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Difference Approximation to Aubry-Mather Sets

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1 PDE Approach to the Aubry-Mather Theory

This paper presents a rough description of the PDE approach to the Aubry-Mather theory and the results of the preprint [9] by Takaaki Nishida and the author on difference approximation to Aubry-Mather sets.

We consider the following non-autonomous Hamiltonian systems with one degree of freedom generated by C^2-functions H:

$$
\begin{aligned}
&x'(s) = H_u(x(s), s, u(s)), \quad u'(s) = -H_x(x(s), s, u(s)),\\
&H(x, s, u) : \mathbb{T} \times \mathbb{T} \times \mathbb{R} \to \mathbb{R}, \;\; \mathbb{T} := \mathbb{R}/\mathbb{Z}.
\end{aligned} \tag{1.1}
$$

A physical example of (1.1) is a forced pendulum with a time-periodic force. Under several conditions, autonomous Hamiltonian systems with two degrees of freedom can be deduced to (1.1) on each energy level set.

Since $H(x, s, u)$ is periodic with respect to s with the period 1, the dynamics of (1.1) can be studied by the iteration of its time-1 map

$$\mu = \phi_H^{0,1} : \mathbb{T} \times \mathbb{R} \ni (x(0), u(0)) \mapsto (x(1), u(1)) \in \mathbb{T} \times \mathbb{R},$$

where $\phi_H^{0,s}$ is the flow of (1.1).

Figure 1 shows a numerical example of the trajectories $(x^k, u^k) = \mu^k(x^0, u^0)$, $k \in \mathbb{N}$.

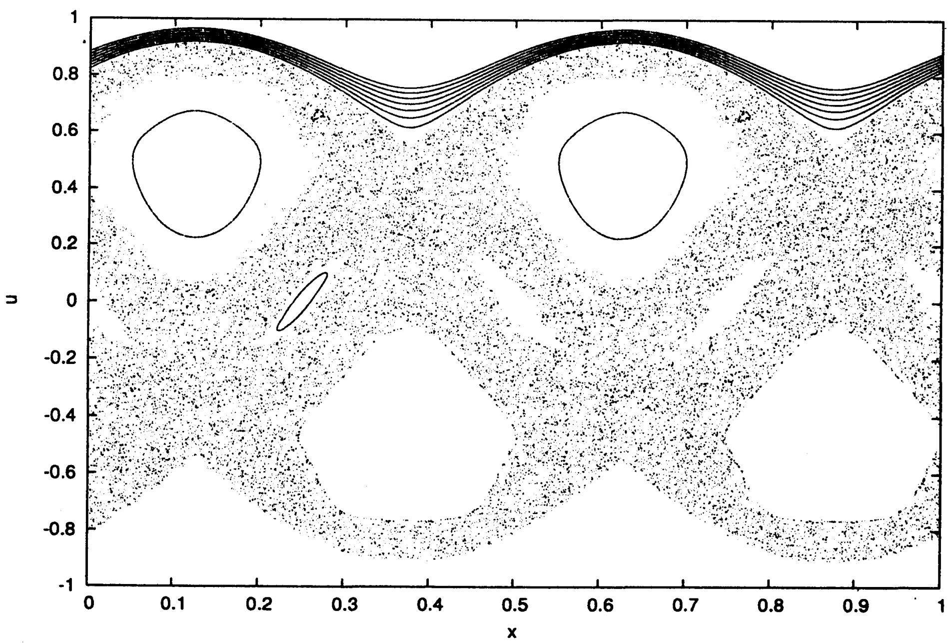

Figure 1.

In Figure 1 we observe the several smooth curves diffeomorphic to $\mathbb{T}$ which trap the trajectories. Such curves are called μ-invariant tori. We also observe the region where only one trajectory seems to move around densely. Such a region is sometimes called a chaotic region.

We focus our attention on the search for μ-invariant tori or, more generally, μ-invariant sets, where $\mathcal{I} \subset \mathbb{T} \times \mathbb{R}$ is said to be μ-invariant, if $\mu(\mathcal{I}) \subset \mathcal{I}$. This is one of the central issues in the theory of Hamiltonian dynamics and has been studied theoretically and numerically for a long time. The celebrated results on the issue are the KAM (Kolmogorove-Arnold-Moser) theory and Aubry-Mather theory for area preserving twist maps on an annulus. Although these theories cover also the flow cases, we will not refer to this here.

The KAM theory applies in particular to slightly perturbed integrable twist maps. Suppose that μ is smooth as needed and is of the form $\mu(x, u) = \mu_0(x, u) + \mu_1(x, u)$ with $\mu_0(x, u) = (x + \rho(u), u)$, $\rho'(u) \neq 0$ and $|\mu_1(x, u)| \ll 1$ on $\mathbb{T} \times [u_1, u_2]$. Then the KAM theory provides a family of μ-invariant tori called KAM tori which occupies a large part of $\mathbb{T} \times [u_1, u_2]$. Each KAM torus carries quasi-periodic trajectories with a common asymptotic slope $\alpha = \lim_{|k|\to\infty} \frac{\tilde{x}^k}{k}$ ($\tilde{x}^k \in \mathbb{R}$ is the lift of $x^k \in \mathbb{T}$) called a rotation number, which is a Diophantine number. The KAM theory is a kind of perturbation theory and requires a severe smallness condition for the perturbation and a number theoretical condition. Numerical studies show that KAM tori disappear and chaotic regions spread as the magnitude of perturbation gets larger, in general. It is an interesting problem to make this process clear.

The Aubry-Mather theory is not based on a perturbation theory but on calculus of variation and does not require the smallness condition nor the number theoretical condition. Suppose that μ satisfies the twist condition, namely, $\mu(x, u) = (f(x, u), g(x, u))$ with $f_u(x, u) \neq 0$. Then the Aubry-Mather theory provides a family of μ-invariant sets called Aubry-Mather sets. Each Aubry-Mather set is either a smooth curve diffeomorphic to $\mathbb{T}$ or a subset of a Lipschitz curve homeomorphic to $\mathbb{T}$ and carries conditionally periodic trajectories with a common rotation number α, namely, if α is rational (irrational), the trajectories are periodic (quasi-periodic). The remarkable fact is that not only for Diophantine numbers but also for any number α there exists the Aubry-Mather set with the rotation number α. We refer to [7] for an interesting review of the Aubry-Mahter theory.

J. Moser pointed out that each area preserving twist map on an annulus is represented as the time-1 map of a certain Hamiltonian system of the form (1.1) with $H(x, s, u)$ which is strictly convex in u [7] (see also its reference [28]). Let us remark that the converse is not true.

A new approach to the Aubry-Mather theory is pioneered independently by A. Fathi [5] and W. E [4] in combination with the analysis of Hamilton-Jacobi equations. Let us call the new approach the "PDE approach". Fathi deals with autonomous Hamiltonian systems with general degrees of freedom; His setting is different from ours. We focus our attention to the results of E, which deal with (1.1) for $H(x, s, u)$ which is strictly convex in u. E shows that for each number α there exists a μ-invariant set with the rotation number α which is a subset of the graph of a solution to a nonlinear PDE and has a structure quite similar to that of Aubry-Mather sets. This result is independent of the twist condition of μ.

The PDE approach can be seen as a generalization of a consequence of the "method of characteristics" for first order nonlinear PDEs. Let us consider the initial value problem to the hyperbolic conservation law with C^1-initial data

$$(1.2)\qquad \begin{cases} u_t(x,t) + H(x,t,u(x,t))_x = 0 \text{ in } \mathbb{R}^2, \\ u(x,0) = u_0(x) \text{ on } \mathbb{R}. \end{cases}$$

Let $(\tilde{x}(s;y), u(s;y)) : \mathbb{R} \to \mathbb{R}^2$ be the solution of

$$\tilde{x}'(s) = H_u(\tilde{x}(s), s, u(s)), \quad u'(s) = -H_x(\tilde{x}(s), s, u(s)), \quad \tilde{x}(0) = y, \quad u(0) = u_0(y).$$

The curve $c(s;y) := (\tilde{x}(s;y), s, u(s;y))$ is called a characteristic curve of (1.2). The idea of the method of characteristics is the following: If for each x, t there exists the unique value $y = y(x,t)$ for which $\tilde{x}(t;y) = x$ and the family of characteristic curves $\{c(s;y)\}_{y\in\mathbb{R}}$ forms a C^1-surface of the (x,t,u)-space represented as $u = u(x,t) := u(t; y(x,t))$, then $u(x,t)$ is the C^1-solution of (1.2). Conversely, if there exists a C^1-solution $u(x,t)$ of (1.2), then the surface defined as the graph of $u(x,t)$ consists of the family of characteristic curves. As a consequence, we have the following statement:

Proposition 1.1 *Suppose that there exists a $\mathbb{Z}^2$-periodic C^1-solution $\bar{u}(x,t)$ of*

$$(1.3)\qquad u_t(x,t) + H(x,t,u(x,t))_x = 0.$$

Then $\mathcal{I}(\bar{u}) := \{(x, \bar{u}(x,0)) \mid x \in \mathbb{T}\}$ is a μ-invariant torus which carries trajectories with a common rotation number.

The proof is simple. Let (x^0, u^0) be any point of $\mathcal{I}(\bar{u})$ and $\tilde{x}(s)$ be the solution of

$$\tilde{x}'(s) = H_u(\tilde{x}(s), s, \bar{u}(\tilde{x}(s), s)), \quad \tilde{x}(0) = x^0.$$

Note that $u^0 = \bar{u}(x^0, 0)$ and $\tilde{x}(s)$ can be defined globally on $\mathbb{R}$. Then using (1.3) we have

$$\begin{aligned} \frac{d}{ds}\bar{u}(\tilde{x}(s), s) &= \bar{u}_t(\tilde{x}(s), s) + \bar{u}_x(\tilde{x}(s), s)\tilde{x}'(s) \\ &= \bar{u}_t(\tilde{x}(s), s) + H_u(\tilde{x}(s), s, \bar{u}(\tilde{x}(s), s))\bar{u}_x(\tilde{x}(s), s) \\ &= -H_x(\tilde{x}(s), s, \bar{u}(\tilde{x}(s), s)). \end{aligned}$$

Therefore $(x(s), u(s)) := (\tilde{x}(s) \mod 1, \bar{u}(\tilde{x}(s), s))$ is a solution of (1.1). By the $\mathbb{Z}^2$-periodicity of $\bar{u}$ we have $\mu^k(x^0, u^0) = (x(k), u(k)) = (x(k), \bar{u}(x(k), 0)) \in \mathcal{I}(\bar{u})$ for any $k \in \mathbb{Z}$. We see also that $X(s) := (x(s), s \mod 1) : \mathbb{R} \to \mathbb{T}^2$ is either periodic or one-to-one. Thus we conclude that there exists $\lim_{|s|\to\infty} \frac{\tilde{x}(s)}{s}$ which is independent of the point $(x^0, u^0) \in \mathcal{I}$, due to the classical result of Poincaré: *Let $y(s) : \mathbb{R} \to \mathbb{R}$ be continuous such that $Y(s) := (y(s) \mod 1, s \mod 1) : \mathbb{R} \to \mathbb{T}^2$ is either periodic or one-to-one. Then there exists the asymptotic slope $\lim_{|s|\to\infty} \frac{y(s)}{s}$ which is finite. If, for another $\tilde{y}(s) : \mathbb{R} \to \mathbb{R}$ satisfying the above condition, $\tilde{Y}(s)$ and $Y(s)$ never intersect, then their asymptotic slopes are the same.*

We cannot always expect the existence of $\mathbb{Z}^2$-periodic C^1-solutions $\bar{u}(x,t)$. The method of characteristics is technically limited to construction of local in time C^1-solutions. It is also mathematically impossible to have always global in time C^1-solutions even in the case where characteristic curves are always defined globally, because the surface formed by the characteristic curves may be eventually folded and it cannot be represented as the graph of a single valued function. In other words, the curves $(\tilde{x}(s;y),s):\mathbb{R}\to\mathbb{R}^2$, which are called the projected characteristic curves, may eventually have intersections with others in finite time. That is why the class of entropy solutions is introduced. Entropy solutions are special weak solutions of (1.3) with an additional condition called the entropy condition.

Now we consider $\mathbb{Z}^2$-periodic solutions of (1.3) in the class of entropy solutions. A function $\bar{u}(x,t)$ is a $\mathbb{Z}^2$-periodic entropy solution of (1.3) with $H(x,s,u)$ which is convex in u, if $\bar{u}$ belongs to $L^1_{loc}(\mathbb{R}^2)$ and satisfies the following:

- $\bar{u}(x+k,t+l)=\bar{u}(x,t)$ for any $x,t\in\mathbb{R}$ and $k,l\in\mathbb{Z}$,
- $\displaystyle\iint_{\mathbb{R}^2}\bar{u}(x,t)\varphi_t(x,t)+H(x,t,\bar{u}(x,t))\varphi_x(x,t)dxdt=0$ for any $\varphi\in C_0^\infty(\mathbb{R}^2)$,
- $\bar{u}(x+h,t)-\bar{u}(x,t)\le e(t)h$ for any $h>0$ and $x,t\in\mathbb{R}$,

where $C_0^\infty(\mathbb{R}^2)$ denotes the set of C^∞-functions defined on $\mathbb{R}^2$ with compact supports and $e(t)$ is a positive valued function. The last condition is the so-called entropy condition for the convex case. Countably many discontinuities are allowed for $\bar{u}(\cdot,t)|_{x\in\mathbb{T}}$ for each fixed t, but they must jump down! A set of points $x_0=x_0(t)$ of discontinuity of $\bar{u}(\cdot,t)$ form a continuous curve for $t\in(t_0,\infty)$, which is called a shock.

The problems here are the following: How to find $\mathbb{Z}^2$-periodic entropy solutions of (1.3)? Is there any result similar to Proposition 1.1 in the class of entropy solutions? For simplicity we consider these problems taking a simple example of

$$H(x,s,u)=\frac{1}{2}u^2-F(x,s).$$

In this case (1.1) is of the form

$$x'(s)=u(s),\quad u'(s)=F_x(x(s),s) \tag{1.4}$$

and (1.3) is the forced Burgers equation with the $\mathbb{Z}^2$-periodic forcing term $F_x(x,t)$

$$u_t(x,t)+u(x,t)u_x(x,t)=F_x(x,t). \tag{1.5}$$

The arguments below hold for general functions $H(x,s,u)$ which is strictly convex and superlinear with respect to u (see e.g. [2], [1]).

H. R. Jauslin, H. O. Kreiss and J. Moser [6] prove that there exist $\mathbb{Z}^2$-periodic entropy solutions of (1.5) through the vanishing viscosity method and conjecture that $\mathcal{I}(\bar{u})$ would contain a μ-invariant set $\mathcal{M}(\bar{u})$ for each $\mathbb{Z}^2$-periodic *entropy solution* $\bar{u}$ of (1.5). Jauslin-Kreiss-Moser take the following steps to obtain $\mathbb{Z}^2$-periodic entropy solutions: First they find $\mathbb{Z}$-periodic in t solutions $\bar{u}^\nu$ of the parabolic equation with the periodic boundary condition

$$u^\nu_t(x,t)+u^\nu(x,t)u^\nu_x(x,t)=F_x(x,t)+\nu u^\nu_{xx}(x,t)\ \text{in}\ \mathbb{T}\times\mathbb{R}_+, \tag{1.6}$$

where the term νu^ν_{xx}, $\nu>0$ is called an artificial viscosity, which yields classical solutions to (1.6). And then they find a sequence $\nu_j\to 0+$ for which $\bar{u}^{\nu_j}$ converges to a $\mathbb{Z}^2$-periodic entropy solution $\bar{u}$ of (1.5). Note that each classical solution of (1.6) conserves its average on $\mathbb{T}$: $\int_0^1 u^\nu(x,t)dx\equiv C$ for $t>0$, and so does each entropy solution of (1.5) in $\mathbb{T}\times\mathbb{R}_+$. This average is called the momentum.

Theorem 1.2 ([6]) *1. Fix $\nu > 0$ arbitrarily. Then for each $C \in \mathbb{R}$, there exists the unique time-periodic C^2-solution $\bar{u}^\nu$ of (1.6) with the momentum C. Any other solutions u^ν of (1.6) with the same momentum C satisfy $\| u^\nu(\cdot,t) - \bar{u}^\nu(\cdot,t) \|_{L^1(\mathbb{T})} \to 0$ as $t \to \infty$.*

2. For each $C \in \mathbb{R}$ there exists a sequence $\nu_j \to 0+$ such that the sequence $\bar{u}^{\nu_j}$ with the momentum C converges to a $\mathbb{Z}^2$-periodic entropy solution $\bar{u}$ of (1.5) in the topology of $C^0(\mathbb{T}; L^1(\mathbb{T}))$, which belongs to $Lip(\mathbb{T}; L^1(\mathbb{T}))$ and has the momentum C. Furthermore $\bar{u}$ is uniformly bounded and $\bar{u}(\cdot,t)|_{x\in\mathbb{T}}$ is a function of bounded variation for each t.

The conjecture of Jauslin-Kreiss-Moser above is proved to hold by E [4]: *There exists a μ-invariant closed subset $\mathcal{M}(\bar{u})$ of $\mathcal{I}(\bar{u})$ carrying the conditionally periodic trajectories with a common rotation number. $\mathcal{I}(\bar{u})$ is backward μ-invariant. The remaining part $\mathcal{I}(\bar{u}) \setminus \mathcal{M}(\bar{u})$ is the unstable set of $\mathcal{M}(\bar{u})$*, namely, any backward trajectories of μ on the graph fall into $\mathcal{M}(\bar{u})$. As is discussed below, the last fact is important also for the computation of $\mathcal{M}(\bar{u})$.

Before stating more details we see that the results are natural consequences of properties of entropy solutions. Let us consider the situation where $\bar{u}$ is piecewise C^1 and $\bar{u}(\cdot,t)|_{x\in\mathbb{T}}$ has a certain finite number of points of discontinuity for each t. The smooth part of graph$(\bar{u}) := \{(x,t,\bar{u}(x,t)) \,|\, x,t \in \mathbb{T},\ \bar{u}(x-0,t) = \bar{u}(x+0,t)\}$ consists of the characteristic curves. The projected characteristic curves have the velocity $(\bar{u}(x(t),t),1)$.

For a point x_0 of discontinuity of $\bar{u}(\cdot,t_0)$, we have a positive number $d > 0$ such that $\bar{u}(x_0-h,t_0) > \bar{u}(x_0+h,t_0)+d$ for any small $h \geq 0$. Hence the two projected characteristic curves through (x_0-h,t_0) and (x_0+h,t_0) necessarily intersect at $s > t_0$ on a shock and these characteristic curves go away from graph$(\bar{u})$. The situation is illustrated on $\mathbb{T}^2$ in Figure 2.

For $s < t_0$, on the other hand, each characteristic curve never runs into any shocks. In other words, two projected characteristic curves never intersect for $s < t_0$, because otherwise the entropy condition is violated at the intersection. Therefore any characteristic curve $c(s) = (\tilde{x}(s), s, u(s))$ through a point of the smooth part of graph$(\bar{u})$ at $s = t_0$ stay on the smooth part for $s < t_0$, namely, $u(s) = \bar{u}(\tilde{x}(s),s)$ for $s < t_0$.

That is why $\mathcal{I}(\bar{u})$ is only backward μ-invariant in general. For each characteristic curve there exist accumulating points x^* of $\{\tilde{x}(-k) \mod 1\}_{k\in\mathbb{N}}$, which are the points of continuity of $\bar{u}(\cdot,0)$. Therefore we conclude that the characteristic curves through $(x^*,0,\bar{u}(x^*,0))$ never run into any shocks in both directions. These special points $(x^*,0,\bar{u}(x^*,0))$ yield a μ-invariant set $\mathcal{M}(\bar{u})$.

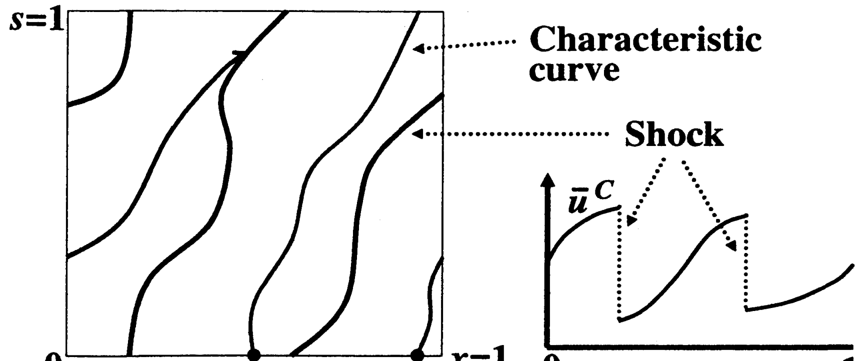

Figure 2.

The above speculation can be justified through the theory of viscosity solutions of the Hamilton-Jacobi equations of the form

$$v_t(x,t) + H(x,t,v_x(x,t)) = \text{const.} \tag{1.7}$$

We briefly refer to the notion of viscosity solutions. If we have a C^1-solution $v(x,t)$ of (1.7) defined on $\mathbb{R}\times\mathbb{R}_+$, then multiplying the derivative φ_x of any test function $\varphi\in C_0^\infty(\mathbb{R}\times\mathbb{R}_+)$ and integrating by parts over $\mathbb{R}\times\mathbb{R}_+$ we have a continuous weak solution $u=v_x$ of (1.3). The regularity of entropy solutions on $\mathbb{R}\times\mathbb{R}_+$ is worse than C^0 in general. Therefore we cannot always expect such a solution of (1.7). Note that the method of characteristics also works for the construction of local in time C^2-solutions of (1.7) with C^2-initial data.

Let $u(x,t)\in L^\infty(\mathbb{R}\times\mathbb{R}_+;\mathbb{R})$ be a weak solution of (1.3) which belongs also to $Lip_{loc}(\mathbb{R}_+;L^1(K;\mathbb{R}))$ for each compact set $K\subset\mathbb{R}$. Then we have a locally Lipschitz function v of the form $v(x,t)=\int_0^x u(y,t)dy+\int_0^t P(s)ds$ with some function $P(s)$ which is defined on $\mathbb{R}\times\mathbb{R}_+$ and satisfies (1.7) almost everywhere. Note that a locally Lipschitz function is differentiable almost everywhere. If u satisfies the entropy condition, then $v(\cdot,t)$ for each fixed t has a special property in its nondifferentiable part which is called semiconcavity. Satisfaction of the equation almost everywhere and semiconcavity are the characterizations of viscosity solution of (1.7), which are also realized by the vanishing viscosity method with the artificial viscosity νv_{xx}. We point out [3], [1] for more details.

Now we see some details of the results by E. Let $\bar{u}(x,t)$ be a $\mathbb{Z}^2$-periodic entropy solution of (1.5) with the momentum C. It follows from the above argument that there exists a $\mathbb{Z}^2$-periodic Lipschitz function $\bar{v}(x,t)$ satisfying $C+\bar{v}_x=\bar{u}$ a.e. such that

$$\bar{v}_t(x,t)+H(x,t,C+\bar{v}_x(x,t))=\bar{H}(C) \quad \text{in } \mathbb{R}^2 \qquad \text{(in the sense of viscosity solution),}$$
$$\bar{H}(C)=\iint_{\mathbb{T}^2} H(x,t,\bar{u}(x,t))dxdt.$$

Let $L^C(x,t,\xi)$ be the Legendre transform of $H(x,t,C+p)$ with respect to p. The well-known representation formula for viscosity solutions yields

$$\bar{v}(x,t)=\inf_{\eta\in AC,\ \eta(t)=x}\{\int_\tau^t L^C(\eta(s),s,\eta'(s))ds+\bar{v}(\eta(\tau),\tau)\}+\bar{H}(C)(t-\tau), \tag{1.8}$$

where AC is the class of absolutely continuous curves $\eta:\mathbb{R}\to\mathbb{R}$ and τ is an arbitrary number less than t. This is a direct generalization of the representation formula for local in time C^2-solutions via the method of characteristics. The variational problem in (1.8) is denoted by $(\mathrm{CV})_\tau^{x,t}$. There exists a C^2-minimizer $\gamma:[\tau,t]\to\mathbb{R}$ of $(\mathrm{CV})_\tau^{x,t}$ satisfying the Euler-Lagrange equation with respect to L^C. Therefore $(x(s),u(s)):=(\gamma(s) \mod 1, C+L_\xi^C(\gamma(s),s,\gamma'(s))):[\tau,t]\to\mathbb{T}\times\mathbb{R}$ satisfies (1.4).

Theorem 1.3 **([4])** *1. Existence of "one-sided minimizers": For each* $(x,t)\in\mathbb{R}^2$ *there exists a curve* $\gamma(s):(-\infty,t]\to\mathbb{R}$ *with* $\gamma(t)=x$ *such that for any interval* $[t_1,t_0]\subset(-\infty,t]$ *the restriction* $\gamma|_{[t_1,t_0]}$ *is a minimizer of* $(CV)_{t_1}^{\gamma(t_0),t_0}$ *and* $\bar{v}$ *is differentiable with respect to* x *at each point* $(\gamma(s),s)$ *with* $s<t$ *satisfying*

$$\bar{v}_x(\gamma(s),s)=L_\xi^C(\gamma(s),s,\gamma'(s)). \tag{1.9}$$

If there exists $\bar{v}_x(x,t)$, then (1.9) holds for $s=t$ and such γ is unique.

2. Existence of "two-sided minimizers": There exist curves $\gamma^*(s):\mathbb{R}\to\mathbb{R}$ such that for any interval $[t_1,t_0]\subset\mathbb{R}$ the restriction $\gamma^*|_{[t_1,t_0]}$ is a minimizer of $(CV)_{t_1}^{\gamma^*(t_0),t_0}$ and $\bar{v}$ is differentiable with respect to x at each point $(\gamma^*(s),s)$ with $s\in\mathbb{R}$ satisfying

$$\bar{v}_x(\gamma^*(s),s)=L_\xi^C(\gamma^*(s),s,\gamma^{*\prime}(s)).$$

3. $\bar{H}(C)$ depends only on C and belongs to $C^1(\mathbb{R})$. $\bar{H}'(C)$ is monotone increasing. Each one and two-sided minimizer γ,γ^* satisfies

$$\lim_{s\to-\infty}\frac{\gamma(s)}{s}=\bar{H}'(C),\quad \lim_{|s|\to\infty}\frac{\gamma^*(s)}{s}=\bar{H}'(C).$$

The following are characteristic curves of (1.5) in $\mathbb{T}^2$:

$$c(s)=(\gamma(s)\bmod 1, s\bmod 1, C+\bar{v}_x(\gamma(s),s)):(-\infty,t]\to\mathbb{T}^2\times\mathbb{R},$$
$$c^*(s)=(\gamma^*(s)\bmod 1, s\bmod 1, C+\bar{v}_x(\gamma^*(s),s)):\mathbb{R}\to\mathbb{T}^2\times\mathbb{R}.$$

They are trapped on graph$(\bar{u})=\{(x,t,C+\bar{v}_x(x,t))\,|\,x,t\in\mathbb{T},$ there exists $\bar{v}_x(x,t)\}$. In other words, if a characteristic curve $c(s)$ of (1.5) in $\mathbb{T}^2$ satisfies $c(t_0)\in$ graph$(\bar{u})$ for some $t_0\in\mathbb{R}$, then $c(s)$ is trapped on graph$(\bar{u})$ for $s\le t_0$. Let $\Omega(\gamma)$ be the set of all the ω-limit points of $\{\gamma(-k) \mod 1\,|\,k\in\mathbb{N},-k\le t_0\}$, where $\gamma:(-\infty,t_0]\to\mathbb{R}$ is a one-sided minimizer of (1.8). Each one-sided minimizer through a point of $\Omega(\gamma)$ can be extended to a two-sided minimizer. We define

$$\Gamma(\bar{u}):=\left\{\Big(\gamma^*(s)\bmod 1, s\bmod 1, C+\bar{v}_x(\gamma^*(s),s)\Big)\,\Big|\,\gamma^*(0)\in\bigcup_\gamma\Omega(\gamma),\ s\in\mathbb{R}\right\},$$

where γ^* are two-sided minimizers. $\Gamma(\bar{u})$ is a subset of graph$(\bar{u})$. It is proved in [4] that for each one-sided minimizer γ there exists a two-sided minimizer γ^* such that

$$|\gamma(s)-\gamma^*(s)|\to 0\quad\text{as } s\to-\infty \tag{1.10}$$

and therefore each $c(s)$ trapped on graph$(\bar{u})$ for $s\le t_0$ falls into $\Gamma(\bar{u})$ as $s\to-\infty$. This is based on the simple fact that, for two different minimizers $\gamma,\tilde{\gamma}$ associated with $\bar{u}$, $(\gamma(s),s)$ and $(\tilde{\gamma}(s),s)$ never intersect. For $s\to+\infty$, each $c(s)$ through a point of graph$(\bar{u})\setminus\Gamma(\bar{u})$ runs into a shock and goes away from graph$(\bar{u})$ in general. $\mathcal{M}(\bar{u}):=\Gamma(\bar{u})\cap(\mathbb{T}\times\{0\}\times\mathbb{R})$ is a μ-invariant set carrying the trajectories of μ with the rotation number $\bar{H}'(C)$. For any $\alpha\in\mathbb{R}$, there exists C such that $\bar{H}'(C)=\alpha$. As is proved in [4], $\mathcal{M}(\bar{u})$ is a closed subset of a Lipschitz curve. We call $\mathcal{M}(\bar{u})$ *the Aubry-Mather set associated with the entropy solution* $\bar{u}$. From now on, the term "Aubry-Mather set" means $\mathcal{M}(\bar{u})$.

2 Difference Approximation to Aubry-Mather sets

Let us consider the computational aspects of the issue. The main interest is the computation of $\mathbb{Z}^2$-periodic entropy solutions of (1.5) and Aubry-Mather sets. As an effective approach to entropy solutions, we have not only the smooth approximation by the vanishing viscosity method but also the difference approximation. The latter approach is convenient also for numerical simulations. The rigorous treatment of the difference approximation to entropy solutions is found in lots of works (e.g. [8]). Many of them are on entropy solutions of initial value problems.

Concerning the difference approximation of periodic entropy solutions, we find in [6] a difference scheme to (1.6) for numerical tests of Theorem 1.2 (but there is no theoretical argument for the scheme). We also point out [10], in which the existence of $\mathbb{Z}^2$-periodic entropy solutions of (1.5) is proved with the Lax-Friedrichs difference scheme and Brouwer's fixed point theorem. The idea is to regard $\mathbb{Z}^2$-periodic difference solutions as fixed points of the time-1 map derived from the semigroup of the difference scheme.

We present two methods which are more constructive and easily simulated [9]: The one is based on the long time behavior of difference solutions derived from the Lax-Friedrichs difference scheme. The other is based on Newton's method for the fixed points of the time-1 map. The convergence of these methods are established. Our results can be extended to general types of the forced Burgers equation (1.3) with $H(x,s,u)$ which is strictly convex in u, assuming additional conditions.

We construct $\mathbb{Z}^2$-periodic entropy solutions of the forced Burgers equation (1.5) with a $\mathbb{Z}^2$-periodic C^2-function F. Our basic tool is the two-step Lax-Friedrichs difference scheme in $\mathbb{T}\times\mathbb{R}_{t\ge0}$. Let N,K be natural numbers. The mesh size is defined as $\Delta x := N^{-1}$, $\Delta t := K^{-1}$. Set $\lambda := \Delta t/\Delta x$, $x_n := n\Delta x \in [0,1]$ $(n=0,1,2,\cdots,N)$, $t_k := k\Delta t \in [0,+\infty)$ $(k=0,1,2,\cdots)$. The solution to the initial value problem

$$\begin{cases} u_t(x,t)+u(x,t)u_x(x,t)=F_x(x,t) \ \text{ in } \mathbb{T}\times\mathbb{R}_+, \\ u(x,0)=g(x) \ \text{ on } \mathbb{T} \end{cases}$$

is replaced with the family of vectors

$$u^k=(u^k_0,u^k_1,\cdots,u^k_{N-1})\in\mathbb{R}^N \quad (k=0,1,2,\cdots),$$

where $u^0=(g(x_0),\cdots,g(x_{N-1}))$. This is called the difference solution with the initial value u^0. Each difference solution u^k with an initial value $u^0\in\mathbb{R}^N$ is given in the following way: Let $\Delta y := \frac{1}{2}\Delta x$, $\Delta\tau := \frac{1}{2}\Delta t$, $y_m := m\Delta y\in[0,1]$ $(m=0,1,2,\cdots,2N)$, $\tau_l := l\Delta\tau\in[0,+\infty)$ $(l=0,1,2,\cdots)$; Define

$$u^k_n := W^{2k}_{2n},$$

where W^l_m are computed for each $l=0,1,2,\cdots$ and each $m\in\{0,1,2,\cdots,2N\}$ satisfying $l+m=$ even through the difference equation with the periodic boundary condition

$$\begin{cases} \dfrac{W^{l+1}_{m+1}-\frac{(W^l_{m+2}+W^l_m)}{2}}{\Delta\tau}+\dfrac{1}{2}\dfrac{(W^l_{m+2})^2-(W^l_m)^2}{2\Delta y}=\dfrac{F(y_{m+2},\tau_l)-F(y_m,\tau_l)}{2\Delta y}, \\ W^l_{m\pm2N}=W^l_m, \quad W^0_{2n}=u^0_n. \end{cases}$$

Set $u^k_{n\pm N}:=u^k_n$. The semigroups $u^0\mapsto u^k$, $W^0\mapsto W^l$ are denoted by $\psi^k:\mathbb{R}^N\to\mathbb{R}^N$, $\Psi^l:\mathbb{R}^N\to\mathbb{R}^N$ respectively. ψ^k,Ψ^l are C^2. Since F is $\mathbb{Z}^2$-periodic, the time-1 map

$$\phi := \psi^K = \Psi^{2K}$$

is well-defined. Note that $\psi^{KT+k} = \psi^k \circ \phi^T$ for each $T \in \mathbb{N}$, where ϕ^T is the T-iteration of ϕ. The following function is called an approximate solution of (1.5) in $\mathbb{T} \times \mathbb{R}_{\geq 0}$:

$$u_\Delta(x,t) := \frac{u^k_{n+1} - u^k_n}{\Delta x}(x - x_n) + u^k_n \quad \text{for } x \in [x_n, x_{n+1}],\ t \in [t_k, t_{k+1}),\ \Delta = (\Delta x, \Delta t).$$

It follows from a simple calculation that the average in x of each difference solution u^k at each k and therefore that of the approximate solution $u_\Delta(x,t)$ is conservative, namely

$$C(u^0) := \sum_{n=0}^{N-1} u^0_n \Delta x \equiv \sum_{n=0}^{N-1} u^k_n \Delta x \equiv \int_0^1 u_\Delta(x,t)dx.$$

The value $C = C(u^0)$ is called the momentum of a solution. $u^k(C), u^C_\Delta(x,t)$ denote $u^k, u_\Delta(x,t)$ with the momentum C. u^k is said to be a periodic difference solution, if $u^{k+K} = u^k$ for all $k = 0, 1, 2, \cdots$. This is equivalent to the relation $\phi(u^0) = u^0$. For each $v = (v_0, \cdots, v_{N-1}) \in \mathbb{R}^N$, the following are introduced:

$$\| v \|_\infty := \max_{0 \leq n \leq N-1} |v_n|, \quad \| v \|_1 := \sum_{n=0}^{N-1} |v_n|, \quad Var.[v] := \sum_{n=0}^{N-1} |v_{n+1} - v_n| \quad (v_N = v_0).$$

Theorem 2.1 ([9]) *Let* $M = \sqrt{\max_{(x,t)\in\mathbb{T}^2} 2F_{xx}(x,t)}$, $r > 0$, $\tilde{r} > M$, $0 < \lambda_0 < (r + \tilde{r})^{-1}$. *Fix any natural numbers* N, K *so that* $\Delta x = N^{-1}$, $\Delta t = K^{-1}$ *satisfy*

$$\lambda_0 \leq \frac{\Delta t}{\Delta x} = \lambda < \left(r + \tilde{r}\right)^{-1}, \quad \tilde{r} < 2\Delta t^{-1}, \quad \Delta t \leq \Delta x. \tag{2.1}$$

Initial values are restricted to the set

$$B_{r,\tilde{r}} := \left\{ v \in \mathbb{R}^N \mid -r \leq \sum_{n=0}^{N-1} v_n \Delta x \leq r,\ \max_{0 \leq n \leq N-1} \frac{v_{n+1} - v_n}{\Delta x} \leq \tilde{r} \quad (v_N = v_0) \right\}.$$

1. For each $u^0 \in B_{r,\tilde{r}}$, *there exists the unique difference solution* $u^k = \psi^k(u^0)$, *which satisfies for any* k

$$\max_{0 \leq n \leq N-1} \frac{u^k_{n+1} - u^k_n}{\Delta x} \leq \tilde{r}, \quad \| u^k \|_\infty \leq |C(u^0)| + \tilde{r}, \quad Var.[u^k] \leq 2\tilde{r}.$$

2. For each $C \in [-r, r]$, *there exists the unique periodic difference solution* $\bar{u}^k(C)$ *with the momentum* C, *which satisfies for any* k

$$\max_{0 \leq n \leq N-1} \frac{\bar{u}^k_{n+1}(C) - \bar{u}^k_n(C)}{\Delta x} \leq M, \quad \| \bar{u}^k(C) \|_\infty \leq |C| + M, \quad Var.[\bar{u}^k(C)] \leq 2M.$$

3. $\bar{u}^k(C)$ *is stable: Any other difference solutions* $u^k(C)$ *with the same momentum* C *satisfy* $\| u^k(C) - \bar{u}^k(C) \|_1 \to 0$ *as* $k \to \infty$.

4. Any two difference solutions $u^k(C), v^k(C)$ *with the same momentum* C *satisfy* $\| u^k(C) - v^k(C) \|_1 \to 0$ *as* $k \to \infty$.

5. The decay rate is exponential: There exist constants $a > 0$ *and* $\rho < 1$ *which may depend on* $\Delta x, \Delta t$ *such that any two difference solutions* $u^k(C), v^k(C)$ *with the same momentum* C *satisfy* $\| u^{TK} - v^{TK} \|_1 = \| \phi^T(u^0) - \phi^T(v^0) \|_1 \leq a\rho^T$ *for each* $T \in \mathbb{N}$.

6. *$\bar{u}^k_n(C)$ is a strictly increasing C^1-function with respect to C for each fixed n, k.*

7. *Newton's method to the equation $\phi(u) = u$ is convergent.*

8. *There exists a sequence $\bar{u}^C_{\Delta_i}$ of $\{\bar{u}^C_\Delta(x,t) \mid \Delta = (\Delta x, \Delta t)$ satisfies (2.1)$\}$, where $\Delta_i \to 0$, which converges to a $\mathbb{Z}^2$-periodic entropy solution $\bar{u}^C$ of (1.5) in $C^0(\mathbb{T}; L^1(\mathbb{T}))$.*

9. *$\bar{u}^C$ belongs to $Lip(\mathbb{T}; L^1(\mathbb{T}))$, has the momentum C and satisfies for a.e. $x, y \in \mathbb{T}$, $x \neq y$ and all $t, \tau \in \mathbb{T}$*

$$\frac{\bar{u}^C(y,t) - \bar{u}^C(x,t)}{y-x} \le M, \quad Var.[\bar{u}^C(\cdot,t)] \le 2M, \quad \| \bar{u}^C(\cdot,t) \|_{L^\infty(\mathbb{T})} \le |C| + M,$$

$$\| \bar{u}^C(\cdot, t+\tau) - \bar{u}^C(\cdot,t) \|_{L^1(\mathbb{T})} \le 2A|\tau|, \quad A = \frac{2M}{\lambda_0} + \max_{(x,t)\in\mathbb{T}^2} |F_x(x,t)|.$$

Although we add no viscosity term, the above assertions are reminiscent of the results on (1.6) found in [6]. The $\| \cdot \|_1$-contraction property is because of the so-called *numerical viscosity* of the difference scheme. Our proof follows Oleinik [**8**], where the Δ-independent one-sided estimate for $\frac{u^k_{n+1} - u^k_n}{\Delta x}$ is established and then the argument on the functions of bounded variation is used. However we need further investigations, since we deal with the long time behavior of the difference scheme with a fixed mesh Δ, namely, we consider the limit $t_k \to \infty$ with a fixed mesh Δ at first and then take the limit $\Delta \to 0$. Note that these two limit processes are not commutable in general, since the decay exponent $\rho < 1$ in 5. may be arbitrarily close to 1 as $\Delta \to 0$.

Now we compute Aubry-Mather sets. A simulation of a KAM torus of μ can be easily made through a computation of an initial value problem of (1.4), with the Runge-Kutta method for instance. Aubry-Mather sets may exist in the region with no KAM tori, filled with chaotic trajectories. In such a region, it is quite difficult to directly compute trajectories on the Aubry-Mather sets, because it is not easy to find the initial values of such trajectories and chaotic motions are very sensitive to initial values. Even if we find the appropriate initial value with the "*double*" accuracy of C Language, the slight error is likely to soon cause a behavior totally different from the theoretically expected one.

It follows from (1.9) that each one-sided minimizer $\gamma : (-\infty, t_0] \to \mathbb{R}$ satisfies

$$\gamma'(s) = \bar{u}(\gamma(s), s). \tag{2.2}$$

We are interested in the uniqueness of the initial value problems for (2.2) and the simulations of the problems for $s \to -\infty$ in order to obtain approximations of Aubry-Mather sets. The uniqueness argument is rather difficult, since $\bar{u}$ may have countably many shocks within $x, t, \in \mathbb{T}$. We state a sufficient condition on the uniqueness, showing a regularity property of $\mathbb{Z}^2$-periodic entropy solutions.

Theorem 2.2 ([9]) *Suppose that there exists an interval $[x_1, x_2] \subset [0,1)$ on which $\bar{u}(\cdot, t)$ is continuous. Then $\bar{u}$ is continuous on the following stripe region of $\mathbb{R}^2$:*

$$G(t) := \bigcup_{s\in(-\infty,t]} [\gamma_1(s), \gamma_2(s)] \times \{s\},$$

where γ_1, γ_2 are one-sided minimizers satisfying $\gamma_1(t) = x_1, \gamma_2(t) = x_2$. Furthermore $\bar{u}$ is Lipschitz continuous with respect to x on $G(t-\varepsilon)$ for $\varepsilon > 0$ with the Lipschitz constant $L = \frac{3}{\delta}$, where $\delta = \min\{\varepsilon, \ (2\max_{(x,t)\in\mathbb{T}^2} |F_{xx}(x,t)|)^{-\frac{1}{2}}\}$.

The proof is based on the simple fact that, for two different minimizers $\gamma, \tilde{\gamma}$ associated with $\bar{u}$, $(\gamma(s), s)$ and $(\tilde{\gamma}(s), s)$ never intersect.

We simulate the initial value problems to (2.2) through an approximation $\bar{u}_\Delta$ of $\bar{u}$ and obtain approximations $c_\Delta(s)$ of $c(s)$. Our difference scheme for the task is the following:

$$\frac{\gamma^{-(k+1)} - \gamma^{-k}}{-\Delta t} = \bar{u}_\Delta(\gamma^{-k}, t_{-k}), \quad k \geq 0.$$

We define $c_\Delta(s) : (-\infty, 0] \to \mathbb{T}^2 \times \mathbb{R}$ as

$$c_\Delta(s) := (\gamma^{-(k+1)} \mod 1, t_{-(k+1)} \mod 1, \bar{u}_\Delta(\gamma^{-(k+1)}, t_{-(k+1)})) \text{ for } s \in [t_{-(k+1)}, t_{-k}).$$

(1.10) implies that $c_\Delta(s)$ are likely to be absorbed as $s \to -\infty$ in a subset $\Gamma(\bar{u}_\Delta)$ of graph$(\bar{u}_\Delta)$. $\Gamma(\bar{u}_\Delta)$ is considered as an approximation of $\Gamma(\bar{u})$. Unfortunately this method is not mathematically justified at this stage. An error estimate between $\mathbb{Z}^2$-periodic difference solutions and genuine entropy solutions should also be established.

Now we show numerical results of $\bar{u}_\Delta(x,t)$, $c_\Delta(s)$, $\mathcal{M}(\bar{u}) := \Gamma(\bar{u}_\Delta) \cap (\mathbb{T} \times \{0\} \times \mathbb{R})$ and the dynamics of the time-1 map μ of (1.4), obtained by the "*double*" accuracy of C Language. (X, Y, Z) denotes the axes of $\mathbb{T}^2 \times \mathbb{R}$, where X corresponds to space and Y to time. We take an example

$$F(x,s) = -\frac{1}{10}\cos(4\pi x)\sin(2\pi s).$$

Figure 3 shows the graphs of $\bar{u}^C_\Delta$ with $C = 0.6$, -0.2, -0.78 on the section $Y = 0$ together with the Aubry-Mather sets associated with them, which lie in the chaotic region. Small scattered (green) dots are formed by one chaotic trajectory of μ computed by the Runge-Kutta method. We observe the following: Each Aubry-Mather set consists of periodic trajectories visualized with large (blue) dots. When C increases, the position of shocks of $C = 0.6$ moves right along with the upper boundary of the chaotic region. Similarly when C decreases the position of shocks of $C = -0.78$ moves left along with the lower boundary of the chaotic region. And finally we have the continuous graphs of $\bar{u}^C_\Delta$ which coincide with the upper and lower boundaries of the chaotic region. The nondifferentiable parts of the boundaries are the Aubry-Mather sets. The Aubry-Mather set associated with $C = -0.2$ consists of two periodic trajectories with the rotation number $-\frac{1}{4}$. The Runge-Kutta method for (1.4) with an initial point on the Aubry-Mather set provides a solution which behaves periodically only within a short span of time. Even though the initial point is additionally re-adjusted with the "*long-double*" accuracy, $|s| \cong 25$ is the longest time for which the solution keeps the periodic state!

Figure 4 indicates the graph of $\bar{u}^C_\Delta$ with $C = -0.17965244477$ on the section $Y = 0$ together with the Aubry-Mather set associated with it. The Aubry-Mather set drawn by (blue) dots consists of the only one periodic trajectory with the rotation number $-\frac{3193}{12774} = -0.249960857992798$. The value of $C = -0.17965244477$ would be close to the critical value at which the rotation number changes from $-\frac{1}{4}$ to an irrational rotation number, since $C = -0.1796524448$ yields the Aubry-Mather set consists of two periodic trajectories with the rotation number $-\frac{1}{4}$. If $\bar{u}^C$ has a shock and $\mathcal{M}(\bar{u}^C)$ has an irrational rotation number, then $\bar{u}^C$ must have infinitely many shocks with the arbitrarily small sizes within $(x,t) \in \mathbb{T}^2$. In such a case the profile of $\bar{u}^C_\Delta$ may be rather different from that of $\bar{u}^C$, because smaller shocks cannot be captured by the difference approximation with fixed $\Delta x, \Delta t$. Here is a question: When $\Delta x, \Delta t$ are fixed and C is varied, does there exist $\mathcal{M}(\bar{u}^C_\Delta)$ having an arbitrarily given rotation number?

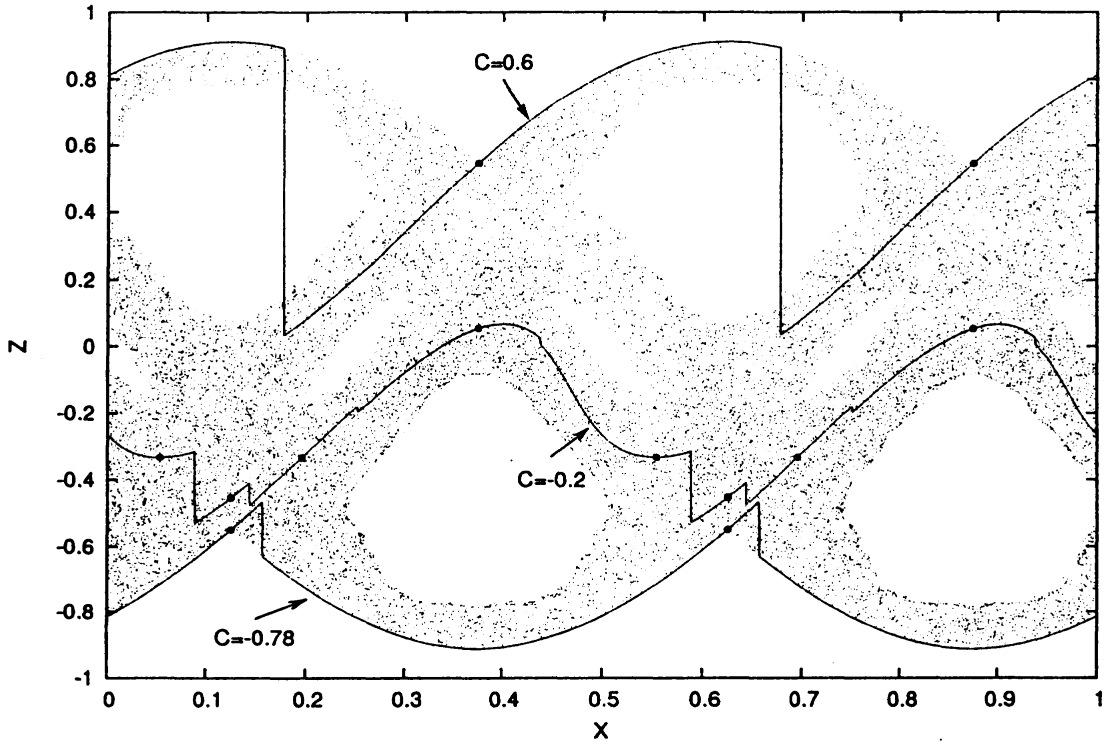

Figure 3.

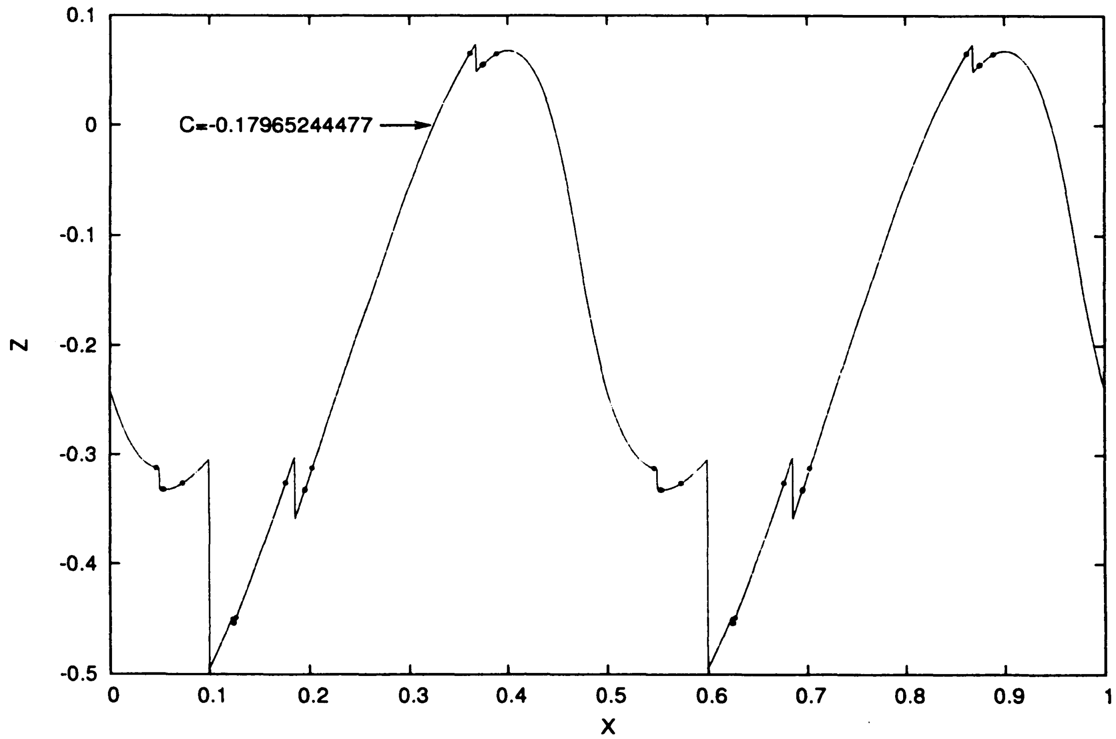

Figure 4.

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