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Periodic Oscillations of a Linear Wave Equation with a Small Time-periodic Potential

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Dedicated to Professor Kenji Nishihara on his 60th birthday

Abstract

We shall consider BVP to a 3-dimensional radially symmetric linear wave equation with a small time-periodic potential

$$\partial_t^2 u - \Delta u + mu + \varepsilon f(x, \omega t)u = 0,\ (x,t) \in D \times R_t^1,$$

where D is the 3-ball, $f(x,\theta)$ is 2π-periodic in θ and smooth in (x,θ), m is a positive constant, ε is a small parameter and ω is a positive constant depending on ε. We shall show that BVP has families of periodic solutions with periods $2\pi/\omega(\varepsilon)$ for $\varepsilon \in \Lambda$, where Λ is contained in a neighborhood of 0, and is uncountable and has Lebesgue measure zero. The solutions bifurcate from each normal mode of $\partial_t^2 u - \Delta u + mu = 0$.

1 Introduction

Let D be a 3-ball with radius a and center at the origin and ∂D be its boundary. Let $\Omega = D \times R^1$ and $\partial\Omega$ be the boundary of Ω. Let Δ be the 3-dimensional Laplacian. Let all functions with space variable x be radially symmetric in the space variable $x \in D$.

Consider a time-periodic BVP for a 3-dimensional linear wave equation with a small time-periodic potential

$$\begin{cases} (\partial_t^2 - \Delta + m)\, u + \varepsilon f(x, \omega t)\, u = 0, & (x,t) \in \Omega, \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega, \\ u(x, t + 2\pi/\omega) = u(x,t), & (x,t) \in \Omega, \end{cases} \tag{P}$$

where ε is a small parameter and $\omega > 0$ is a constant depending on ε determined later. Here we assume that m is a positive constant *i.e.,* we deal with the nondegenerate case ($m \neq 0$).

Yamaguchi [Ya1] treated the degenerate case ($m = 0$), D is a bounded interval $(0, \pi)$ and f is a sufficiently smooth function of only θ, *i.e.* $f = f(\omega t)$, and showed that for any small ε every solution of IBVP (BVP (P) with initial condition) is almost periodic in t, provided that the eigenvalues of $-\Delta$ and the periods of f satisfy some Diophantine condition. Note that in this case the frequencies ω are independent of ε, while the frequencies except ω of the almost periodic solutions are the smooth functions of ε perturbed from the eigenvalues μ_l^2 of $-\Delta$. This statement holds for almost all frequencies $\omega > 0$, since the above Diophantine condition holds for almost all $\omega \in R^1$. This is proved by using the reduction theory of one-dimensional Schrödinger equations with a quasiperiodic potential based on KAM method (Parashuk [Pa]). However this effective method is not able to be applied to the case where f depends on x as well as t. Moreover it is pointed out ([Ya1]) that even in case f depends on only t, there exist ω such that every nontrivial solution in some family of solutions of IBVP ((P) with IC) is unbounded (hence not periodic) in $t \in R^1$. Hence there exist no periodic solutions.

Consider the eigenvalue problem to $-\Delta + m$

$$\begin{cases} (-\Delta + m)\, \phi(x) = \mu^2\, \phi(x), & x \in D, \\ \phi(x) = 0, \quad x \in \partial D. \end{cases} \tag{EP}$$

Let $\{\mu_l^2\}$ and $\{\phi_l(x)\}$ be the sequences of the eigenvalues and the corresponding eigenfunctions of (EP). It is well-known that $\{\mu_l\}$ is written in the form $\mu_l = \sqrt{(l\pi/a)^2 + m}$. $\{\phi_l(x)\}$ is taken as a CONS in $L^2_{rad}(D)$ and then it also turns out to be a complete and orthogonal system in $H^1_{0,rad}(D)$. Here $L^2_{rad}(D)$ and $H^1_{0,rad}(D)$ are the subspaces of the usual Lebesgue and Sobolev spaces $L^2(D)$ and $H^1_0(D)$ respectively whose elements are radially symmetric in D. In this paper we set $a = \pi$ for simplicity. Then μ_l is of the form

$\sqrt{l^2+m}$. Later we study number-theoretic property of μ_l that is essentially important in the existence of periodic solutions.

Consider the case where Eq. has no potential *i.e.*, $\varepsilon = 0$. In this case BVP (P) has infinitely many normal modes

$$\cos \mu_l t\, \phi_l(x), \quad l = 1, 2, \cdots$$

with the period $2\pi/\mu_l$. The purpose of this paper is *to show that for each normal mode there exists a family of periodic solutions of BVP* (P) *that bifurcates from the normal mode.* It is shown that each family of the periodic solutions and the correspondind periods ω are parametrized by ε contained in a suitable uncountable and Lebesgue measure zero set in a neighborhood of 0. The solution and its period tend to the normal mode of the form $\cos \mu_j t\, \phi_j(x)$ and the period $2\pi/\mu_j$ of the normal mode respectively as $\varepsilon \to 0$.

Assumptions on $f(x, \theta)$ and m

We assume the following condition on the time-periodic potential $f(x, \theta)$. Let s be a positive integer.

(A) $f(x, \theta)$ is nonnegative and of C^s-class in $(x, \theta) \in D \times R^1$, and 2π-periodic and even in $\theta \in R^1$.

From now on without loss of generality we assume that $f(x, \theta)$ is not identically zero.

Next we assume a number-theoretic condition on m in the same way as in [Ya3].

Remark 1.1 As is seen below, the condition $m \neq 0$ (the nondegeneracy) is essential in our argument from number-theoretic point of view.

Let V be the set of the continued fractions $[0; c_1, c_2, \cdots]$ which satisfy

$$c_1 \geq \max_{i \geq 2} c_i + 3. \tag{1.1}$$

V is contained in some right neighborhood $[0, \gamma)$ of 0 and uncountable, and has the Lebesgue measure zero in R^1 (Khinchin [Kh]). 0 is an accumulating point of V from right. Let $j \in N$. Define W_j and W by

$$W_j = \{2jc + c^2; c \in V\}, \quad W = \bigcup_{j=1}^{\infty} W_j. \tag{1.2}$$

W_j and W are uncountable and have an accumulating point 0 from right. For more properties of W_j and W, see [Ya3].

We assume the following number-theoretic condition on m.

(M) m belongs to W.

From (M) there exists $j \in N$ such that m belongs to W_j. Note that if $m \in W_j$, then there exists $[0; c_1, c_2, \cdots] \in V$such that

$$\mu_j = j + [0; c_1, c_2, \cdots] = [j; c_1, c_2, \cdots].$$

Remark 1.2 The nonnegativity of $f(x,\theta)$ in (A) can be weakened to the condition

$$\int_{D\times(0,2\pi)} f(x,\theta)\,(\cos\theta\,\phi_j(x))^2 d\theta dx \neq 0$$

for the above $j \in N$ (see the proof of Proposition 2.4).

Notation and Definitions

Let O be any open set in R^n. Let $L^2(O)$ and $H^s(O)$, $H_0^1(O)$ be the usual Lebesgue and Sobolev spaces respectively. We denote the inner products of $L^2(O)$ and $H^s(O)$ by $(\cdot,\cdot)$ and $(\cdot,\cdot)_{H^s(O)}$ respectively.

Let $\Gamma = D\times(0,2\pi)$. Let $H^s_{rad}(\Gamma)$ be the subspace of $H^s(\Gamma)$ whose elements are radially symmetric in the space variable. Let $\tilde{H}^1_{0,rad}(\Gamma)$ be the subspace of $H^1_{rad}(\Gamma)$ whose elements vanish at $\partial D\times(0,2\pi)$ almost everywhere. In this paper we take the following spaces as the basic function spaces

$$X^s = \{h \in H^s_{rad}(\Gamma)\cap \tilde{H}^1_{0,rad}(\Gamma);\, h(x,\theta+2\pi) = h(x,\theta) = h(x,-\theta)\},$$
$$X^0 = \{h \in L^2_{rad}(\Gamma);\, h(x,\theta+2\pi) = h(x,\theta) = h(x,-\theta)\}$$

for $s \in N$. We define norm $|\cdot|_s$ of X^s by $|\cdot|_{H^s(\Gamma)}$ for $s \in Z_+$.

Main Theorem

In order to study the problem, we shall transform BVP (P) to the following periodic BVP by changing the variable t to θ by $\theta = \omega t$

$$\begin{cases} (\omega^2\,\partial_\theta^2 - \Delta + m)\,u + \varepsilon f(x,\theta)\,u = 0, & (x,\theta)\in\Omega, \\ u(x,\theta) = 0, \quad (x,\theta)\in\partial\Omega, \\ u(x,\theta+2\pi) = u(x,\theta), \quad (x,\theta)\in\Omega, \end{cases} \tag{TP}$$

where $\omega > 0$ is regarded as a parameter depending on ε. The solution of (TP) corresponds to $2\pi/\omega$-periodic solution of (P).

When $\varepsilon = 0$, BVP (TP) has normal modes $\cos k\theta\, \phi_j(x)$ for any fixed $(j,k) \in N \times N$, provided that $k\omega = \mu_j$ holds. We shall look for a family of 2π-periodic solutions of (TP) that bifurcates from each normal mode $\cos k\theta\, \phi_j(x)$, $(j,k) \in N \times N$. In this paper, for simplicity we shall treat only the normal mode $\cos\theta\, \phi_j(x)$, *i.e.*, $k = 1$. We shall be able to deal with other normal modes in the same way.

In order to show the existence of solutions of BVP (TP), we shall apply the Lyapunov-Schmidt decomposition to BVP (TP). We decompose BVP (TP) into the normal mode direction and its orthogonal direction in the above space X^0, and we solve those two systems.

We formulate our theorem. From now on throughout this paper, we fix $j \in N$. We denote by X_j one-dimensional linear space spanned by the normal mode $\cos\theta\, \phi_j(x)$ in X^0 and by $X_j^\perp$ its orthogonal complement in X^0 . We denote the projectors of X^0 to X_j and $X_j^\perp$ by P and $P^\perp$ respectively. We set $v(x,\theta) = \cos\theta\, \phi_j(x)$ for brevity.

We have the following main theorem.

Theorem 1.1 *Assume* (A) *and* (M). *Then there exist* $\varepsilon_0 > 0$, *a set* $\Lambda \subset [0, \varepsilon_0)$ *and a monotone increasing function* $\omega(\varepsilon)$ *defined on* Λ *such that for any* $\varepsilon \in \Lambda$ BVP (TP) *has a family of* 2π*-periodic solutions in* X^s. *The solutions are of the form* $\cos\theta\, \phi_j(x) + \varepsilon w$, *where* $w \in X^s \cap X_j^\perp$. Λ *is uncountable, accumulates to* 0 *and has the Lebesgue measure zero.* $\omega(\varepsilon)$ *is represented by an asymptotic formula*

$$\omega(\varepsilon)^2 = \mu_j^2 + \varepsilon \int_\Gamma f(x,\theta) v(x,\theta)^2 d\theta dx + o(\varepsilon) \quad (\varepsilon \to 0). \tag{1.3}$$

Remark 1.3 The regularity of the solutions coincides with the differentiability of the potential $f(x,\theta)$.

Remark 1.4 If $s \geq 4$, the solutions are of C^2-class in $D \times R^1$ by the Sobolev Lemma.

From this theorem we obtain one-parameter family of periodic solutions of BVP (P).

Corollary 1.1 *Let* $s \geq 4$. *Under* (A) *and* (M) *BVP* (P) *has a periodic solution with the period* $2\pi/\omega(\varepsilon)$ *of* C^2 *for every* $\varepsilon \in \Lambda$.

2 Proof of Main Theorem

Consider BVP (TP) and apply the Lyapunov-Schmidt decomposition. We decompose BVP (TP) into BVPs for a system of linear wave equations as follows. We look for the solution u in the form

$$u = v + \varepsilon\, w \equiv Pu + \varepsilon P^{\perp} u, \tag{2.1}$$

where $w \in X_j^{\perp}$. We operate P and $P^{\perp}$ to BVP (TP). Then ω, ε and w satisfy the following

$$A_\omega\, v + \varepsilon P(f(x,\theta)v) = P(-\varepsilon^2 f(x,\theta)w), \quad (x,\theta) \in \Omega, \tag{2.2}$$

$$\begin{cases} A_\omega\, w = P^{\perp}(-\varepsilon f(x,\theta)w - f(x,\theta)v), & (x,\theta) \in \Omega, \\ w(x,\theta) = 0, \quad (x,\theta) \in \partial\Omega, \\ w(x,\theta+2\pi) = w(x,\theta), \quad (x,\theta) \in \Omega. \end{cases} \tag{2.3}$$

Here $A_\omega = \omega^2 \partial_\theta^2 - \Delta + m$. We solve (2.2) and (2.3) for unknowns (w, ω, ε) so that Theorem 1.1 follows.

First we deal with BVP (2.3). We fix ω so as to satisfy the Diophantine condition (see (N) below). Then we show the existence of periodic solutions of BVP (2.3) with small ε. We apply the contraction mapping principle to BVP (2.3). To this end we basically need to solve the following BVP to a linear wave equation in $X_j^{\perp}$

$$\begin{cases} A_\omega\, w = h(x,\theta), \quad (x,\theta) \in \Omega, \\ w(x,\theta) = 0, \quad (x,\theta) \in \partial\Omega, \\ w(x,\theta+2\pi) = w(x,\theta), \quad (x,\theta) \in \Omega, \end{cases} \tag{2.4}$$

where $h(x,\theta) \in X_j^{\perp}$, *i.e.* $h(x,\theta)$ is 2π-periodic and even in θ and orthogonal to the normal mode v.

For evolution equations $d^2u(t)/dt^2 + A\,u = f(t,u)$ (A is an elliptic operator) like wave equations, beam equations and so on with time periodic terms $f(t,u)$, the Diophantine conditions on the eigenvalues of A and the periods $2\pi/\omega$ play an essential role in the existence of periodic solutions. In this paper we assume the following Diophantine conditions of the weak Poincaré type.

(N) $\{\mu_l\}$ and ω satisfy the following Diophantine inequality : There exists a constant $C > 0$ dependent on ω such that

$$|\mu_l - k\omega| \geq \frac{C}{k} \tag{2.5}$$

for all $(l, k) \in (N \setminus \{j\}) \times N$.

Let S be a set of ω in R^1_+. The following condition is called a uniform Diophantine condition for the set S.

(NU) $\{\mu_l\}$ and any $\omega \in S$ satisfy the Diophantine inequality : There exists a constant $C > 0$ independent of ω such that

$$|\mu_l - k\omega| \geq \frac{C}{k} \tag{2.6}$$

for all $(l, k) \in (N \setminus \{j\}) \times N$. We say that S satisfies (NU).

The following proposition will be used to construct the set of ω satisfying (NU) so that we may construct Λ of ε in Theorem 1.1.

Proposition 2.1 *Assume* (M). *Let $j \in N$ be fixed so as to satisfy $m \in W_j$. Then there exists a set B_{μ_j} of ω in a right neighborhood Ξ_j of μ_j that satisfies* (NU). *B_{μ_j} is uncountable and has the Lebesgue measure zero, and accumulates to μ_j from right.*

Proof. From (M) μ_j has the continued fraction with bounded elements. Therefore applying Proposition 5.1 in [Ya3] to μ_j, we construct the uncountable set B_{μ_j} of ω contained in a right neighbourhood of μ_j that satisfies (NU), has the Lebessgue measure 0 and accumulates to μ_j from right.

We show the existence of periodic solutions of the linear BVP (2.4) in the following proposition.

Proposition 2.2 *Let $s \in Z_+$. Assume that h belongs to $X^s \cap X_j^\perp$ and* (N) *holds. Then* BVP (2.4) *has a solution w unique in $X^s \cap X_j^\perp$. w satisfies*

$$|w|_s \leq C_s \frac{a^2}{C} |h|_s, \tag{2.7}$$

where $C_s > 0$ is a constant dependent on s, and C is the same constant in (N).

If $\omega \in B_{\mu_j}$, the constant C is taken uniformly with respect to $\omega \in B_{\mu_j}$, where B_{μ_j} is seen in Proposition 2.1.

The proposition is proved in the same way as the proof of Proposition 4.1 in [Ya2], showing the existence of the weal periodic solutions by the Fourier expansion method and then obtaining the regularity of the weak solutions by the elliptic regulality technique together with the bootstrap method.

Now we are in position to solve BVP (2.3). We shall show the following proposition.

Proposition 2.3 *Assume that* (A) *and* (M) *hold and* $\omega \in B_{\mu_j}$. *Then there exists* $\varepsilon_1 > 0$ *dependent on* $\max_{\alpha+|\beta|\leq s} \sup_{x,\theta} |\partial_\theta^\alpha \partial_x^\beta f(x,\theta)|$ *and* C *in* (NU) *and independent of* ω *such that for any* ε, $|\varepsilon| \leq \varepsilon_1$ *BVP* (2.3) *has a solution* w *in* $X^s \cap X_j^\perp$. w *satisfies*

$$|w|_s \leq c_1, \quad \left|\frac{\partial w}{\partial \varepsilon}\right|_s \leq c_2, \tag{2.8}$$

where $c_i > 0$ *depend on* ε_1 *and are independent of* ε *and* ω.

Proof. Since B_{μ_j} satisfies (NU) by Proposition 2.1, it follows from Proposition 2.2 that A_ω has the inverse A_ω^{-1} in $X^s \cap X_j^\perp$. Define an integral operator F_ε related to BVP (2.3) by

$$F_\varepsilon(w) = A_\omega^{-1} \circ (P^\perp(-\varepsilon f(x,\theta)w - f(x,\theta)v)).$$

Let $R > 0$ be a constant $\geq 2|fv|_s$ and set $B(R) = \{w \in X_s; |w|_s \leq R\}$. We apply the contraction mapping principle in $X^s \cap X_j^\perp$ to F_ε. By using (A) and the estimate (2.7) in Proposition 2.2, it follows that there exists $\bar{\varepsilon} > 0$ independent of $\omega \in B_{\mu_j}$ such that for any ε, $|\varepsilon| \leq \bar{\varepsilon}$

$$F_\varepsilon(w) \in B(R),$$
$$|F_\varepsilon(w_1) - F_\varepsilon(w_2)|_s \leq \hat{c}\,|w_1 - w_2|_s$$

for w, $w_i \in B(R)$, where $\hat{c}$ is a positive constant less than 1. Hence F_ε has a fixed point $w \in B(R) \subset X_s \cap X_j^\perp$, whence BVP (2.3) has a solution w satisfying the first estimate of (2.8) for any ε, $|\varepsilon| \leq \bar{\varepsilon}$.

We show that w is differentiable with respect to ε and derive the second estimate of (2.8). Consider the following BVP obtained by differentiating BVP (2.3) formally with respect to ε

$$\begin{cases} A_\omega w_\varepsilon = P^\perp(-\varepsilon f(x,\theta)w_\varepsilon - f(x,\theta)w), & (x,\theta) \in \Omega, \\ w_\varepsilon(x,\theta) = 0, \quad (x,\theta) \in \partial\Omega, \\ w_\varepsilon(x,\theta+2\pi) = w_\varepsilon(x,\theta), \quad (x,\theta) \in \Omega. \end{cases} \tag{2.9}$$

Then we can show in the same way as in the above proof that there exists $\hat{\varepsilon} > 0$ such that for any ε, $|\varepsilon| \le \hat{\varepsilon}$ BVP (2.9) has a solution w_ε in $X^s \cap X_j^\perp$ satisfying $|w_\varepsilon|_s \le c_2$. w_ε is unique in the ball. We write the solution $w(x,\theta)$ as $w(x,\theta;\varepsilon)$, briefly $w(\varepsilon)$ as a function of ε. Also we set $\bar{w}(\varepsilon;h) = (w(\varepsilon + h) - w(\varepsilon))/h - w_\varepsilon$. Then from (2.3) $\bar{w}(\varepsilon;h)$ satisfies the following BVP

$$\begin{cases} A_\omega \, \bar{w}(\varepsilon;h) = -P^\perp f(x,\theta)\,\{(\varepsilon+h)\,\bar{w}(\varepsilon;h) + hw_\varepsilon\} \\ \bar{w}(\varepsilon;h) = 0, \quad (x,\theta) \in \partial D \times R^1, \\ \bar{w}(x,\theta+2\pi;\varepsilon;h) = \bar{w}(x,\theta;\varepsilon;h), \quad (x,\theta) \in \Omega. \end{cases} \tag{2.10}$$

By applying Propoosition 2.2 to (2.10), it follows that

$$|\bar{w}(\varepsilon;h)|_s \le c\,|h|\,|w_\varepsilon|_s.$$

The right hand side tends to 0 as $h \to 0$. This means that $w(\varepsilon)$ is differentiable in ε in X^s and $\dfrac{\partial w}{\partial \varepsilon} = w_\varepsilon$ holds. From the above argument we obtain the second estimate of (2.8) for any ε, $|\varepsilon| \le \hat{\varepsilon}$. In order to obtain the conclusion, we have only to take $\varepsilon_1 = \min(\bar{\varepsilon}, \hat{\varepsilon})$.

As the second step we solve the problem (2.2) regarding as the equation with respect to ω and ε for given solutions $w(\varepsilon,\omega)$ of BVP (2.3). We have the following proposition.

Proposition 2.4 *Assume* (A) *and* (M). *There exist $\varepsilon_2 > 0$ with $\varepsilon_2 \le \varepsilon_1$, a set $\Lambda \subset [0,\varepsilon_2)$ and a monotone increasing function $\omega(\varepsilon)$ defined in Λ such that $(\omega(\varepsilon),\varepsilon)$ solve* (2.2). *Here ε_1 is the same constant in Proposition* 2.3. *Λ is uncountable, has the Lebesgue measure* 0 *and accumulates to* 0.

Proof. Let ε_1 be the same constant as in Proposition 2.3 and let $\varepsilon \in [0,\varepsilon_1)$. Then BVP (2.3) has the solution $w(\varepsilon,\omega)$ for ε with $|\varepsilon| \le \varepsilon_1$. Taking inner product of (2.2) with v, we obtain

$$\omega^2 = \mu_j^2 + \varepsilon(f(x,\theta)v,v) + \varepsilon^2(f(x,\theta)w(\varepsilon,\omega),v). \tag{2.11}$$

This is an equation with respect to ω and ε. It follows from (A) and (2.8) in Proposition 2.3 that $(f(x,\theta)v,v) > 0$, and also $|(f(x,\theta)w,v)| \le \tilde{c}$ and $|(f(x,\theta)w_\varepsilon,v)| \le \tilde{c}$ hold, where $\tilde{c}$ is independent of ε and ω. Hence applying the implicit function theorem to (2.11), we can take $\varepsilon_2 > 0$, $\varepsilon_2 \le \varepsilon_1$ such that for any fixed $\omega \in B_{\mu_j}$ there exists a unique solution $\varepsilon \in [0,\varepsilon_2)$ of (2.11).

$\varepsilon = \varepsilon(\omega)$ is monotone increasing as a function of $\omega \in B_{\mu_j}$. Therefore there exists the inverse monotone increasing function $\omega(\varepsilon)$ defined in $\varepsilon(B_{\mu_j}) \cap [0, \varepsilon_2)$.

It is clear from Proporsitions 2.3 and 2.4 that the conclusions of Theorem 1.1 follows.

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