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Lefschetz elements of Artinian Gorenstein algebras and Hessians of homogeneous polynomials

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This article is based on my joint work with Junzo Watanabe [8]. The Lefschetz property is a ring-theoretic abstraction of the Hard Lefschetz Theorem for compact Kähler manifolds. The following are fundamental problems on the study of the Lefschetz property for Artinian graded algebras:

Problem 0.1. For a given graded Artinian algebra A, decide whether or not A has the strong (or weak) Lefschetz property.

Problem 0.2. When a graded Artinian algebra A has the strong Lefschetz property, determine the set of Lefschetz elements in the part A_1 of degree one.

In this work, we give a characterization of the Lefschetz elements in Artinian Gorenstein rings over a field k of characteristic zero in terms of the higher Hessians. As an application, we give new examples of Artinian Gorenstein rings which do not have the strong Lefschetz property.

1 Lefschetz properties

Definition 1.1. Let $A=\oplus_{d=0}^{D}A_d$, $A_D\neq 0$, be a graded Artinian algebra.
(1) We say that A has the strong Lefschetz property if there exits an element $L\in A_1$ such that the multiplication map

$$\times L^d : A_i \rightarrow A_{i+d}$$

is of full rank (i.e. injective or surjective) for all $0 \leq i \leq D$ and $0 \leq d \leq D-i$. We call $L \in A_1$ with this property a strong Lefschetz element.
(2) If we assume the existence of $L \in A_1$ such that

$$\times L : A_d \longrightarrow A_{d+1}$$

is of full rank for $d = 0, \ldots, D-1$, we say that A has the weak Lefschetz property.

In the following sections, we mainly investigate the strong Lefschetz property for Artinian Gorenstein algebras.

2 Artinian Gorenstein algebra

Definition 2.1. (See [10, Chapter 5, 6.5].) A finite-dimensional graded k-algebra $A = \oplus_{d=0}^{D} A_d$ is called the Poincaré duality algebra if $\dim_k A_D = 1$ and the bilinear pairing

$$A_d \times A_{D-d} \longrightarrow A_D \cong k$$

is non-degenerate for $d = 0, \ldots, [D/2]$.

We will need two characterizations of the Artinian Gorenstein algebra in order to show our main theorem in the next section. The proofs of the following propositions also can be found in [8].

Proposition 2.1. (See [3].) *A graded Artinian k-algebra A is a Poincaré duality algebra if and only if A is Gorenstein.*

Proposition 2.2. (See [1],[2],[4].) *Let I be an ideal of $Q = k[X_1, \ldots, X_n]$ and $A = Q/I$ the quotient algebra. Denote by $\mathfrak{m}$ the maximal ideal $(X_1, \ldots, X_n)$ of Q. Then $\sqrt{I} = \mathfrak{m}$ and the k-algebra A is Gorenstein if and only if there exists a polynomial $F \in R = k[x_1, \ldots, x_n]$ such that $I = \mathrm{Ann}_Q F$.*

3 Characterization of Lefschetz elements

In this section we discuss the set of the Lefschetz elements for graded Artinian Gorenstein rings $A = k[X_1, \ldots, X_n]/\mathrm{Ann}_Q F$ with the standard grading.

Definition 3.1. Let G be a polynomial in $k[x_1, \dots, x_n]$. When a family $\mathbf{B}_d = \{\alpha_i^{(d)}\}_i$ of homogeneous polynomials of degree $d > 0$ is given, we call the polynomial

$$\det\left((\alpha_i^{(d)}(X)\alpha_j^{(d)}(X)G(x))_{i,j=1}^{\#\mathbf{B}_d}\right) \in k[x_1, \dots, x_n]$$

the d-th Hessian of G with respect to $\mathbf{B}_d$, and denote it by $\mathrm{Hess}_{\mathbf{B}_d}^{(d)}G$. We denote the d-th Hessian simply by $\mathrm{Hess}^{(d)}G$ if the choice of $\mathbf{B}_d$ is clear.

Theorem 3.1. ([11, Theorem 4], [8, Theorem 3.1]) *Fix an arbitrary k-linear basis $\mathbf{B}_d$ of A_d for $d = 1, \dots, [D/2]$. An element $L = a_1X_1 + \cdots + a_nX_n \in A_1$ is a strong Lefschetz element of $A = Q/\mathrm{Ann}_Q F$ if and only if $F(a_1, \dots, a_n) \neq 0$ and*

$$(\mathrm{Hess}_{\mathbf{B}_d}^{(d)}F)(a_1, \dots, a_n) \neq 0$$

for $d = 1, \dots, [D/2]$.

The proof is easy if we take the propositions in the previous section for granted. Fix the identification $[\ \ ] : A_D \xrightarrow{\sim} k$ by $[\omega(X)] := \omega(X)F(x)$ for any $\omega(X) \in A_D$. Note that $\omega(X)F(x) \in k$, because $\deg\omega = \deg F = D$. Since A is a Poincaré duality algebra, the necessary and sufficient condition for $L = a_1X_1 + \cdots + a_nX_n \in A_1$ to be a strong Lefschetz element is that the bilinear pairing

$$\begin{array}{ccccc} A_d \times A_d & \to & A_D & \cong & k \\ (\xi, \eta) & \mapsto & L^{D-2d}\xi\eta & \mapsto & [L^{D-2d}\xi\eta] \end{array}$$

is non-degenerate for $d = 0, \dots, [D/2]$. Therefore L is a Lefschetz element if and only if the matrix

$$(L^{D-2d}\alpha_i^{(d)}(X)\alpha_j^{(d)}(X)F(x))_{ij}$$

has nonzero determinant. For a homogeneous polynomial $G(x_1, \dots, x_n) \in k[x_1, \dots, x_n]$ of degree d, we have the formula

$$(a_1X_1 + \cdots + a_nX_n)^d G(x_1, \dots, x_n) = d!G(a_1, \dots, a_n),$$

so

$$L^{D-2d}\alpha_i^{(d)}(X)\alpha_j^{(d)}(X)F(x) = (D-2d)!\alpha_i^{(d)}(X)\alpha_j^{(d)}(X)F(x)|_{(x_1,\dots,x_n)=(a_1,\dots,a_n)}.$$

This completes the proof of the theorem.

4 Set of Lefschetz elements

In this section we discuss the set of Lefschetz elements for some simple examples of Gorenstein algebras with the strong Lefschetz property based on Corollary 3.1.

Example 4.1. Let us consider the Gorenstein ring $A = k[X_1, \dots, X_n]/\mathrm{Ann}_Q F$ associated to the Fermat type polynomial

$$F = \sum_{i=1}^{n} x_i^n - n(n-1)s \prod_{i=1}^{n} x_i,$$

where $s \in k$ is a parameter. One can check that A has the strong Lefschetz property for any $s \in k$ by computation of Hessians. Let us see the explicit condition for the Lefschetz element for $n = 3$. For the polynomial $F = x^3 + y^3 + z^3 - 6s \cdot xyz$, A has the following structure:

$$\begin{array}{ll} \text{Case } s^3 \neq 0, 1, & A \cong k[X, Y, Z]/(sX^2 + YZ, sY^2 + XZ, sZ^2 + XY), \\ \text{Case } s = 0, & A \cong k[X, Y, Z]/(X^3 - Y^3, X^3 - Z^3, XY, YZ, XZ), \\ \text{Case } s^3 = 1, & A \cong k[X, Y, Z]/(sX^2 + YZ, sY^2 + XZ, sZ^2 + XY, XZ^2, YZ^2). \end{array}$$

The Hilbert function of A is $\mathrm{Hilb}(A) = (1, 3, 3, 1)$ for all $s \in k$. The condition for $L = aX + bY + cZ \in A_1$ to be a strong Lefschetz element is that

$$a^3 + b^3 + c^3 - 6s \cdot abc \neq 0$$

and

$$s^2a^3 + s^2b^3 + s^2c^3 - (1 - 2s^3)abc \neq 0.$$

This means that the projectivization of the set of non-Lefschetz elements in $\mathbb{P}(A_1) \cong \mathbb{P}^2$ is the union of two elliptic curves intersectiong at each other's inflection points.

Example 4.2. In [7], the set of the Lefschetz elements for the coinvariant algebra of the finite Coxeter group is determined except for type H_4. Let V be the standard reflection representation of the finite irreducible Coxeter group W. Then W acts on the polynomial ring $R = \mathrm{Sym}_{\mathbb{R}} V^*$ and the W-invariant subalgebra R^W is generated by the fundamental W-invariants $f_1, \dots, f_r$, $r = \dim V$. The coinvariant algebra R_W is defined as the quotient algebra

$R/(f_1, \ldots, f_r)$. It is known that R_W is Gorenstein (see e.g. [10, Theorem 7.5.1]). When W is crystallographic, R_W is isomorphic to the cohomology ring of the corresponding flag variety. In [7], it was shown that the set of Lefschetz elements in $V^* = (R_W)_1$ is the complement of the union of the reflection hyperplanes. For crystallographic case, their argument is based on the ampleness criterion for the $\mathbb{R}$-divisors on the flag variety, so it is applicable only when the field k of coefficients is the field $\mathbb{R}$ of real numbers.

Let us consider the case $W = S_3$ and

$$R_W = \mathbb{R}[X, Y, Z]/(X + Y + Z, XY + YZ + ZX, XYZ).$$

The algebra R_W is also given by $R_W = \mathbb{R}[X, Y, Z]/\mathrm{Ann}\,\Delta$ with $\Delta = (x - y)(x - z)(y - z)$. The degree one part $(R_W)_1$ has a linear basis $\mathbf{B}_1 = \{X, Y\}$. Then we have

$$\mathrm{Hess}^{(1)}_{\mathbf{B}_1}\Delta = -4(x^2 + y^2 + z^2 - xy - yz - zx),$$

which is a negative definite quadratic form. Hence the set of the Lefschetz elements is given by

$$\{(x, y, z) \mid \Delta(x, y, z) \neq 0\} \subset V^*.$$

If we work in $V^*_{\mathbb{C}}$, we have to take care of the condition $x^2 + y^2 + z^2 - xy - yz - zx \neq 0$, too.

Remark 4.1. Recently, a purely algebraic proof of the strong Lefschetz property for coinvariant algebras of finite Coxeter groups has been given by McDaniel [6] except for types E_8 and H_4.

Example 4.3. Let us consider the toric variety $\mathbf{X}_\Sigma$ associated to a fan Σ of the lattice $N = \mathbb{Z}^r$. We assume that $\mathbf{X}_\Sigma$ is compact and nonsingular. Let $\Sigma(1) = \{\rho_1, \ldots, \rho_k\}$ be the set of 1-dimensional cones (rays) of Σ. Each ray $\rho \in \Sigma(1)$ determines a torus invariant divisior $D(\rho)$ on $\mathbf{X}_\Sigma$. Denote by $n(\rho) \in N$ the primitive generator of a ray $\rho \in \Sigma(1)$. Let Q be a polynomial ring in the variables X_ρ, $\rho \in \Sigma(1)$. We defines the ideals $I, J \subset Q$ by

$$I := \Big(\sum_{\rho\in\Sigma(1)} \langle m, n(\rho)\rangle X_\rho \mid m \in M\Big),$$

and

$$J := (X_{\rho_1} \cdots X_{\rho_s} \mid \rho_1, \ldots, \rho_s \text{ are distinct and do not generate a cone in } \Sigma).$$

Then the cohomology ring of $\mathbf{X}_\Sigma$ has the following structure:

$$H^*(\mathbf{X}_\Sigma) \cong Q/I + J.$$

If we define a polynomial $F(x)$ as the intersection number

$$F(x) := \left(\sum_{i=1}^{k} x_k D(\rho_i) \right)^r,$$

then we have a presentation of the cohomology ring of $\mathbf{X}_\Sigma$ as follows:

$$H^*(\mathbf{X}_\Sigma) = Q/\mathrm{Ann}_Q F.$$

I. (Hirzebruch surfece)
For an integer a, consider the complete fan of $N = \mathbb{Z}n_1 + \mathbb{Z}n_2$ determined by the rays

$$\rho_1 := \mathbb{R}_+ n_1, \rho_2 := \mathbb{R}_+ n_2, \rho_3 := \mathbb{R}_+(-n_1 + an_2), \rho_4 := \mathbb{R}_+(-n_2).$$

The corresponding toric surface is the Hirzebruch surface $\mathcal{F}_a$, which has the structure of a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$:

$$\mathcal{F}_a = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(a)) \to \mathbb{P}^1.$$

The cohomology ring $H^*(\mathcal{F}_a)$ is generated by the classes of the fiber $f := D(\rho_1)$ and the section $s := D(\rho_2)$. We can see that the intersection form on $H^2(\mathcal{F}_a)$ is given by

$$f^2 = 0, fs = 1, s^2 = -a.$$

Then it is easy to see the following:
(i) The element $\alpha f + \beta s \in H^2(\mathcal{F}_a)$ is ample $\Leftrightarrow$ $\alpha > a\beta$ and $\beta > 0$.
(ii) The element $\alpha f + \beta s \in H^2(\mathcal{F}_a)$ is Lefschetz $\Leftrightarrow$ $\beta \neq 0$ and $2\alpha \neq a\beta$.
Though the classes of form $\alpha(f + as)$, $\alpha > 0$, belong to the boundary of the ample cone, they are Lefschetz. In this example, we observe that points on the wall of the ample cone may be the Lefschetz element.

II. (non-projective toric 3-fold.)
The simplest example of non-projective compact toric manifolds is due to Miyake and Oda [9]. Let $N = \mathbb{Z}n_1 + \mathbb{Z}n_2 + \mathbb{Z}n_3$ be the lattice of rank 3. We

define a complete fan Σ of N consisting of the following 3-dimensional cones and their faces:

$$\begin{array}{lll} \mathbb{R}_+n_1+\mathbb{R}_+n_2+\mathbb{R}_+n_3, & \mathbb{R}_+n_1+\mathbb{R}_+n_2+\mathbb{R}_+n_1', & \mathbb{R}_+n_2+\mathbb{R}_+n_3+\mathbb{R}_+n_2', \\ \mathbb{R}_+n_1+\mathbb{R}_+n_3+\mathbb{R}_+n_3', & \mathbb{R}_+n_1+\mathbb{R}_+n_1'+\mathbb{R}_+n_3', & \mathbb{R}_+n_2+\mathbb{R}_+n_2'+\mathbb{R}_+n_1', \\ \mathbb{R}_+n_3+\mathbb{R}_+n_2'+\mathbb{R}_+n_3', & \mathbb{R}_+n_0+\mathbb{R}_+n_1'+\mathbb{R}_+n_2', & \mathbb{R}_+n_0+\mathbb{R}_+n_2'+\mathbb{R}_+n_3', \\ \mathbb{R}_+n_0+\mathbb{R}_+n_1'+\mathbb{R}_+n_3', & & \end{array}$$

where

$$n_0 := -n_1-n_2-n_3, n_1' := n_0+n_1, n_2' := n_0+n_2, n_3' := n_0+n_3.$$

There exist no strictly upper convex Σ-linear support functions, so the fan Σ gives an example of compact non-projective toric manifold $\mathbf{X}_\Sigma$. Let us denote by $\rho_i := \mathbb{R}_+n_i$, $i=0,1,2,3$, the rays of Σ. The cohomology ring of $\mathbf{X}_\Sigma$ has the presentation

$$H^*(\mathbf{X}_\Sigma) = \mathbb{C}[X_0, X_1, X_2, X_3]/\mathrm{Ann}F,$$

where

$$\begin{aligned} F &= (x_0D(\rho_0)+x_1D(\rho_1)+x_2D(\rho_2)+x_3D(\rho_3))^3 \\ &= -4x_0^3-x_1^3-x_2^3-x_3^3+3x_1x_2x_3. \end{aligned}$$

Since the Hessian of F is given by

$$\mathrm{Hess}(F) = -1296x_0(x_1^3+x_2^3+x_3^3-3x_1x_2x_3),$$

we can see that $H^*(\mathbf{X}_\Sigma)$ has the strong Lefschetz property, though $\mathbf{X}_\Sigma$ is non-projective. The set of non-Lefschetz elements in $H^2(\mathbf{X}_\Sigma, \mathbb{C})$ consists of four hyperplanes and one cubic hypersurface.

5 Gorenstein algebras which do not have the strong Lefschetz property

The result in Section 3 shows that a polynomial F gives an example of Gorenstein algebra which does not have the strong Lefschetz property if one of the higher Hessians of F is identically zero. The polynomial $F = x_0u^2+x_1uv+x_2v^2$ is the simplest example whose Hessian vanishes, but no variables can be eliminated by a linear transformation of the variables (see [11, Example 1]).

Here we give examples of forms F such that Hess $F \neq 0$ and $\mathrm{Hess}^{(2)} F = 0$. By using these forms we can also give examples of Gorenstein algebras $A = Q/\mathrm{Ann}_Q F$ which do not satisfy the strong Lefschetz property.

Example 5.1. Let us consider the polynomial

$$F := \sum_{j=0}^{n} x_j^2 u^{n-j} v^j \in k[u, v, x_0, \dots, x_n]$$

and the corresponding algebra $A = Q/\mathrm{Ann}_Q F$, where $Q = k[U, V, X_0, \dots, X_n]$, $U = \partial/\partial u$, $V = \partial/\partial v$ and $X_i = \partial/\partial x_i$. The Hessian of F with respect to the basis $U, V, X_0, \dots, X_n$ of degree one is expressed as follows:

$$\begin{aligned} &\text{Hess } F = \\ &2^{n+1}(uv)^{\frac{n(n-1)}{2}} \left\{ \left(\sum_{j=0}^{n-1} (n-j)(n-j+1) x_j^2 u^{n-j-1} v^j\right)\left(\sum_{j=1}^{n} j(j+1) x_j^2 u^{n-j} v^{j-1}\right) \right. \\ &\left. - uv \left(\sum_{j=1}^{n-1} j(n-j) x_j^2 u^{n-j-1} v^{j-1}\right)^2 \right\} \neq 0. \end{aligned}$$

On the other hand, we can see that the second Hessian is identically zero, i.e. $\mathrm{Hess}^{(2)} F = 0$ in $k[u, v, x_0, \dots, x_n]$. This means that the algebra A does not have the strong Lefschetz property.

Example 5.2. There exists an example of a polynomial F of degree 5 with 5 variables such that Hess $F \neq 0$ and $\mathrm{Hess}^{(2)} F = 0$. Let us choose

$$F = x^2u^3 + xyu^2v + y^2uv^2 + z^2v^3 \in k[u, v, x, y, z].$$

Then

$$U, V, X, Y, Z \in A = k[U, V, X, Y, Z]/\mathrm{Ann}_Q F$$

are linearly independent. So we have

$$\begin{aligned} \text{Hess } F = 48u^3v^3 \big(&u^5x^4 + 8u^4vx^3y + 16u^3v^2x^2y^2 + 19u^2v^3x^2z^2 \\ &+9u^2v^3xy^3 + 13uv^4xyz^2 + 2uv^4y^4 + 4v^5y^2z^2\big) \neq 0. \end{aligned}$$

We also see that $\mathrm{Hess}^{(2)} F = 0$, the algebra A does not have the weak Lefschetz property.

Example 5.3. The following example is due to Ikeda [5]. Let us choose the polynomial $F = w^3xy + wx^3z + y^3z^2$. Then the corresponding algebra $A = Q/\mathrm{Ann}_Q F$ has the Hilbert function $(1, 4, 10, 10, 4, 1)$. The Hessian is given as follows:

$$\begin{aligned}\mathrm{Hess}\, F = 8(&3w^7xy^4 + 8w^6x^6 - 27w^5x^3y^3z + 27w^4y^6z^2 \\ &- 45w^3x^5y^2z^2 - 54w^2x^2y^5z^3 + 9wx^7yz^3 + 27x^4y^4z^4).\end{aligned}$$

In this case, we again have $\mathrm{Hess}^{(2)} F$.

Remark 5.1. It is still open whether the Artinian Gorenstein algebra with $\dim A_1 = 3$ has the strong (or weak) Lefschetz property.

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