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THE LIE MODULE OF THE SYMMETRIC GROUP

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The Lie module of the symmetric group $\mathfrak{S}_n$ appears in many contexts; in particular it is closely related to the free Lie algebra. One possible approach is to view it as the right ideal of the group algebra $F\mathfrak{S}_n$, generated by the 'Dynkin-Specht-Wever element'

$$\omega_n := (1-c_2)(1-c_3)\cdots(1-c_n)$$

where c_k is the k-cycle $(1,2,\ldots,k)$. We write $\mathrm{Lie}(n)=\omega_n F\mathfrak{S}_n$ for this Lie module.

Our main motivation comes from the work of Selick and Wu [SW1]. Their problem is to find natural homotopy decompositions of the loop suspension of a p-torsion suspension where p is a prime. In [SW1] it is proved that this problem is equivalent to the algebraic problem of finding natural coalgebra decompositions of the primitively generated tensor algebras over the field with p elements. They determine the finest coalgebra decomposition of a tensor algebra (over arbitrary fields), which can be described as a functorial Poincarè-Birkhoff-Witt theorem [SW1, Theorem 6.5]. In order to compute the factors in this decomposition, one must know a maximal projective submodule, called $\mathrm{Lie}^{\max}(n)$, of the Lie module $\mathrm{Lie}(n)$.

The projective modules for the symmetric groups over fields of positive characteristic are not known. Their structure depends on the decomposition matrices for symmetric groups, and the determination of the latter is a famous open problem. According to [SW2], it would be interesting to know, even if the modules cannot be computed precisely, how quickly the dimensions grow, and whether or not the growth rate is exponential. Evidence in [SW2], for small cases in characteristic 2, is that $\mathrm{Lie}^{\max}(n)$ is relatively large compared with $\mathrm{Lie}(n)$ and this would correspond to factors in the functorial PBW theorem being relatively small.

Given a finite group G, and a finite-dimensional FG-module V, we have a decomposition $V=V_{pr}\oplus V_{pf}$, where V_{pr} is projective and V_{pf} does not have any projective summand. If P is a subgroup of G, then one may consider the restriction $\mathrm{Res}^G_P V$. Then $\mathrm{Res}^G_P(V_{pr})$ is a direct summand of $(\mathrm{Res}^G_P V)_{pr}$ and therefore

$$\dim(V_{pr})\le \dim(\mathrm{Res}^G_P V)_{pr}\le \dim V-\dim(\mathrm{Res}^G_P V)_{pf}.$$

Thus, when $G=\mathfrak{S}_n$ and $V=\mathrm{Lie}(n)$, we have

$$\dim(\mathrm{Lie}^{\max}(n))=\dim(\mathrm{Lie}(n)_{pr})\le (n-1)!-\dim(\mathrm{Res}^{\mathfrak{S}_n}_P \mathrm{Lie}(n))_{pf}.$$

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Here, we take P to be a Sylow p-subgroup of $\mathfrak{S}_n$, and consider the case where $n = kp$ with $p \nmid k$. In this case, Lie(pk) has been studied in [ES] via a module known as the p-th symmetrisation of Lie(k), denoted as $S^p(\mathrm{Lie}(k))$.

To define $S^p(\mathrm{Lie}(k))$, we first define the subgroups $\Delta_k\mathfrak{S}_p$ and $\mathfrak{S}_k^{[p]}$ of $\mathfrak{S}_{kp}$, which are isomorphic to $\mathfrak{S}_p$ and $\mathfrak{S}_k$ respectively. For $\tau \in \mathfrak{S}_p$, define $\Delta_k\tau \in \mathfrak{S}_{kp}$ to be the permutation that permutes each of the k successively blocks of size p in $\{1, \ldots, pk\}$ according to τ. For $\sigma \in \mathfrak{S}_k$, define $\sigma^{[p]} \in \mathfrak{S}_{kp}$ to be the permutation that permutes the k successively blocks of size p in $\{1, \ldots, pk\}$ according to σ. Then $\Delta_k\mathfrak{S}_p = \{\Delta_k\tau \mid \tau \in \mathfrak{S}_p\}$ and $\mathfrak{S}_k^{[p]} = \{\sigma^{[p]} \mid \sigma \in \mathfrak{S}_k\}$. We note that these two subgroups commute with each other.

Let $D = \Delta_k\mathfrak{S}_p \times \mathfrak{S}_k^{[p]}$. Let $\Lambda_k = F \boxtimes \mathrm{Lie}(k)$; this is a D-module where $\Delta_k\mathfrak{S}_p$ acts trivially, while the action of $\mathfrak{S}_k^{[p]}$ on Λ_k is equivalent to that of $\mathfrak{S}_k$ on Lie(k). Then we have $S^p(\mathrm{Lie}(k)) = \mathrm{Ind}_D^{\mathfrak{S}_{pk}} \Lambda_k$.

The first author and Schocker proved the following result.

Theorem 1. [ES, Theorem 10] *Let $n = pk$ with $p \nmid k$. Then there is a short exact sequence of right $F\mathfrak{S}_n$-modules*

$$0 \to \mathrm{Lie}(n) \to eF\mathfrak{S}_n \to S^p(\mathrm{Lie}(k)) \to 0$$

where e is an idempotent in $\mathfrak{S}_n$.

As a corollary, we have $\Omega(S^p(\mathrm{Lie}(k))) \cong \mathrm{Lie}(n)_{pf}$. Here, and hereafter, Ω denotes the Heller operator. Applying the exact restriction functor to the short exact sequence also yields

$$(\mathrm{Res}_P^{\mathfrak{S}_n} \mathrm{Lie}(n))_{pf} \cong \Omega(\mathrm{Res}_P^{\mathfrak{S}_n} S^p(\mathrm{Lie}(k))) = \Omega((\mathrm{Res}_P^{\mathfrak{S}_n} S^p(\mathrm{Lie}(k)))_{pf}).$$

By Mackey's formula, we have

$$\mathrm{Res}_P^{\mathfrak{S}_n} S^p(\mathrm{Lie}(k)) \cong \mathrm{Res}_P^{\mathfrak{S}_n} \mathrm{Ind}_D^{\mathfrak{S}_n} \Lambda_k = \bigoplus_{x \in D/\mathfrak{S}_n \backslash P} \mathrm{Ind}_{D^x \cap P}^P (\Lambda_k \otimes x).$$

Proposition 2 ([ET, Proposition 3.2]).

(1) *If $(\Delta_k\mathfrak{S}_p)^x \cap P = 1$, then $\mathrm{Ind}_{D^x \cap P}^P(\Lambda_k \otimes x)$ is projective.*
(2) *If $(\Delta_k\mathfrak{S}_p)^x \cap P \neq 1$, then $\mathrm{Ind}_{D^x \cap P}^P(\Lambda_k \otimes x)$ has no projective summand.*

In view of Proposition 2, let S be the set of all double coset representatives in $D/\mathfrak{S}_n \setminus P$ such that $(\Delta_k\mathfrak{S}_p)^x \cap P \neq 1$. Then we have

Corollary 3 ([ET, Corollary 3.3]). *Let $k \in \mathbb{Z}^+$ with $p \nmid k$. Then*

$$(\mathrm{Res}_P^{\mathfrak{S}_{kp}} S^p(\mathrm{Lie}(k)))_{pf} \cong \bigoplus_{x \in S} \mathrm{Ind}_{D^x \cap P}^P (\Lambda_k \otimes x).$$

Lemma 4 ([ET, Lemma 3.4]). *For $x \in S$ we have*

$$\Omega(\mathrm{Ind}_{D^x \cap P}^P(\Lambda_k \otimes x)) \cong \mathrm{Ind}_{D^x \cap P}^P((\Omega(F) \boxtimes \mathrm{Lie}(k)) \otimes x),$$

and $\Omega(F)$ has dimension $p - 1$.

Theorem 5 ([ET, Theorem 3.5]). *Let $k \in \mathbb{Z}^+$ with $p \nmid k$. We have*
$$\dim((\mathrm{Res}_P^{\mathfrak{S}_{kp}} \mathrm{Lie}(kp))_{pf}) = (p-1)(k-1)! \sum_{x \in S} [P : D^x \cap P].$$

A simple argument using group action yields the following:

Corollary 6 ([ET, Corollary 3.6]). *Let $k \in \mathbb{Z}^+$ with $p \nmid k$. We have*
$$\dim((\mathrm{Res}_P^{\mathfrak{S}_{kp}} \mathrm{Lie}(kp))_{pf}) = (p-1)(k-1)!N,$$
where N is the number of cosets Dx such that $(\Delta_k \mathfrak{S}_p)^x \cap P \neq 1$.

In order to proceed, we introduce a combinatorial object which we call *p-compositions* to help us study the elements x such that $(\Delta_k \mathfrak{S}_p)^x \cap P \neq 1$, and obtain a transversal to the right cosets Dx such that $(\Delta_k \mathfrak{S}_p)^x \cap P \neq 1$. We refer the interested reader to [ET] for details.

For $m \in \mathbb{Z}_{\geq 0}$, define a_m recursively as follows:
$$a_0 = p(p-1), \qquad a_m = a_{m-1}^p + p^{2p^m - 1}(p-1).$$

Theorem 7 ([ET, Theorem 6.6 and Corollary 6.7]). *Let $k \in \mathbb{Z}^+$ with $p \nmid k$. Let $k - 1 = \sum_{i=1}^l p^{\kappa_i}$ where $\kappa_i \in \mathbb{Z}_{\geq 0}$ such that each p-power does not occur p times or more in the sum. Then the number of cosets Dx such that $(\Delta_k \mathfrak{S}_p)^x \cap P \neq 1$ equals*
$$\prod_{i=1}^{l} a_{\kappa_i}.$$

Thus, $\dim((\mathrm{Res}_P^{\mathfrak{S}_{kp}} \mathrm{Lie}(kp))_{pf}) = (p-1)(k-1)! \prod_{i=1}^l a_{\kappa_i}$.

Theorem 8 ([ET, Theorem 6.9]). *Let $k \in \mathbb{Z}^+$ with $p \nmid k$.*

(1) *The dimension of* $(\mathrm{Res}_P \mathrm{Lie}(kp))_{pf}$*, and hence of* $\mathrm{Lie}(kp)_{pf}$*, grows exponentially with k.*
(2) $\dim((\mathrm{Res}_P \mathrm{Lie}(kp))_{pf}) / \dim(\mathrm{Lie}(kp)) \to 0$ *as $k \to \infty$.*

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