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A remark on hyperfocal subalgebras of blocks of finite groups

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1 The hyperfocal subalgebra of a block

Let G be a finite group and P be a Sylow p-subgroup of G. Moreover set $Q = O^p(G) \cap P$, which is called the hyperfocal subgroup in [12]. We have

$$Q = \langle\ [O^p(N_G(U)),\ U] \mid U \leq P\ \rangle$$

(see [1], Lemma 2.2 for a proof). I thank Koshitani who informed me of [1]. In particular $Q = 1$ if and only if G is p-nilpotent. If P is abelian, then $Q = [N_G(P), P]$.

Let $(\mathcal{K}, \mathcal{O}, k)$ be a sufficiently large p-modular system such that k is algebraically closed. Let G be a finite group and b be a block of $\mathcal{O}G$ and let P_γ be a defect pointed group of a pointed group $G_{\{b\}}$ on $\mathcal{O}G$, that is, P_γ is a maximal local pointed group contained in $G_{\{b\}}$. Let

$$Q = \langle\ [O^p(N_G(U_\delta)),\ U] \mid U_\delta \in \mathcal{S}_{\mathcal{L}}(P_\gamma)\ \rangle.$$

where $\mathcal{S}_{\mathcal{L}}(P_\gamma)$ is the set of local pointed groups on $\mathcal{O}G$ contained in P_γ. Following [12], Q is called the hyperfocal subgroup of P_γ. Let $j \in \gamma$ and let $B = j\mathcal{O}Gj$. B is a source algebra of b and j is called a source idempotent of b. By [12],Theorem 1.8, [13], §13 and §14, there exists a unique P-stable unitary subalgebra D of B, up to $(B^P)^\times$-conjugation, which satisfies

$$D \cap Pj = Qj \quad \text{and} \quad B = \bigoplus_{u \in P/Q} Du \cong D \otimes_{\mathcal{O}Q} \mathcal{O}P,$$

where $(B^P)^\times$ is the group of invertible elements of B^P. D is called a hyperfocal subalgebra of b. D becomes an interior Q-algebra with a group homomorphism $q \in Q \to qj \in D^\times$. By [12] or [13], Corollary 13.13, $Q = 1$ if and only if f b is nilpotent, and in that case D is $\mathcal{O}$-simple, that is, D is isomorphic to a full matrix algebra over $\mathcal{O}$

We set $\mathcal{R} = \mathcal{O}$ or k. Let $\mathbf{A}$ be an $\mathcal{R}$-algebra and $\mathbf{B}$ be an interior $\mathbf{A}$-algebra, that is, $\mathbf{B}$ is an $\mathcal{R}$-algebra which is an $\mathbf{A}$-bimodule satisfying $(xa)y = x(ay)$ for $a \in \mathbf{A}$, $x, y \in \mathbf{B}$. Let $\mu_{\mathbf{B}} : \mathbf{B} \otimes_{\mathbf{A}} \mathbf{B} \to \mathbf{B}$ denote the map of $\mathbf{B}$-bimodules satisfying $\mu(x \otimes y) = xy$ for $x, y \in \mathbf{B}$. Following [6], we say $\mathbf{B}$ is a separable interior $\mathbf{A}$-algebra if $\mu_{\mathbf{B}}$ splits as a map of $\mathbf{B}$-bimodules. By [6], Lemma 4, B is a separable interior $\mathcal{O}P$-algebra.

Theorem 1 ([18], Theorem 1) *D is a separable interior $\mathcal{O}Q$-algebra.*

Corollary 1 ([18], Corollary 1) *Let N be a finitely generated (left) D-module. Then N is a direct summand of $D \otimes_{\mathcal{O}Q} N$ as a D-module. In particular $\bar{D} = D \otimes_{\mathcal{O}} k$ is of finite representation type if Q is cyclic.*

We recall that if P is abelian and Q is cyclic, then the number of isomorphism classes of irreducible $\bar{D}$-modules is equal to $|N_G(P_\gamma)/C_G(P)|$ by Theorem in [17].

2 Fan's question

Assume that P is abelian. Then we have $Q = [P, N_G(P_\gamma)]$ ([18]). Let $L = C_P(N_G(P_\gamma))$. Then we have

$$P = Q \times L$$

as is well known. For $x \in \mathcal{O}G$ and $X \subseteq \mathcal{O}G$, we denote by $\bar{x}$ and $\bar{X}$ the images in kG by the canonical homomorphism from $\mathcal{O}G$ onto kG. Now $G_{\{b\}}$ is Q-locally controlled by P_γ in the sense of Fan [2].

Question 1 (Fan [2], p. 789) *As interior P-algebras*

$$B \cong D' \otimes_{\mathcal{O}} \mathcal{O}L$$

for some interior P-algebra D'.

This question is true if P is normal in G, or G is p-solvable (see Remark 1 below). Also Okuyama showed that the question is true for $\bar{B} = B \otimes_{\mathcal{O}} k$.

Theorem 2 ([18], Theorem 2) *With the above notations, there is a group homomorphism $\rho : P \to \bar{D}^\times$ such that $\rho(q) = q\bar{j}$ for any $q \in Q$ and that $d^u = d^{\rho(u)}$ for any $d \in \bar{D}$ and $u \in L$. Moreover, then, there is an interior P-algebra isomorphism $\bar{B} \cong \bar{D} \otimes_k kL$ mapping du on $d\rho(u) \otimes u$ for any $d \in \bar{D}$ and $u \in L$ where $\bar{D}$ is regarded as an interior P-algebra with ρ as structural map.*

(See also [16].) We will show that if Q is normal in G, then Fan's question is true.

3 The case where Q is normal in G

Assume that P_γ is associated with the maximal b-Brauer pair (P, b_P). We have $N_G(P, b_P) = N_G(P_\gamma)$. Set $b_0 = (b_P)^{N_G(P)}$. Then b_0 is a Brauer correspondent of b. Let B be a source algebra of b defined in the above and let B_0 be a source algebra of b_0. Let $E = N/C_G(P)$ be a p-complement of $N_G(P_\gamma)/C_G(P)$ and we denote by $[E]$ a set of representatives for the $C_G(P)$-cosets in N. For $a \in (\mathcal{O}G)^P$, we set $a' = \mathrm{Br}_P(a)$. Recall that $ga'g^{-1} = (gag^{-1})'$ $(g \in N_G(P))$.

Proposition 1 *With the above notations, assume that there exists a normal p-subgroup $\mathbf{Q}$ of G such that $\mathbf{Q} \subseteq Z(P)$ and $(b_P)^{C_G(\mathbf{Q})}$ is nilpotent.*

(i) *$B \cong S \otimes_{\mathcal{O}} B_0$ as interior P-algebras, where S is a (primitive) (interior) Dade P-algebra.*

(ii) *If P is abelian, then $B \cong D \otimes_{\mathcal{O}} \mathcal{O}L$ as interior P-algebras, where $L = C_P(N_G(P_\gamma))$.*

(iii) *b and b_0 are basic Morita equivalent* (See [11] for the definition of basic Morita equivalence).

Remark 1 *If G is p-solvable and P is abelian, then the above theorem holds without the assumption by Remark 3.6 in [3].*

Remark 2 *From the proof of the proposition, if b is a principal block of G, then $B \cong B_0$.*

For a p-subgroup X of G, we denote by $\mathcal{LP}_{\mathcal{R}G}(X)$ the set of local point of X on $\mathcal{R}G$.

Lemma 1 *Let* $\mathbf{Q}$ *be a normal* p*-subgroup of* G *and set* $C = C_G(\mathbf{Q})$*. Let* X *be a* p*-subgroup of* G *containing* $\mathbf{Q}$*. Then any* $\epsilon \in \mathcal{LP}_{\mathcal{R}C}(X)$ *is contained a uniquely determined* $\epsilon' \in \mathcal{LP}_{\mathcal{R}G}(X)$*. Moreover the map* $\epsilon \in \mathcal{LP}_{\mathcal{R}C}(X) \mapsto \epsilon' \in \mathcal{LP}_{\mathcal{R}G}(X)$ *is a bijection.*

Proof. Since there is a natural bijection between $\mathcal{LP}_{\mathcal{O}G}(X)$ and $\mathcal{LP}_{kG}(X)$, we may assume $\mathcal{R} = k$. Let $\epsilon \in \mathcal{LP}_{kC}(X)$ and let $i \in \epsilon$. Suppose that

$$i = i_1 + i_2, \quad i_1 i_2 = i_2 i_1 = 0$$

for some idempotents i_1, i_2 in $(kG)^X$. Since $\mathbf{Q} \leq X$, we have $i = \mathrm{Br}_{\mathbf{Q}}(i_1) + \mathrm{Br}_{\mathbf{Q}}(i_2)$. Since $\mathrm{Br}_{\mathbf{Q}}(i_1)$, $\mathrm{Br}_{\mathbf{Q}}(i_2) \in (kC)^X$ and since i is primitive in $(kC)^X$, we may assume that $i = \mathrm{Br}_{\mathbf{Q}}(i_1)$ and $\mathrm{Br}_{\mathbf{Q}}(i_2) = 0$. So $i_2 \in \mathrm{Ker}(\mathrm{Br}_{\mathbf{Q}}) = \sum_{Y<\mathbf{Q}} (kG)_Y^{\mathbf{Q}}$. Since $\mathbf{Q}$ is a normal p-subgroup of G, $\mathrm{Ker}(\mathrm{Br}_{\mathbf{Q}})$ is contained in the radical of kG. Therefore $i_2 = 0$. This implies i is primitive in $(kG)^X$. Since $C_C(X) = C_G(X)$ and since there is a canonical bijection between $\mathcal{LP}_{kG}(X)$ and the set of points of $kC_G(X)$, the lemma easily follows. So the proof is complete. ■

Proof of Proposition 1

(i) Set

$$b_{\mathbf{Q}} = (b_P)^{C_G(\mathbf{Q})} \text{ and } C = C_G(\mathbf{Q}).$$

Then b is a unique block of G which covers $b_{\mathbf{Q}}$ and (P, b_P) is a maximal $b_{\mathbf{Q}}$-Brauer pair. In order to prove (i), we may assume $b_{\mathbf{Q}}$ is G-invariant. By the Frattini argument $G = CN_G(P, b_P) = CN$. Since $b_{\mathbf{Q}}$ is nilpotent, $C \cap N = C_G(P)$. Let P_δ be a defect pointed group of $C_{\{b_{\mathbf{Q}}\}}$ on $\mathcal{O}C$. By Lemma 1, we also may assume $\delta \subseteq \gamma$. Let $i \in \delta$ and set $B_{\mathbf{Q}} = i\mathcal{O}Ci$, a source algebra of $b_{\mathbf{Q}}$. Note that we may assume $B = i\mathcal{O}Gi$. Let S be a hyperfocal subalgebra of $b_{\mathbf{Q}}$ contained in $B_{\mathbf{Q}}$ and set $C_B(S) = \{x \in B \mid xs = sx \ (\forall s \in S)\}$. Then $C_B(S)$ is P-stable because S is P-stable. We will observe that $C_B(S)$ is a crossed product of $C_{B_{\mathbf{Q}}}(S)$ over E, then $C_B(S) \cong B_0$ as interior P-algebras.

By [10], Theorem 1.6, S is a (primitive) Dade P-algebra. Moreover by [10], 1.8, there is a unique group homomorphism $\iota : P \to S^\times$ lifting the action of P on S such that $\det(\iota(u)) = 1$ for any $u \in P$. Set $z_u = \iota(u^{-1})u = u\iota(u^{-1})$. We have $z_u z_v = z_{uv}$ and $z_u \in (C_{B_{\mathbf{Q}}}(S))^P$ $(u \in Z(P))$. Hence $C_B(S)$ becomes an interior P-algebra. Moreover

$$B_{\mathbf{Q}} = \bigoplus_{u \in P} Su = \bigoplus_{u \in P} Sz_u.$$

Since S is $\mathcal{O}$-simple,

$$C_{B_{\mathbf{Q}}}(S) = \bigoplus_{u \in P} \mathcal{O}z_u \cong \mathcal{O}P.$$

Let $g \in N$. Since P_δ is N-invariant, there is $x_g \in ((\mathcal{O}C)^P)^\times$ such that $gig^{-1} = x_g i x_g^{-1}$. Set $a_g = (x_g^{-1}g)i = i(x_g^{-1}g) \in B \cap \mathcal{O}Cg$. Then $(g^{-1}x_g)i = i(g^{-1}x_g)$ is the inverse of a_g in B (cf. [15], (44.2)). It is easy to see that

$$(1) \qquad {}^{a_g}u = a_g u (a_g)^{-1} = (gug^{-1})i \ (\forall u \in P).$$

Here we note we can take $x_{cg} = cx_g$ and hence $a_{cg} = a_g$ for any $c \in C_G(P)$. From (1), ${}^{a_g}S$ is a hyperfocal subaglgebra of $b_{\mathbf{Q}}$. By [12], 13.3, S is unique up to $((B_{\mathbf{Q}})^P)^\times$-conjugation, and hence we may assume that $S = {}^{a_g}S$ by replacing x_g by $x_g(y_g + (1 - i))$

where $y_g \in ((B_{\mathbf{Q}})^P)^\times$. On the other hand, since S is $\mathcal{O}$-simple, there exists $t_g \in S^\times$ such that

$$^{a_g}s = {}^{t_g}s \quad (\forall s \in S)$$

by a theorem of Skolem-Noether. We may assume $t_g = t_{cg}$ for any $c \in C_G(P)$. Since $\iota(u^g)s\iota((u^g)^{-1}) = u^g s (u^g)^{-1}$, we can see

$$^{a_g}\iota(u^g)s(^{a_g}(\iota((u^g)^{-1})))) = usu^{-1}.$$

Note $\det(^{a_g}\iota(u)) = \det(^{t_g}\iota(u)) = 1$. Hence, by the uniqueness of ι, we have

(2) $$\iota(u^g) = \iota(u)^{a_g} = \iota(u)^{t_g}.$$

Now we can see

(3) $$B = \bigoplus_{g \in [E]} B_{\mathbf{Q}} a_g = \bigoplus_{g \in [E]} (B \cap \mathcal{O}Cg).$$

Set $c_g = {t_g}^{-1} a_g \in C_B(S) \cap \mathcal{O}Cg$. We may assume $c_g = c_{cg}$ for any $c \in C_G(P)$. Moreover $(a_g)^{-1} t_g$ is the inverse of c_g in B. From (1) and (2) we can see

(4) $$^{a_g}z_u = z_{^g u}, \ ^{c_g}z_u = z_{^g u} \quad (g \in N, \ u \in P).$$

Moreover

$$c_g c_h (c_{gh})^{-1} \in (C_{B_{\mathbf{Q}}}(S))^\times.$$

Since we have

$$B = \bigoplus_{g \in [E]} \bigoplus_{u \in P} S z_u c_g,$$

(5) $$C_B(S) = \bigoplus_{g \in [E], u \in P} \mathcal{O} z_u c_g.$$

Thus $C_B(S)$ is a crossed product of E over $C_{B_{\mathbf{Q}}}(S)$. From (4) and [4], Lemma M, $C_B(S)$ is a twisted group algebra of $P \rtimes E$ over $\mathcal{O}$ (see [7] and [5]). In fact, by replacing c_g by $c_g \epsilon_g$ for some $\epsilon_g \in i + J(Z(\mathcal{O}\tilde{P})) \subseteq (\mathcal{O}C)^P$ if necessary, where $\tilde{P} = \{z_u \mid u \in P\}$, we have for some 2-cocycle $\alpha \in Z^2(E, \mathcal{O}^\times)$

(6) $$c_g c_h = \alpha(g,h) c_{gh} \ (g, h \in N).$$

Hence by replacing x_g by $\tilde{x}_g := x_g(^{a_g}(\epsilon_g^{-1}) + 1 - i)$, we may assume (6) holds. Then note that we have $S = {}^{(\tilde{x}_g^{-1} g) i} S$.

Since S is $\mathcal{O}$-simple,

$$B \cong S \otimes_{\mathcal{O}} C_B(S)$$

as interior P-algebras. In order to complete the proof of (i), by [10], Lemma 7.8, it suffices to show $C_B(S) \cong B_0$ as interior P-algebras assuming $\mathcal{R} = k$.

Set $N_S(P) = \{t \in S^\times \mid t.P = t\iota(P) = \iota(P)t = P.t\}$. By [9], (e) and [10], Theorem 1.6, there is a group homomorphism $f : N_{S^\times}(P) \to S(P)^\times = k^\times i'$ which extends $\mathrm{Br}_P|_{(S^P)^\times}$. Since $t_g \in N_{S^\times}$ from (2) we set

$$f(t_g) = \delta_g i' \ (g \in N, \ \delta_g \in k^\times).$$

Now since $gig^{-1} = x_g i x_g^{-1}$ we have

$$gi'g^{-1} = x'_g \delta_g i' \delta_g^{-1} {x'_g}^{-1}.$$

We set

$$\mathbf{a}_g = ({\delta_g}^{-1} x_g'^{-1} g) i' = i' ({\delta_g}^{-1} x_g'^{-1} g) \in (i'kN_G(P_\gamma)i')^\times.$$

We may assume $\mathbf{a}_g = \mathbf{a}_{cg}$ for any $c \in C_G(P)$. Moreover we have

(7) $$^{\mathbf{a}_g}(ui') = {}^g ui' \ (g \in N,\ u \in P).$$

From (6) we have

$$\alpha(g,h)i' = \mathrm{Br}_P(c_{gh}^{-1} c_g c_h) = (gh)^{-1} \mathrm{Br}_P(x_{gh} t_{gh} {t_g}^{-1} x_g^{-1} (g {t_h}^{-1} x_h^{-1} g^{-1})) gh$$

$$= (gh)^{-1} x'_{gh} i' \delta_{gh} {\delta_g}^{-1} {x'_g}^{-1} (g {\delta_h}^{-1} {x'_h}^{-1} g^{-1}) gh = {\mathbf{a}_{gh}}^{-1} \mathbf{a}_g \mathbf{a}_h,$$

and hence

(8) $$\mathbf{a}_g \mathbf{a}_h = \alpha(g,h) \mathbf{a}_{gh} \ (g,h \in N).$$

Since $B_0 = i'kN_G(P_\gamma)i' = \bigoplus_{g \in [E]} \bigoplus_{u \in P} k(ui')\mathbf{a}_g$, from (4), (6), (7) and (8), $B_0 \cong C_B(S)$ as interior P-algebras. This proves (i).

(ii) Since Q is $N_G(P_\gamma)$-invariant, from (1), $D = \bigoplus_{g \in [E]} \bigoplus_{u \in Q} Sua_g = \bigoplus_{g \in [E]} \bigoplus_{u \in Q} Sz_u c_g$ is P-stable, and we see D is a hyperfocal subalgebra of b. On the other hand $\bigoplus_{r \in L} \mathcal{O} z_r$ is contained in the center $Z(B)$ and $B = \bigoplus_{r \in L} Dz_r$. This implies (ii).

(iii). Let e be a primitive idempotent of S and set $V = Se$. Then V becomes an endo-permutation $\mathcal{O}P$-module with $p \nmid \mathrm{rank}_{\mathcal{O}} V$ by [10], Theorem 1.6. Now from (i) and [8], Theorem 3.4, the $(\mathcal{O}Gb, \mathcal{O}N_G(P)b_0)$-bimodule

$$\mathcal{M} = \mathcal{O}Gi \otimes_{B \cong S \otimes_{\mathcal{O}} B_0} (V \otimes_{\mathcal{O}} B_0) \otimes_{B_0} \mathcal{O}N_G(P)$$

and the $(\mathcal{O}N_G(P)b_0, \mathcal{O}Gb)$-bimodule

$$\mathcal{N} = \mathcal{O}N_G(P) \otimes_{B_0} (B_0 \otimes_{\mathcal{O}} V^*) \otimes_{B_0 \otimes_{\mathcal{O}} S \cong B} i\mathcal{O}G$$

induce a Morita equivalence between b and b_0. We notice that $\mathcal{N} \cong \mathcal{M}^*$. In fact $\mathcal{N} \cong \mathrm{Hom}_A(\mathcal{M}, A) \cong \mathcal{M}^*$ because A is symmetric, where $A = \mathcal{O}Gb$) (Auslander-Fuller, 22.1). We can see

$$\mathcal{M} \mid \mathcal{O}Gi \otimes_{\mathcal{O}P} (V \otimes_{\mathcal{O}} B_0) \otimes_{\mathcal{O}P} \mathcal{O}N_G(P), \quad V \otimes_{\mathcal{O}} B_0 \mid {}_{\mathcal{O}P}\mathcal{M}_{\mathcal{O}P}$$

because B and B_0 are, respectively, separable interior $\mathcal{O}P$-algebras. Since B_0 is a permutation $\mathcal{O}(P \times P)$- module and V is an endo-permutation $\mathcal{O}P$-module, $V \otimes_{\mathcal{O}} B_0$ is an endo-permutation $\mathcal{O}(P \times P)$-module. This implies b and b_0 are basic Morita equivalent. Recall that any indecomposable component of B_0 is isomorphic to $\mathrm{Ind}_{P_x}^{P \times P}(\mathcal{O})$ for some $x \in G$, where P_x denotes the subgroup $\{(u, {}^{x^{-1}}u) \in P \times P \mid u \in P \cap {}^x P\}$. Since $|P| \,\|\, \mathrm{rank}_{\mathcal{O}}(V \otimes_{\mathcal{O}} B_0)$, we can see $\Delta P = \{(u,u) | u \in P\}$ is a vertex of $\mathcal{M}$. ■

In the above proposition assume that P is abelian and $C_G(\mathbf{Q}) \cap N_G(P_\gamma) = C_G(\mathbf{Q}) \cap N_G(P, b_P) = C_G(P)$. Then $b_{\mathbf{Q}}$ is nilpotent.

Corollary 2 *Assume that P is abelian and let $\mathbf{Q}$ be a normal p-subgroup of $N_G(P_\gamma)$ such that $C_G(\mathbf{Q}) \cap N_G(P_\gamma) = C_G(P)$. Then $(b_P)^{N_G(\mathbf{Q})}$ and b_0 are basic Morita equivalent.*

Proof. Set

$$c = (b_P)^{N_G(\mathbf{Q})}, \quad d = (b_P)^{N_G(\mathbf{Q}) \cap N_G(P)}.$$

By the above theorem c and d are basic Morita equivalent. On the other hand $d\mathcal{O}N_G(P)$ realizes a (splendid) Morita equivalent between d and b_0. This implies that c and b_0 are basic Morita equivalent. ■

$$\begin{array}{c}
N_G(\mathbf{Q}) \quad c \\
\uparrow \quad \textit{basic Morita eq.} \\
N_G(\mathbf{Q}) \cap N_G(P) \quad d \\
\uparrow \\
C_G(P) \quad b_P
\end{array}$$

Corollary 3 *Assume that P is abelian. Then $\hat{b}_Q = (b_P)^{N_G(Q)}$ and b_0 are basic Morita equivalent. In particular, b and b_0 are derived equivalent if and only if b and $\hat{b}_Q$ are derived equivalent.*

Corollary 4 (see [14]) *Assume that P is abelian and suppose that Q is cyclic, and let Q_1 be a non-trivial subgroup of Q. Then $(b_P)^{N_G(Q_1)}$ and b_0 are basic Morita equivalent.*

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