*) = (\alpha_1, \alpha_2)$. Hence $\alpha_1 \leq \alpha^*$ and $\alpha_2 < \kappa^+$. Take a look at $\langle C^\alpha_{\alpha^*} \mid \delta \in \dot{E}^{\alpha_{\alpha^*}} \rangle$ in the current universe $V[\mathbb{G}_{\kappa}][O_{\alpha \ast}]$. If $\langle C^\alpha_{\alpha^*} \mid \delta \in \dot{E}^{\alpha_{\alpha^*}} \rangle$ happens to be a non-reflecting ladder system in $V[\mathbb{G}_{\kappa}][O_{\alpha \ast}]$, then let $C^\ast_{\alpha, \delta} = \langle C^\alpha_{\alpha^*} \mid \delta \in \dot{E}^{\alpha_{\alpha^*}} \rangle$. If $\langle C^\alpha_{\alpha^*} \mid \delta \in \dot{E}^{\alpha_{\alpha^*}} \rangle$ does not happen to be non-reflecting in $V[\mathbb{G}_{\kappa}][O_{\alpha \ast}]$, then we switch $\Phi(P^*_\alpha)$ and let $\langle C^\ast_{\alpha, \delta} \mid \delta \in \dot{E}^{\ast}_{\alpha^*} \rangle = \Phi(P^*_\alpha)$. In either case this specifies a non-reflecting ladder system $\langle C^\ast_{\alpha, \delta} \mid \delta \in \dot{E}^{\ast}_{\alpha^*} \rangle$.

Claim. $1 \not\Vdash_{\mathcal{I}} \text{"FRP}(\omega_2)$ holds.

Proof. Let $O_{\alpha^*}$ be any $P^*_\alpha$-generic filter over $V[\mathbb{G}_{\kappa}]$. Let us suppose on the contrary that $\langle C^\alpha_{\alpha^*} \mid \delta \in \dot{E} \rangle$ were a non-reflecting ladder system in $V[\mathbb{G}_{\kappa}][O_{\alpha^*}^{\kappa}]$. Since $P^*_\alpha$ has the $\kappa^+$-c.c., we have $\alpha_1 < \kappa^+$ such that $\langle C^\alpha_{\alpha^*} \mid \delta \in \dot{E}^{\alpha_{\alpha^*}} \rangle$ happens to be a non-reflecting ladder system in $V[\mathbb{G}_{\kappa}][O_{\alpha \ast}]$. Then let $C^\ast_{\alpha, \delta} = \langle C^\alpha_{\alpha^*} \mid \delta \in \dot{E}^{\alpha_{\alpha^*}} \rangle$. If $\langle C^\alpha_{\alpha^*} \mid \delta \in \dot{E}^{\alpha_{\alpha^*}} \rangle$ does not happen to be non-reflecting in $V[\mathbb{G}_{\kappa}][O_{\alpha \ast}]$, then we switch $\Phi(P^*_\alpha)$ and let $\langle C^\ast_{\alpha, \delta} \mid \delta \in \dot{E}^{\ast}_{\alpha^*} \rangle = \Phi(P^*_\alpha)$. In either case this specifies a non-reflecting ladder system $\langle C^\ast_{\alpha, \delta} \mid \delta \in \dot{E}^{\ast}_{\alpha^*} \rangle$.

$\square$

§3. Proof part one.

Proof of lemma (Successor) We have our iteration $\langle P^*_\beta, \beta^* \leq \alpha^* + 1 \rangle$ such that $\langle P^*_\beta, \beta^* \leq \alpha^* \rangle$ is wonderful. We want to show that $\langle P^*_\beta, \beta^* \leq \alpha^* + 1 \rangle$ is wonderful.

Now let $N \in N$ with $P^*_\beta \in N[G_{\kappa}]$. Let $p^* \in P^*_\beta \cap N[G_{\kappa}]$. We want a $P^*_\beta \in N[G_{\kappa}]$-generic sequence $\langle q^*_k \mid k < \lambda \rangle \in V[G_{\lambda}]$ below $p^*$ such that $\langle \pi(p^*_k) \mid k < \lambda \rangle \in V[G_{\lambda}]$. Since $P^*_\beta \in N[G_{\kappa}]$ and $P^*_\beta \in N[G_{\kappa}] \cap \alpha^* \in P^*_\alpha$, we have a $P^*_\beta \in N[G_{\kappa}]$-generic sequence $\langle q^*_k \mid k < \lambda \rangle \in V[G_{\lambda}]$ below $p^* \in P^*_\alpha$ such that $\langle \pi(q^*_k) \mid k < \lambda \rangle \in V[G_{\lambda}]$.

We denote $p = \pi(p^*)$, $q_k = \pi(q^*_k)$, $\alpha = \pi(\alpha^*)$, $\beta = \pi(\beta^*) \leq \alpha^* + 1$ and $\langle q_k \mid k < \lambda \rangle = \{ q_k \in P_\alpha \mid 3k y \geq q_k \in P_\alpha \}$. Then in $V[G_{\lambda}]$, it is routine to show that $\langle q_k \mid k < \lambda \rangle$ is a $P_\alpha$, $M[G_{\lambda}]$-generic sequence and so $\langle q_k \mid k < \lambda \rangle \in V[G_{\lambda}]$ is a $P_\alpha$-generic filter over $M[G_{\lambda}]$ with $p$ in it. We have seen that there exists $q^*_k \in P^*_\beta$ below the $q^*_k$'s. Hence $q^*_k$ is $\langle P^*_\beta, N[G_{\kappa}] \rangle$-generic.

Let $O_{\alpha^*}$ be any $P^*_\alpha$-generic filter over $V[\mathbb{G}_{\kappa}]$ with $q^*_k \in O_{\alpha^*}$. Let $Q^*_k$ be the interpretation of $Q^*_k$ by $O_{\alpha^*}$. Let $\langle C^\delta_{\delta^*} \mid \delta \in \dot{E}^{\alpha_{\alpha^*}} \rangle$ be (omitting $\alpha^*$ and *') the interpretation of $\langle C^\alpha_{\alpha^*} \mid \delta \in \dot{E}^{\alpha_{\alpha^*}} \rangle$. Then $\langle C^\delta_{\delta^*} \mid \delta \in \dot{E}^{\alpha_{\alpha^*}} \rangle$ is a non-reflecting ladder system and the associated $Q^*_k$ shoots a club off $E$ over $V[G_{\kappa}][O_{\alpha^*}]$. Note we have $Q^*_k \in N[G_{\kappa}][O_{\alpha^*}]$ and $\langle C^\delta_{\delta^*} \mid \delta \in \dot{E} \rangle \in N[G_{\kappa}][O_{\alpha^*}]$.

Then in the generic extension $V[G_{\kappa}][O_{\alpha^*}]$, the collapse $\pi : N[G_{\kappa}] \longrightarrow M[G_{\lambda}]$ gets extended to $\pi : N[G_{\kappa}][O_{\alpha^*}] \longrightarrow M[G_{\lambda}][\langle q_k \mid k < \lambda \rangle]$. This is because $\{ \pi(x) \mid x \in O_{\alpha^*} \cap N[G_{\kappa}] \} = \{ \pi(x) \mid x \in N[G_{\kappa}], \exists k < \lambda x \geq q^*_k \text{ in } P^*_\alpha \} = \{ y \in P_\alpha \mid \exists k < \lambda y \geq q^*_k \text{ in } P_\alpha \}$.
We denote $\langle Q_{\beta} \mid \beta \leq \alpha \rangle = \pi(\langle Q_{\beta}^* \mid \beta^* \leq \alpha^* \rangle)$. Let $Q_{\alpha}$ be the interpretation of $Q_{\alpha}$ by $\langle q_k \mid k < \lambda \rangle$. Then we have $\langle C_{\delta} \mid \delta \in E \cap \lambda \rangle = \pi(\langle C_{\delta} \mid \delta \in E \rangle)$ and $Q_{\alpha} = \pi(Q_{\alpha}^*)$. Hence $\langle C_{\delta} \mid \delta \in E \cap \lambda \rangle \in M[G_{\lambda}] [\{q_k \mid k < \lambda \}]$ is a non-reflecting ladder system and $Q_{\alpha}$ is the associated p.o. set shooting a club off $E \cap \lambda$ over $M[G_{\lambda}] [\{q_k \mid k < \lambda \}]$.

**Claim.** $E \cap \lambda$ is not stationary in $V[G_{\lambda}]$.

**Proof.** Suppose not. Then $\langle C_{\delta} \mid \delta \in E \cap \lambda \rangle$ is a ladder system in the intermediate $V[G_{\lambda}]$. Hence it gets a filtration $\{X_i \mid i < \omega_1 \}$ on $\lambda$ in $V[G_{\lambda}+1]$. Then due to this filtration the original ladder system $\langle C_{\delta} \mid \delta \in E \rangle$ gets reflected in $V[G_{\alpha}][O_{\alpha}]$. This would be a contradiction.

Let $C \in V[G_{\lambda}]$ be a closed cofinal subset of $\lambda$ such that $C \cap (E \cap \lambda) = C \cap E = \emptyset$. By making use of this $C$, we construct a $(P_{\alpha+1}, M[G_{\lambda}])$-generic sequence $(l \mapsto q_{k_{l}}(\tau_{l}) | l < \lambda)$ below $p \in P_{\alpha+1}$ in the intermediate $V[G_{\lambda}]$. We first see that this suffices. Let $p_{l}^* \in P_{\alpha+1}$ be the preimage of $q_{k_{l}}(\tau_{l}) \in P_{\alpha+1}$ under $\pi: N[G_{\alpha}] \rightarrow M[G_{\lambda}]$. Then it is routine to show that this $(p_{l}^* \mid l < \lambda) \in V[G_{\alpha}]$ is a $(P_{\alpha+1}, \lambda)$-generic sequence below $p^*$.

Now we begin to construct $q_{k_{l}}(\tau_{l}) \in V[G_{\lambda}]$. Let $(D_l \mid l < \lambda)$ enumerate the dense open subsets $D$ of $P_{\alpha+1}$ with $D \in M[G_{\lambda}]$. The crucial fact is that $V[G_{\lambda}] \cap (\kappa M[G_{\lambda}] \subseteq M[G_{\lambda}]$. This means that the initial segments constructed are all in $M[G_{\lambda}]$. Hence we may make use of the initial segments as sequences of conditions in $M[G_{\lambda}]$ and so may give rise to conditions in $P_{\alpha+1} \in M[G_{\lambda}]$.

$I = 0$: Since $q_0 \leq p[\alpha]$ in $P_{\alpha}$, let $\tau_0 = p(\alpha)$. Then $q_{0}^{-}(\tau_{0}) \leq p$ in $P_{\alpha+1}$. Let $k_0 = 0$.

$I \to I + 1$: Suppose we have constructed $q_{k_{l}}^{-}(\tau_{l}) \in P_{\alpha+1}$. Pick $q_{k''} \leq q_{k_{l}}$ so that $q_{k''}$ decides the value of $\tau_{l}$ to be $s$. This is possible as $P_{\alpha}$ is $\lambda$-Baire in $M[G_{\lambda}]$ and the $q_{k}$'s form a $(P_{\alpha}, M[G_{\lambda}])$-generic sequence. Pick $e \in C$ with $\sup(s) < e < \lambda$. Then $q_{k''}^{-}(s \cup \{e\}) \in P_{\alpha+1}$. Since $\{a \in P_{\alpha} \mid a \leq x[\alpha]$ for some $x \in D_{l}$ with $x \leq q_{k''}^{-}(s \cup \{e\})$ or $a$ is incompatible with $q_{k''}$ in $P_{\alpha}$) is dense open subset of $P_{\alpha}$ and belongs to $M[G_{\lambda}]$, we may pick $q_{k_{l+1}}^{-}(\tau_{l+1}) \in D_{l}$ such that $q_{k_{l+1}}^{-}(\tau_{l+1}) \leq q_{k''}^{-}(s \cup \{e\}) \leq q_{k''}^{-}(\tau_{l})$ in $P_{\alpha+1}$.

(Limit $I$): Suppose we have constructed $q_{k_{l}}^{-}(\tau_{l}) \mid l' < l \rangle$. Pick $q_{k_{l}}$ so that for all $l' < l$, we have $q_{k_{l}} \leq q_{k_{l'}}$. Then $q_{k_{l}}$ decides the value of $\sup(\{\tau_{l} \mid l' < l\})$ to be some limit $e' < \lambda$. Then $e' \in C$ and so $e' \notin E \cap \lambda$. Remember $E \cap \lambda$ is the relevant non-reflecting ladder system here in $M[G_{\lambda}] [\{q_k \mid k < \lambda \}]$. Hence we may further assume $q_{k_{l}}^{-}(\cup(\{\tau_{l} \mid l' < l\}) \cup \{e\}) \in P_{\alpha+1}$. Let $q_{k_{l}^{-}} M[G_{\lambda}][\tau_{l}] = (\cup(\{\tau_{l} \mid l' < l\}) \cup \{e\}).$

Then for all $l' < l$, we have $q_{k_{l}}^{-}(\tau_{l}) \leq q_{k_{l}}^{-}(\tau_{l})$. This completes the construction.

**§4. Proof part two**

Proof of lemma (Limit). Let $(P_{\alpha}^*, \mid \beta^* \leq \alpha^*)$ be our iteration such that $\alpha^*$ is limit and for all $\gamma^* < \alpha^*$, we assume that $(P_{\gamma}^*, \mid \beta^* \leq \gamma^*)$ are wonderful. We want to show that $(P_{\gamma}^*, \mid \beta^* \leq \gamma^*)$ is wonderful. We have seen that $P_{\alpha}^* \in (H_{\alpha+1} V[G_{\alpha}]$. Let $N \in \mathcal{N}$ such that $P_{\alpha}^* \in N[G_{\alpha}]$. Let $p^* \in P_{\alpha}^* \cap N[G_{\alpha}]$.

We denote $p = \pi(p^*)$, $\alpha = \pi(\alpha^*)$, $(P_{\beta, \alpha} \mid \beta \leq \alpha) = \pi((P_{\beta} \mid \beta^* \leq \alpha^*))$, $(Q_{\beta} \mid \beta < \alpha) = \pi((Q_{\beta}^* \mid \beta^* \leq \alpha^*))$, $\langle \langle C_{\delta} \mid \delta \in E_{\beta} \rangle \mid \beta < \alpha \rangle = \pi(\langle \langle C_{\delta} \mid \delta \in E_{\beta} \rangle \mid \beta^* \leq \alpha^* \rangle)$. We want a $(P_{\alpha}, M[G_{\lambda}])$-generic sequence in $V[G_{\lambda}]$.

For the rest of this section, we argue in the intermediate $V[G_{\lambda}]$. Recall that $(\omega_1)^{V[G_{\lambda}]} = \omega_1^V$ and $(\omega_2)^{V[G_{\lambda}]} = \lambda$.

**Claim.** We have $\diamond(S_{\beta}^*)$ in $V[G_{\lambda}]$.

**Proof.** Suppose that $A = (\dot{A})_{G_{\lambda}} \subseteq \lambda$ and $(\dot{G})_{G_{\lambda}}$ is a club in $\lambda$. In $V$, we may represent $\dot{A}$ as $\langle A_{\alpha} \mid \alpha < \lambda \rangle$ such that $A_{\alpha}$ is an anti-chain in $L(V, \omega_2)$ and so $|A_{\alpha}| < \lambda$. We assume $\alpha \in A$ iff $A_{\alpha} \cap G_{\lambda} \neq \emptyset$.
In $V$, let $C = \{ \xi < \lambda \mid \forall \alpha < \xi A_\alpha \subset \text{L}v(\xi, \omega_1) \}$. Then this $C$ is a club. Now in $V[G_\lambda]$, pick $\xi \in (\bar{C})G_\lambda \cap C$ with $\text{cf}(\xi) = \omega_1$. Then $A \cap \xi \in \mathcal{P}(\xi) \cap V[G_\xi]$ and $|\mathcal{P}(\xi) \cap V[G_\xi]| \leq \omega_1$. Hence $(\mathcal{P}(\xi) \cap V[G_\xi]| \xi \in S^2_1)$ is a $\diamond(S^2_1)$-sequence.

In view of $|M[G_\lambda]| = \lambda$ and $P_\alpha \cup \{ P_\alpha \} \subset M[G_\lambda]$, we may fix $(i \mapsto ((q_{ij} \mid j < i), D(i)) \mid i \in S^2_1)$ such that

- $(q_{ij} \mid j < i)$ is a descending sequence of elements of $P_\alpha$.
- $D(i) \subseteq P_\xi$, for some $\xi \leq \alpha$ and $D(i) \in M[G_\lambda]$.
- For any descending sequence $(p_i \mid i < \lambda)$ of elements of $P_\alpha$ and any $D \subseteq P_\xi$ for some $\xi \leq \alpha$ with $D \in M[G_\lambda]$, the following
  \[ \{ i \in S^2_1 \mid (q_{ij} \mid j < i) = (p_j \mid j < i) \text{ and } D(i) = D \} \]

is stationary.

We make use of this form of guessing to construct a $(P_\alpha, M[G_\lambda])$-generic sequence below $p$. We first take the greatest lower bound of $(q_{ij} \mid j < i)$ as much as possible (i.e. $q^0_\alpha$). Hence sort of $q^0_\alpha \equiv (q_{ij}[\alpha(i) \mid j < i)$ and no more. Then we hit the possible dense open subset $D(i)$ below the lower bound in advance (i.e. $q^1_\alpha$). Hence $q^1_\alpha \leq q^0_\alpha$ in $P_\alpha(i)$ and if $D(i)$ is dense open in $P_\xi$, with some $\xi \leq \alpha(i)$, then $q^1_{\alpha(i)}|\xi \subseteq D(i)$. Therefore as long as guessing succeed, we would have taken care of every relevant dense open subset. This way we cover shortages of steps compared to the number of relevent dense open subsets (i.e. $\omega, \omega_1$ vs. $\omega_2$).

**Definition.** We associate $(i \mapsto (q^0_i, q^1_i, \alpha(i)) \mid i \in S^2_1)$ such that

- $\alpha(i) \leq \alpha$ and $q^0_i, q^1_i \in P_\alpha(i)$.
- For any $j < i$ and any $\eta < \alpha(i)$, we have $q^0_\alpha[\eta \leq q_{ij}[\eta$ in $P_\eta$ and $q^0_\alpha[\eta$ forces (over $M[G_\lambda]$) the following;
  \[ q^0_\alpha(\eta) = \overline{\{q_{ij}(\eta) \mid j < i\}}, \]

where $\overline{s}$ denotes the closure of $s$. Therefore, $q^0_\alpha[\eta$ forces the disjunction of the following (1) or (2);

(1) $\exists j < i \ q_{ji}(\eta) \neq \emptyset$ and $\sup(\bigcup\{q_{ij}(\eta) \mid j < i\}) \notin E_\eta$ and

\[ q^0_\alpha(\eta) = (\bigcup\{q_{ij}(\eta) \mid j < i\}) \cup \{ \sup(\bigcup\{q_{ij}(\eta) \mid j < i\}) \}. \]

(2) $\forall j < i \ q_{ij}(\eta) = \emptyset$ and

\[ q^0_\alpha(\eta) = \emptyset. \]

- If $\alpha(i) < \alpha$, then $q^0_\alpha$ fails to force the disjunction of (1) or (2) as above.
- If $D(i)$ is a dense open subset of $P_\xi$, with some $\xi \leq \alpha(i)$, then $q^1_{\alpha(i)}[\xi \subseteq D(i)$ and $q^1_{\alpha(i)} \leq q^0_\alpha$ in $P_\alpha(i)$. Otherwise, $q^1_{\alpha(i)} = q^0_\alpha$.

Note that we have $(q_{ij} \mid j < i) \in M[G_\lambda]$ and $\sup(q^0_\alpha) \subseteq \bigcup\{\sup(q_{ij}) \mid j < i\}$ and so of size $< \lambda$.

**Definition.** Let $\phi(\xi, (p_i \mid i < \lambda), C, a)$ stands for the following;

- $\xi \leq \alpha$.
- $(p_i \mid i < \lambda)$ is a descending sequence of elements in $P_\xi$ below $a \in P_\xi$ and $C$ is a club in $\lambda$.
- For any $i \in C \cap S^2_1$ and any $\eta < \xi$, $p_i[\eta$ forces (over $M[G_\lambda]$) the following;

\[ p_i(\eta) = \overline{\{p_j(\eta) \mid j < i\}}. \]
For any $i \in C \cap S_1^2$, if $\langle p_j \mid j < i \rangle \equiv \langle q_{ij} \mid j < i \rangle$, then we have

$$p_{i+1} \leq q_i^0 \lceil \xi$$

in $P_{\xi}$,

where $\langle p_j \mid j < i \rangle \equiv \langle q_{ij} \mid j < i \rangle$ means $\forall j < i \exists j' < i \ q_{ij'} \lceil \xi \leq p_j$ in $P_{\xi}$ and conversely $\forall j < i \exists j' < i \ p_{j'} \leq q_{ij} \lceil \xi$ in $P_{\xi}$. Hence these two sequences are not required to be literally equal but share the same strength.

We may abbreviate the third condition in the above as $p_i \equiv \langle p_j \mid j < i \rangle$.

**Proposition.** If $\phi(\xi, (p_i \mid i < \lambda), C, w)$, $i \in C \cap S_1^2$ and $\langle p_j \mid j < i \rangle \equiv \langle q_{ij} \mid j < i \rangle$, then $\xi \leq \alpha(i)$ and $p_i \equiv q_i^0 \lceil \xi$.

**Proof.** It is routine to show $p_i[\eta] \equiv q_i^0[\eta]$ by induction on $\eta \leq \xi$.

Note that we did not make use of the 4th condition of $\phi(\xi, (p_i \mid i < \lambda), C, w)$ in the proof. And by this proposition, the 4th condition makes sense.

**Proposition.** If $\phi(\xi, (p_i \mid i < \lambda), C, w)$, then $\langle p_i \mid i < \lambda \rangle$ is a $(P_{\xi}, M[G_{\lambda}])$-generic sequence below $w$.

**Proof.** Let $D$ be any dense open subset of $P_{\xi}$ with $D \in M[G_{\lambda}]$. By assumption on $\langle\langle q_{ij} \mid j < i \rangle, D(i) \rangle \mid i \in S_1^2\rangle$, we may pick $i \in C \cap S_1^2$ such that $D = D(i)$ and $\langle p_{j'}^2 \mid j < i \rangle = \langle q_{ij} \mid j < i \rangle$. Hence $\langle p_j \mid j < i \rangle \equiv \langle q_{ij} \mid j < i \rangle$ and $D(i)$ is dense open in $P_{\xi}$. Hence $p_{i+1} \leq q_i^0 \lceil \xi$ and $q_i^0 \lceil \xi \in D(i)$. Hence $p_{i+1} \in D(i) = D$.

**Definition.** Let $\phi(\eta, (p_i^0 \mid i < \lambda), C^\eta, a)$ and $\phi(\xi, (p_i^\xi \mid i < \lambda), C^\xi, b)$. We write

$$(\eta, (p_i^0 \mid i < \lambda), C^\eta, a) R (\xi, (p_i^\xi \mid i < \lambda), C^\xi, b),$$

if

- $\eta < \xi, C^\eta \supseteq C^\xi$ and $a = b[\eta]$.
- $\forall i < \lambda \exists j \geq i \ p_i^\eta \lceil \eta = p_j^\eta$.
- There exists a club $C_{\eta\xi}$ in $\lambda$ such that
  1. $C_{\eta\xi} \subseteq C^\eta \cap C^\xi$.
  2. $\forall i \in C_{\eta\xi} \cap S_1^2 \ p_i^\eta = p_i^\xi[\eta]$.

**Proposition.** $R$ is transitive.

**Proof.** $(\eta_1, (p_i^1 \mid i < \lambda), C_{1, a_1}) R (\eta_2, (p_i^2 \mid i < \lambda), C_{2, a_2}) R (\eta_3, (p_i^3 \mid i < \lambda), C_{3, a_3})$ implies $(\eta_1, (p_i^4 \mid i < \lambda), C_{4, a_4})$.

Work in $V[G_{\lambda}]$. By induction on $\xi \leq \alpha = \pi(\alpha^*)$, we show the following IH($\xi$):

$\forall \eta < \xi \forall (p_i^\eta \mid i < \lambda) \forall C^\eta \forall w \in P_{\xi}$, if $\phi(\eta, (p_i^\eta \mid i < \lambda), C^\eta, w[\eta])$, then there exists $(p_i^\xi \mid i < \lambda), C^\xi$ such that

- $\phi(\xi, (p_i^\xi \mid i < \lambda), C^\xi, w)$.
- $(\eta, (p_i^\eta \mid i < \lambda), C^\eta, w[\eta]) R (\xi, (p_i^\xi \mid i < \lambda), C^\xi, w)$.

In particular, let $\eta = 0, \xi = \alpha$ and $w = p = \pi(p^*) \in P_{\alpha}$. Since $\phi(0, (\emptyset \mid i < \lambda), \lambda, w[0])$ holds, we have $(p_i^\eta \mid i < \lambda), C^\alpha)$ such that $\phi(\alpha, (p_i^{\alpha} \mid i < \lambda), C^\alpha, p)$. Hence $(p_i^\eta \mid i < \lambda) \in V[G_{\lambda}]$ is a $(P_{\alpha}, M[G_{\lambda}])$-generic sequence below $p$. This completes the proof of lemma (Limit).
§ 5. Proof part three

Proof of IH(ξ) by induction.

IH(0): IH(0) is vacuously true.

We have two remaining cases.

IH(ξ) implies IH(ξ + 1): Since R is transitive, we may assume that η = ξ. Suppose \( \phi(ξ, (p^ξ_i | i < \lambda), C^ξ, w[ξ]) \) and \( w \in P_{ξ+1} \). We want \( (p^ξ_{i+1} | i < \lambda) \) and \( C^{ξ+1} \) such that \( \phi(ξ + 1, (p^ξ_{i+1} | i < \lambda), C^{ξ+1}, w) \) and \( \langle\xi, (p^ξ_i | i < \lambda), C^ξ, w[ξ]\rangle R (ξ + 1, (p^ξ_{i+1} | i < \lambda), C^{ξ+1}, w) \).

Remember that we have the transitive collapse \( \pi : N[G_\kappa] \rightarrow M[G_\lambda] \). Let \( \pi(ξ^* ) = \xi \) and \( \pi(p^ξ_i) = p^\kappa_i \) for each \( i < \lambda \). Hence \( \pi(P^ξ_\kappa) = P^\kappa_\xi \). Since \( (p^ξ_i | i < \lambda) \) is a \( (P^ξ_\kappa, M[G_\lambda]) \)-generic sequence, its pointwise preimages \( (p^\kappa_{i'} | i' < \lambda) \in V[G_\kappa] \) is a \( (P^\kappa_\xi, N[G_\kappa]) \)-generic sequence with \( cf(\lambda) = \omega_1 \) in \( V[G_\kappa] \). We know that there exists a lower bound \( q' \in P^\kappa_\xi \) of the \( p^\kappa_i \)'s. This \( q' \) is \( (P^\kappa_\xi, N[G_\lambda]) \)-generic.

Let \( O_\xi \) be \( P^\kappa_\xi \)-generic over \( V[G_\kappa] \) with \( q' \in O_\xi \cdot \). Then in the generic extension \( V[G_\kappa][O_\xi \cdot] \), we have the extension \( \pi : N[G_\kappa][O_\xi \cdot] \rightarrow M[G_\lambda]([p^\kappa_i | i < \lambda]) \).

Let \( \langle C_\delta | \delta \in E \rangle \) be the interpretation of \( \langle C^*_\delta \eta = \delta \in E^*_\eta \rangle \) by \( O_\xi \cdot \). Then \( (C_\delta | \delta \in E) \in N[G_\kappa][O_\xi \cdot] \) and
\[
\pi((C_\delta | \delta \in E)) = (C_\delta | \delta \in E \cap \lambda) \Rightarrow (C_\delta | \delta \in E \cap \lambda) \in V[G_\lambda].
\]
This is because if \( E \cap \lambda \) were stationary in \( V[G_\lambda] \). Then \( (C_\delta | \delta \in E \cap \lambda) \) gets a filtration on \( \lambda \) which reflets \( (C_\delta | \delta \in E \cap \lambda) \) in \( V[G_{\lambda+1}] \). This filtration remains up \( V[G_\kappa] \) and further up \( V[G_\kappa][O_\xi \cdot] \). This contradicts that \( (C_\delta | \delta \in E) \) is non-reflecting in this last \( V[G_\kappa][O_\xi \cdot] \).

Since \( E \cap \lambda \) is not stationary in \( V[G_\lambda] \), we may pick a club \( C \in V[G_\lambda] \) such that \( C \cap (E \cap \lambda) = \emptyset \). Let \( \dot{E}_\xi = \pi(\dot{E}_\xi^*) \). Then this \( \dot{E}_\xi \) is a \( P_\kappa \)-name in \( M[G_\lambda] \) such that \( E \cap \lambda \) is the interpretation of \( \dot{E}_\xi \) by \( (p^\kappa_i | i < \lambda) \).

We work in \( V[G_\lambda] \). The crucial point was \( V[G_\lambda] \cap \langle C^\lambda \rangle \subset M[G_\lambda] \) and \( P_{\lambda+1} \in M[G_\lambda] \). We construct \( p^\kappa_i \mapsto(\tau_k) | k < \lambda \) by recursion on \( k < \lambda \).

Case \( (k = 0) \): Let \( p^\kappa_i \mapsto(\tau_0) \leq w \in P_\kappa \).

Case \( (k < k + 1) \): Suppose we have constructed \( p^\kappa_i \mapsto(\tau_k) \in P_{k+1} \). Want \( p^\kappa_{k+1} \mapsto(\tau_{k+1}) \in P_{k+1} \).

Subcase 1. \( k \) is either 0 or successor: Pick a large \( i_{k+1} < \lambda \) and \( \tau_{k+1} \) such that \( p^\kappa_{i_{k+1}} \mapsto(\tau_{k+1}) \leq M[G_\lambda]\langle(\tau_k) < w \leq(\tau_{k+1}) \rangle \) for some \( w \in C \).

Subcase 2. \( k \) is limit: We have two cases.

Subsubcase 2.1. \( i < k \in C^\kappa \cap S^\kappa_\lambda \) and \( p^\kappa_i \mapsto(\tau_k') \mapsto(\tau_k') | k' < k \equiv q_{kk'}[(\xi + 1) | k' < k) \) Then we have \( \xi + 1 \leq \alpha(k) \) and \( p^\kappa_k \equiv q_k^{\xi}(\xi) \). By subcase 2 below, we have \( p^\kappa_k \mapsto M[G_\lambda]^\alpha(\tau_k) \equiv (\bigcup\{\tau_{k'} | k' < k\}) \cup \{\sup(\bigcup\{\tau_{k'} | k' < k\})\} = q_k^{\xi}(\xi) \) and \( q_k^{\xi} \leq q_k^{\xi} \) in \( P(k) \) and \( p^\kappa_{k+1} \leq q_k^{\xi}(\xi) \) holds. Let us take \( \tau_{k+1} = q_k^{\xi}(\xi) \).

Then \( p^\kappa_{k+1} \mapsto(\tau_{k+1}) \leq q_k^{\xi}(\xi + 1) \) holds. Let \( i_{k+1} = k + 1 \). Hence \( p^\kappa_{i_{k+1}} \mapsto(\tau_{k+1}) = p^\kappa_{k+1} \mapsto(\tau_{k+1}) \).

Subsubcase 2.2. Otherwise: Take \( p^\kappa_{i_{k+1}} \mapsto(\tau_{k+1}) \leq p^\kappa_{i_{k+1}} \mapsto(\tau_k) \) in Subcase 1.

Case \( (k \) is limit). We have constructed \( p^\kappa_i \mapsto(\tau_k) \) for all \( k' < k \). We want \( p^\kappa_i \mapsto(\tau_k) \).

Subcase 1. \( cf(k) = \omega \): Pick \( i_k < \lambda \) so that for all \( k' < k, i_k < i_k \) Then for all \( k' < k \), we have \( p^\kappa_k \leq p^\kappa_{i_k} \).

Since \( E \cap \lambda \equiv \{\nu < \lambda | \exists \nu \in P_\kappa \mapsto M[G_\lambda]^{\nu}(\tau_k) \} \), we may assume that \( p^\kappa_k \mapsto M[G_\lambda]^{sup(\bigcup\{\tau_k | k' < k\})} \in \dot{E}_\xi \). Hence we may pick \( \tau_k \) so that \( p^\kappa_k \mapsto M[G_\lambda]^{\nu}(\tau_k) = (\bigcup\{\tau_k | k' < k\}) \cup \{\sup(\bigcup\{\tau_k | k' < k\})\} \in \dot{Q}_\xi \).}

Subcase 2. $\operatorname{cf}(k) = \omega_1$: Let $i_k = \sup\{i_{k'} \mid k' < k\}$. Then for all $k' < k$, we have $p_{i_k}^\xi \leq p_{i_{k'}}^\xi$ and $p_{i_k}^\xi \nleq \sup_{i_{k'}}^\xi (\cup\{\tau_{k'} \mid k' < k\}) \in S_1^{2\xi}$. Hence may take $\tau_k$ to be such that $p_{i_k}^\xi \nleq \sup_{i_{k'}}^\xi (\cup\{\tau_{k'} \mid k' < k\}) \cup (\cup\{\tau_{k} \mid k' < k\}) = \check{Q}_\xi$.

This completes the construction of $(p_{i_k}^\xi \rightarrow \tau_k) \mid k < \lambda$.

Let $C^{\xi+1} = C^{\xi} \cap \{k < \lambda \mid \forall k' < k \ i_{k'} < k\}$. Then this $C^{\xi+1} \in V[G_{\lambda}]$ is a club in $\lambda$.

Claim. If $k \in C^{\xi+1} \cap S_1^\xi$, then $i_k = k$ holds.

Proof. Since $\langle i_{k} \mid k < \lambda \rangle$ is strictly increasing, we have $k \leq i_k$. Since $i_{k'} < k$ for all $k' < k$ and $\operatorname{cf}(k) = \omega_1$, we have $i_k = \sup\{i_{k'} \mid k' < k\} \leq k$. Hence $i_k = k$.

Now for each $k < \lambda$, let us set

$$p_k^{\xi+1} = p_{i_k}^\xi \rightarrow \tau_k.$$  

We want to show $\phi(\xi + 1, (p_k^{\xi+1} | k < \lambda), C^{\xi+1}, w)$ and $(\xi, (p_k^{\xi} | k < \lambda), C^{\xi}, w | \xi) R (\xi + 1, (p_k^{\xi+1} | k < \lambda), C^{\xi+1}, w)$.

By construction we have that $(p_k^{\xi+1} | k < \lambda)$ is descending below $w$ in $P_{\xi+1}$ and that $C^{\xi+1}$ is a club in $\lambda$.

Let $k \in C^{\xi+1} \cap S_1^\xi$. Then we have $k = i_k$. It is routine to check that for any $\eta < 1 + 1$, $p_k^{\xi+1} \mid \eta$ forces the following;

$$p_k^{\xi+1} \mid \eta = \bigcup\{p_k^{\eta+1}(\eta) \mid k' < k\}.$$

(details) Let $\eta < \xi$. Then $p_k^{\xi+1} \mid \eta = p_k^\xi \mid \eta$ which forces the disjunction of (1) or (2);

1. $\exists \eta < k \ p_k^{\xi+1} \mid \eta = p_k^\xi \mid \eta \neq \emptyset$, sup$(\cup\{p_k^{\xi+1} \mid \eta \mid k' < k\}) = \sup(\cup\{p_k^\xi \mid \eta \mid k' < k\}) \notin \check{E}_\eta$ and $p_k^{\xi+1} \mid \eta = p_k^\xi \mid (\cup\{p_k^\xi \mid \eta \mid k' < k\}) \cup \{\sup(\cup\{p_k^\xi \mid \eta \mid k' < k\})\} = (\cup\{p_k^\xi \mid \eta \mid k' < k\}) \cup \{\sup(\cup\{p_k^\xi \mid \eta \mid k' < k\})\}.$

2. $\forall \eta' < k \ p_k^{\xi+1} \mid \eta' = p_k^\xi \mid \eta' = \emptyset$ and $p_k^{\xi+1} \mid \eta = p_k^\xi \mid \eta = \emptyset$.

Next let $\eta = \xi$. Then $p_k^{\xi+1} \mid \xi = p_k^\xi$ which forces the following (1);

1. $\exists \eta' < k \ p_k^{\xi+1} \mid \eta' = \tau_k \neq \emptyset$, sup$(\cup\{p_k^{\xi+1} \mid \eta' \mid k' < k\}) = \sup(\cup\{p_k^\xi \mid \eta' \mid k' < k\}) \notin \check{E}_\eta$ and $p_k^{\xi+1} \mid \eta' = \tau_k = (\cup\{\tau_k \mid k' < k\}) \cup \{\sup(\cup\{\tau_k \mid k' < k\})\} = (\cup\{p_k^\xi \mid \eta' \mid k' < k\}) \cup \{\sup(\cup\{p_k^{\xi+1} \mid \eta' \mid k' < k\})\}.$

Next suppose $k \in C^{\xi+1} \cap S_1^\xi$ and that $(p_k^{\xi+1} \mid \eta \mid k' < k) = (q_{kk}^{\xi+1} \mid (\eta + 1) \mid k' < k)$. Then $i_k = k \in C^{\xi} \cap S_1^\xi$ and $(p_k^\xi \rightarrow \tau_k) \mid k' < k) = (q_{kk} \mid (\xi + 1) \mid k' < k)$. Hence $p_k^{\xi+1} = p_k^{\xi+1} \mid \tau_k \mid k+1 = q_k^\xi \mid (\xi + 1), k + 1 = i_{k+1}$. Therefore we have $\phi(\xi + 1, (p_k^{\xi+1} \mid k < \lambda), C^{\xi+1}, w)$.

Lastly, $(\xi, (p_i^\xi \mid i < \lambda), C^{\xi}, w | \xi) R (\xi + 1, (p_k^{\xi+1} \mid k < \lambda), C^{\xi+1}, w)$ holds.

(details) $\xi \leq \eta + 1, C^{\xi} \supseteq C^{\xi+1}, w | \xi = w | \xi$. 

$\forall k < \lambda \exists i_k \geq k p_{i_k}^\xi = p_k^{\xi+1} \mid \eta$.

Let $C_{\xi+1} = C^{\xi+1}$. Then for $k \in C_{\xi+1} \cap S_1^\xi$, we have $p_{i_k}^\xi = p_{i_k}^\xi \mid \eta$, as $k \in C^{\xi+1} \cap S_1^\xi$ implies $i_k = k$.

This completes IH$(\xi)$ implies IH$(\xi + 1)$.

§6. Proof part four

$\gamma$ limit, $(\forall \xi < \gamma \text{ IH}^*(\xi))$ implies $\text{IH}^*(\gamma)$: We still work in $V[G_{\lambda}]$. Let $\gamma \leq \alpha$ and $\gamma$ be limit. We show that $(\forall \xi < \gamma \text{ IH}^*(\xi))$ implies $\text{IH}^*(\gamma)$. We have two cases according to $\text{cf}(\gamma) = \omega, \omega_1$ and to $\text{cf}(\gamma) = \omega_2$. 

Case. \( \text{cf}(\gamma) = \omega, \omega_1 \): Let \( \eta < \gamma, w \in P_\eta \) and \( \phi(\eta, (p^\gamma_{\eta} | i < \lambda), C^\gamma, w[\eta]) \). We want \( (p^\gamma_{\eta} | i < \lambda) \) and \( C^\gamma \) such that \( \phi(\gamma, (p^\gamma_{\eta} | i < \lambda), C^\gamma, w) \) and \( \phi(\eta, (p^\gamma_{\eta} | i < \lambda), C^\gamma, w[\eta]) R(\gamma, (p^\gamma_{\eta} | i < \lambda), C^\gamma, w) \).

To this end let \( \gamma_k \) be a strictly \( < \)-increasing continuous sequence of ordinals such that \( \gamma_0 = \eta \) and \( \gamma_{\text{cf}(\gamma)} = \gamma \). It suffices construct \( (p^\gamma_{\eta} | i < \lambda) \) and \( C^\gamma \) by recursion on \( k \leq \text{cf}(\gamma) \) such that \( \phi(\gamma_k, (p^\gamma_{\eta} | i < \lambda), C^\gamma, w[\gamma_k]) \) and for all \( l < k \), we have \( (\gamma, (p^\gamma_{\eta} | i < \lambda), C^\gamma, w[\gamma_l]) R(\gamma_k, (p^\gamma_{\eta} | i < \lambda), C^\gamma, w[\gamma_k]) \).

\( k = 0 \): Let \( (p^\gamma_{\eta} | i < \lambda) = (p^\gamma_{\eta} | i < \lambda) \) and \( C^\gamma = G \).

\( k \) to \( k + 1 \): Suppose we have constructed \( (p^\gamma_{\eta} | i < \lambda) \) and \( C^\gamma \) such that \( \phi(\gamma_k, (p^\gamma_{\eta} | i < \lambda), C^\gamma, w[\gamma_k]) \). By IH \( (\gamma_{k+1}) \), we have \( (p^\gamma_{\eta+1} | i < \lambda) \) and \( C^\gamma_{k+1} \) such that \( \phi(\gamma_{k+1}, (p^\gamma_{\eta+1} | i < \lambda), C^\gamma_{k+1}, w[\gamma_{k+1}]) \) and \( (\gamma_k, (p^\gamma_{\eta} | i < \lambda), C^\gamma_{k+1}, w[\gamma_k]) R(\gamma_{k+1}, (p^\gamma_{\eta+1} | i < \lambda), C^\gamma_{k+1}, w[\gamma_{k+1}]) \).

k limit: Let

\[
C^\gamma = \bigcap \{C_{\gamma_m} | m < k \}
\]

and for each \( i \in S^2_\gamma \cap C^\gamma \), let

\[
p^\gamma_i = \bigcup\{p^\gamma_l | l < k \}.
\]

Then \( p^\gamma_i \in P_\gamma \) as \( V[G] \cap C^\gamma \subseteq M[G] \) and \( |\text{supp}(p^\gamma_i)| \leq \omega_1 \).

Let \( f: \lambda \to S^2_\gamma \cap C^\gamma \) be the \( \epsilon \)-isomorphism and let \( C(f) = \{i < \lambda | \forall j < i f(j) < j\} \). Let

\[
C^f = C^\gamma \cap C(f)
\]

and for each \( i < \lambda \), let

\[
p^\gamma_i = p^\gamma_{f(i)}.
\]

Note that if \( i \in S^2_\gamma \cap C^\gamma \), then \( f(i) = i \) holds.

Claim. We have that \( \phi(\gamma_i, (p^\gamma_{\eta} | i < \lambda), C^\gamma, w[\gamma_i]) \) and for all \( l < k \), we have

\[
(\gamma, (p^\gamma_{\eta} | i < \lambda), C^\gamma, w[\gamma_l]) R(\gamma_i, (p^\gamma_{\eta} | i < \lambda), C^\gamma, w[\gamma_i]).
\]

Proof. Some details. \( (p^\gamma_{\eta} \) are descending): Let \( i_1 < i_2 \). Then \( p^\gamma_{i_1} = p^\gamma_{f(i_1)} \geq p^\gamma_{f(i_2)} = p^\gamma_{i_2} \).

(For \( i \in S^2_\gamma \cap C^\gamma \), \( p^\gamma_i \equiv (p^\gamma_j | j < i) \)): Let \( i \in S^2_\gamma \cap C^\gamma \). Let \( \rho < \gamma_i \). Want \( p^\gamma_i[\rho] \) forces the following;

\[
p^\gamma_i(\rho) = \bigcup\{p^\gamma_j(\rho) | j < i\}.
\]

To see this, pick \( l < k \) such that \( \rho < \gamma_l \). By \( \phi(\gamma_l, (p^\gamma_{k_l} | k' < \lambda), C^\gamma, w[\gamma_l]) \) and \( i \in S^2_\gamma \cap C^\gamma \), we have that \( f(i) = i, p^\gamma_i[\gamma_l] = p^\gamma_{f(i)}[\gamma_l] = p^\gamma_i \) and for all \( j < i, p^\gamma_{f(i)}[\gamma_l] = p^\gamma_{f(i)}[\gamma_l] = p^\gamma_{f(i)} \). Hence we have \( p^\gamma_{f(i)}[\rho] = p^\gamma_i[\rho] \) and \( p^\gamma_{f(i)}[\rho] \) forces the following;

\[
p^\gamma_i(\rho) = \bigcup\{p^\gamma_j(\rho) | j < i\}.
\]

But \( p^\gamma_i(\rho) = p^\gamma_i(\rho) \) and \( \bigcup\{p^\gamma_j(\rho) | j < i\} = \bigcup\{p^\gamma_{f(i)}(\rho) | j < i\} = \bigcup\{p^\gamma_i(\rho) | j < i\} \). as \( f(i) = i \in C(f) \).

Hence we are done.

\[
(i \in S^2_\gamma \cap C^\gamma \text{ and } (p^\gamma_j | j < i) \equiv (q^\gamma_j | \gamma_l | j < i) \text{ implies } p^\gamma_{i+1} \leq q^\gamma_{i+1}. \]

Let \( i \in S^2_\gamma \cap C^\gamma \) and \( (p^\gamma_j | j < i) \equiv (q^\gamma_j | \gamma_l | j < i) \). Let \( l < k \). It suffices to show \( p^\gamma_{i+1}[\gamma_l] \leq q^\gamma_{i+1}[\gamma_l] \).

But \( (p^\gamma_j | j < i) \equiv (p^\gamma_{f(i)} | j < i) \equiv (p^\gamma_{f(i)}[\gamma_l] | j < i) = (p^\gamma_{f(i)}[\gamma_l] | j < i) \equiv (q^\gamma_j[\gamma_l] | j < i) \). Let \( i \in S^2_\gamma \cap C^\gamma \). Hence \( p^\gamma_{i+1}[\gamma_l] = p^\gamma_{f(i)}[\gamma_l] \leq q^\gamma_{i+1}[\gamma_l] \).

(For all \( l < k \), we have \( (\gamma_l, (p^\gamma_{\eta} | i < \lambda), C^\gamma, w[\gamma_l]) R(\gamma_i, (p^\gamma_{\eta} | i < \lambda), C^\gamma, w[\gamma_k]) \)): For each \( i < \lambda \), we have \( p^\gamma_i[\gamma_l] = p^\gamma_{f(i)}[\gamma_l] = p^\gamma_i[\gamma_l] \) and \( f(i) \geq i \) holds.

Next let \( i \in S^2_\gamma \cap C^\gamma \). Then \( p^\gamma_i = p^\gamma_{f(i)} = p^\gamma_i \). Hence \( p^\gamma_i[\gamma_l] = p^\gamma_i[\gamma_l] \).
Case. \( \text{cf}(\gamma) = \lambda = \omega^{V[G_{\lambda}]} \). Let \( \eta < \gamma \) and \( w \in P_{\gamma} \). We may assume, by increasing \( \eta \), that \( \text{supp}(w) \subseteq \eta \).

Let \( (\gamma_{k} | k < \lambda) \) be a sequence of ordinals which is continuously \( < - \) increasing, \( \gamma_{0} = \eta \), cofinal in \( \gamma \) and for each \( i \in S_{1}^{2} \), we make sure that
\[
\text{supp}(q_{i}^{1}[\min(\gamma, \alpha(i))]) \subseteq \gamma_{i+1}.
\]
Hence if \( \gamma \leq \alpha(i) \), then \( \text{supp}(q_{i}^{1}[\gamma]) \subseteq \gamma_{i+1} \). If \( \alpha(i) < \gamma \), then \( \text{supp}(q_{i}^{1}) \subseteq \gamma_{i+1} \). This is possible as the supports are of size at most \( \omega_{1} \).

We construct \( (p_{i}^{\gamma_{k}} | i < \lambda) \) and \( C^{\gamma_{k}} \) by recursion on \( k < \lambda \).

\( k = 0 \): Let \( (p_{i}^{70} | i < \lambda) = (p_{i}^{7} | i < \lambda) \) and let \( C^{70} = C^{7} \). Then we have \( \phi(\gamma_{0}, (p_{i}^{70} | i < \lambda), C^{70}, 1) \).

\( k \) to \( k + 1 \): Suppose we have \( \phi(\gamma_{k}, (p_{i}^{7k} | i < \lambda), C^{7k}, 1) \). We have \( \phi(\gamma_{k+1}, (p_{i}^{7k+1} | i < \lambda), C^{7k+1}, 1) \) such that \( (\gamma_{k}, (p_{i}^{7k} | i < \lambda), C^{7k}, 1) R (\gamma_{k+1}, (p_{i}^{7k+1} | i < \lambda), C^{7k+1}, 1) \). We just make sure to take care of the following situation. If \( \text{cf}(k) = \omega_{1}, \gamma \leq \alpha(k) \) and \( p_{k+1}^{\gamma_{k}} \leq q_{k}^{1}[\gamma_{k}] \), then consider
\[
\phi(\gamma^{k}, (p_{i}^{7k} | 0 \leq i \leq k+1) \cap (p_{i}^{7k} | k+1 < i < \lambda), C^{7k} \cap (k+\omega, \lambda), w'),
\]
where \( w' = p_{k+1}^{7k} \cap q_{k}^{1}[\gamma_{k}, \gamma_{k+1}] \in P_{\gamma_{k+1}} \). Let \( (p_{i}^{7k+1} | i < \lambda), C^{7k+1}) \) be such that
\[
\phi(\gamma_{k+1}, (p_{i}^{7k+1} | i < \lambda), C^{7k+1}, w')
\]
and
\[
(\gamma_{k}, (p_{i}^{7k} | 0 \leq i \leq k+1) \cap (p_{i}^{7k} | k+1 < i < \lambda), C^{7k} \cap (k+\omega, \lambda), p_{k+1}^{7k}) R (\gamma_{k+1}, (p_{i}^{7k+1} | i < \lambda), C^{7k+1}, w').
\]
We have that
\[
p_{0}^{7k+1} \leq q_{k}^{1}[\gamma_{k+1}]
\]
and that
\[
(\gamma_{k}, (p_{i}^{7k} | i < \lambda), C^{7k}, 1) R (\gamma_{k+1}, (p_{i}^{7k+1} | i < \lambda), C^{7k+1}, 1).
\]

\( k \) limit: We have \( \text{cf}(k) < \lambda = \omega^{V[G_{\lambda}]} \). Hence there exists \( (p_{i}^{7k} | i < \lambda) \) and \( C^{7k} \) such that \( \phi(\gamma_{k}, (p_{i}^{7k} | i < \lambda), C^{7k}, 1) \) and that for all \( l < k \), \( (\gamma_{l}, (p_{i}^{7l} | i < \lambda), C^{7l}, 1) R (\gamma_{k}, (p_{i}^{7k} | i < \lambda), C^{7k}, 1) \).

This completes the construction of \( (p_{i}^{7k} | i < \lambda), C^{7k} \). Now we begin a sort of diagonal construction. Let
\[
C^{70} = \{ k < \lambda | k \in C_{m, \gamma_{m}} \text{ for all } l < m < k \}.
\]

For each \( i \in S_{1}^{2} \cap C^{70} \), let
\[
p_{i}^{70} = (\bigcup \{ p_{i}^{7l} | l < i \})^{-1} 1 \in P_{\gamma}.
\]

Let \( f : \lambda \longrightarrow S_{1}^{2} \cap C^{70} \) be the \( \varepsilon \)-isomorphism. Let
\[
C^{7} = C^{70} \cap C(f)
\]
and for each \( i < \lambda \), let
\[
p_{i}^{7} = p_{f(i)}^{70}.
\]

Want \( \phi(\gamma, (p_{i}^{7} | i < \lambda), C^{7}, w) \) and that \( (\eta, (p_{i}^{7} | i < \lambda), C^{7}, w[\eta]) R (\gamma, (p_{i}^{7} | i < \lambda), C^{7}, w) \).

Some details.

\( (p_{i}^{7} \) is descending): \( j < i \) implies \( f(j) < f(i) \). Hence for any \( l < f(j) \), we have \( p_{j}^{7}[\eta] = p_{f(l)}^{7l} \leq p_{f(j)}^{7l} = p_{j}^{7}[\eta]. \) Hence \( p_{j}^{7}[\gamma f(j)] \leq p_{j}^{7}[\gamma f(j)] \) and so \( p_{i}^{7} \leq p_{j}^{7} \).
$(i \in S_1^2 \cap C^\gamma)$ implies $(p_i^\gamma \equiv \langle p_j^\gamma \mid j < i \rangle)$: Let $\rho < \gamma$. We first assume that $\rho < \gamma_i$. Then for any $l < i$ such that $\rho < \gamma_l$, we have $p_i^\gamma | \rho = p_i^{\gamma_l} | \rho$ and $p_i^{\gamma_l} | \rho$ forces the following:

$$p_i^{\gamma_l} (\rho) = \bigcup \{ p_j^{\gamma_l} (\rho) \mid j \in i \}.$$ 

But $p_i^\gamma (\rho) = p_i^{\gamma_l} (\rho)$ and $(p_j^\gamma (\rho) \mid j < i) \equiv (p_{f(j)}^{\gamma_l} (\rho) \mid l < f(j), j < i) \equiv (p_j^{\gamma_l} (\rho) \mid j < i)$, as $f(i) = i \in C(f)$.

Hence $p_i^\gamma | \rho$ forces the following:

$$p_i^\gamma (\rho) = \bigcup \{ p_j^\gamma (\rho) \mid j < i \}.$$ 

We next assume $\gamma_i \leq \rho$. Then for all $j \leq i$, we have $p_j^\gamma (\rho) = \emptyset$.

References


miyamoto@nanzan-u.ac.jp

Mathematics

Nanzan University

18 Yamazato-cho, Showa-ku, Nagoya

466-8673 Japan