数理解析研究所講究録
第 1690 巻 2010 年 71-83

Stationary waves for viscous heat-conductive fluid in half space

川島 秀一 (Shuichi Kawashima)[1] 中村 徹 (Tohru Nakamura)[1]
西畑 伸也 (Shinya Nishibata)[2] Peicheng Zhu[3]

[1] 九大・数理 (Faculty of Mathematics, Kyushu University)
[2] 東工大・情報理工 (Department of Mathematical and Computing Sciences, Tokyo Institute of Technology)
[3] Basque Center for Applied Mathematics

Dedicated to Professor Kenji Nishihara on his 60th birthday

1 Introduction

This article is a survey of the papers [3, 8] on a stability of a stationary solution to an ideal polytropic model of compressible, viscous and heat-conductive gases in one-dimensional half space $\mathbb{R}_+ := (0, \infty)$,

$$\rho_t + (\rho u)_x = 0, \tag{1.1a}$$

$$(\rho u)_t + \left(\rho u^2 + p(\rho,\theta)\right)_x = \mu u_{xx}, \tag{1.1b}$$

$$\left\{\rho\left(c_{\mathrm{v}}\theta + \frac{u^2}{2}\right)\right\}_t + \left\{\rho u\left(c_{\mathrm{v}}\theta + \frac{u^2}{2}\right) + p(\rho,\theta)u\right\}_x = \left(\mu u u_x + \kappa\theta_x\right)_x. \tag{1.1c}$$

Here $\rho = \rho(t,x)$, $u = u(t,x)$ and $\theta = \theta(t,x)$ are unknown functions standing for a mass density, a fluid velocity and an absolute temperature, respectively. The pressure $p = p(\rho,\theta)$ is given by $p(\rho,\theta) := R\rho\theta$ due to the Boyle–Charles law, where $R > 0$ is a gas constant. Positive constants μ, κ and c_{v} mean a viscosity coefficient, a thermal conductivity and a specific heat at constant volume, respectively. For the ideal polytropic model, c_{v} is given by $c_{\mathrm{v}} = R/(\gamma - 1)$, where $\gamma > 1$ is an adiabatic constant. We put an initial condition

$$(\rho,u,\theta)(0,x) = (\rho_0,u_0,\theta_0)(x), \tag{1.2a}$$

$$\lim_{x\to\infty}(\rho_0,u_0,\theta_0)(x) = (\rho_+,u_+,\theta_+), \tag{1.2b}$$

$$\inf_{x\in\mathbb{R}_+}\rho_0(x) > 0, \quad \inf_{x\in\mathbb{R}_+}\theta_0(x) > 0, \tag{1.2c}$$

where $\rho_+ > 0$, u_+ and $\theta_+ > 0$ are constants. The main purpose of this article is to summarize the results in [3, 8] which show the existence and the asymptotic stability of the stationary solution for an outflow and an inflow problem to the equations (1.1). Here the outflow problem and the inflow problem are formulated by imposing the following boundary conditions (i) and (ii), respectively:

(i) Outflow boundary condition:

$$u(t,0) = u_{\mathrm{b}} < 0, \quad \theta(t,0) = \theta_{\mathrm{b}} > 0, \tag{1.3}$$

(ii) Inflow boundary condition:

$$\rho(t,0)=\rho_b>0, \quad u(t,0)=u_b>0, \quad \theta(t,0)=\theta_b>0, \tag{1.4}$$

where ρ_b, u_b and θ_b are constants.

For the one-dimensional half space problem for an isentropic model, Matsumura in [5] considered a classification of asymptotic states of solutions. It was expected that asymptotic states of solutions are classified into more than twenty cases subject to the boundary data and the spatial asymptotic data. Several problems in this classification have been already studied. For instance, Matsumura and Nishihara in [7] proved the asymptotic stability of stationary solutions, rarefaction waves and superposition of them for the inflow problem. The research [4] by Kawashima, Nishibata and Zhu showed the asymptotic stability of the stationary solution for the outflow problem. For this stability result, the convergence rate was obtained by Nakamura, Nishibata and Yuge in [9] by using a weighted energy method developed by Kawashima, Matsumura and Nishihara in [2, 6, 10] considering the asymptotic stability of a traveling wave for a scalar viscous conservation law. In the paper [12], the convergence rate toward a degenerate stationary solution was also considered by Ueda, Nakamura and Kawashima.

For the half space problem for the ideal polytropic model (1.1), Kawashima, Nakamura, Nishibata and Zhu [3] proved the existence and the asymptotic stability of the stationary solution for the outflow problem. The convergence rate was also obtained in [3] for a supersonic case and a transonic case. For the inflow problem, Nakamura and Nishibata in [8] proved the asymptotic stability of the stationary solution.

Notations. For constants $p\in[1,\infty)$ and $\alpha\in\mathbb{R}$, the space $L^p_\alpha(\mathbb{R}_+)$ denotes the algebraically weighted L^p space defined by $L^p_\alpha(\mathbb{R}_+):=\{u\in L^p_{\mathrm{loc}}(\mathbb{R}_+)\ ;\ \|u\|_{L^p_\alpha}<\infty\}$ equipped with the norm

$$\|u\|_{L^p_\alpha}:=\Big(\int_{\mathbb{R}_+}(1+x)^\alpha|u(x)|^p\,dx\Big)^{1/p}.$$

The space $H^s_\alpha(\mathbb{R}_+)$ denotes the algebraically weighted H^s space corresponding to $L^2_\alpha(\mathbb{R}_+)$ defined by $H^s_\alpha(\mathbb{R}_+):=\{u\in L^2_\alpha(\mathbb{R}_+)\ ;\ \partial_x^k u\in L^2_\alpha(\mathbb{R}_+)\text{ for }k=0,\dots,s\}$, equipped with the norm

$$\|u\|_{H^s_\alpha}:=\Big(\sum_{k=0}^{s}\|\partial_x^k u\|^2_{L^2_\alpha}\Big)^{1/2}.$$

For $\alpha\in(0,1)$, the space $\mathcal{B}^\alpha(\mathbb{R}_+)$ denotes the set of Hölder continuous functions over $\mathbb{R}_+$ with the Hölder exponent α with respect to x. For a non-negative integer k, $\mathcal{B}^{k+\alpha}(\mathbb{R}_+)$ denotes the space of functions satisfying $\partial_x^i u\in\mathcal{B}^\alpha(\mathbb{R}_+)$ for an arbitrary $i=0,\dots,k$. For $\alpha,\beta\in(0,1)$ and $T>0$, the space $\mathcal{B}^{\alpha,\beta}([0,T]\times\mathbb{R}_+)$ denotes the set of Hölder continuous functions over $[0,T]\times\mathbb{R}_+$ with the Hölder exponents α and β with respect to t and x, respectively. For non-negative integers k and ℓ, $\mathcal{B}_T^{k+\alpha,\ell+\beta}:=\mathcal{B}^{k+\alpha,\ell+\beta}([0,T]\times\mathbb{R}_+)$ denotes the space of functions satisfying $\partial_t^i u$, $\partial_x^j u\in\mathcal{B}^{\alpha,\beta}([0,T]\times\mathbb{R}_+)$ for arbitrary $i=0,\dots,k$ and $j=0,\dots,\ell$.

2 Stationary waves

In the present section, we summarize the existence result of the stationary solution for the outflow problem and the inflow problem discussed in [3, 8]. We also show that the geometric property of local invariant manifolds around an equilibrium point is completely characterized by the Prandtl number. This observation is summarized in Section 2.2.

2.1 Existence of stationary waves

The stationary solution $(\tilde\rho,\tilde u,\tilde\theta)(x)$ is defined as a solution to the system (1.1) independent of time variable t. Thus $(\tilde\rho,\tilde u,\tilde\theta)$ satisfies a system of equations

$$(\tilde\rho\tilde u)_x = 0, \tag{2.1a}$$

$$\big(\tilde\rho\tilde u^2 + p(\tilde\rho,\tilde\theta)\big)_x = \mu\tilde u_{xx}, \tag{2.1b}$$

$$\Big\{\tilde\rho\tilde u\Big(c_{\mathrm{v}}\tilde\theta + \frac{\tilde u^2}{2}\Big) + p(\tilde\rho,\tilde\theta)\tilde u\Big\}_x = \big(\mu\tilde u\tilde u_x + \kappa\tilde\theta_x\big)_x. \tag{2.1c}$$

It is assumed that the stationary solution satisfies the same conditions as (1.2b) and (1.2c):

$$\lim_{x\to\infty}(\tilde\rho(x),\tilde u(x),\tilde\theta(x)) = (\rho_+,u_+,\theta_+), \tag{2.2}$$

$$\inf_{x\in\mathbb{R}_+}\tilde\rho(x) > 0, \quad \inf_{x\in\mathbb{R}_+}\tilde\theta(x) > 0. \tag{2.3}$$

Moreover we prescribe the same boundary conditions as (1.3) and (1.4):

(i)' Outflow boundary condition:

$$\tilde u(0) = u_{\mathrm{b}} < 0, \quad \tilde\theta(0) = \theta_{\mathrm{b}} > 0, \tag{2.4}$$

(ii)' Inflow boundary condition:

$$\tilde\rho(0) = \rho_{\mathrm{b}} > 0, \quad \tilde u(0) = u_{\mathrm{b}} > 0, \quad \tilde\theta(0) = \theta_{\mathrm{b}} > 0. \tag{2.5}$$

To discuss a solvability of the above boundary value problem, we employ the Mach number M_+, a sound speed c_+ and a strength δ of the boundary data as follows:

$$M_+ := \frac{|u_+|}{c_+}, \quad c_+ := \sqrt{R\gamma\theta_+}, \quad \delta := |(u_+,\theta_+) - (u_{\mathrm{b}},\theta_{\mathrm{b}})|.$$

Integrating (2.1a) over (x,∞), we get the relation

$$\tilde\rho\tilde u = \rho_+u_+. \tag{2.6}$$

Especially, substituting $x = 0$ in (2.6), we have $\tilde\rho(0)u_{\mathrm{b}} = \rho_+u_+$. Thus, for the outflow problem, the spatial asymptotic state u_+ of the velocity must be negative

$$u_+ < 0. \tag{2.7}$$

On the other hand, for the inflow problem, we have a necessary condition for the existence of the solution as follows:

$$u_+ > 0 \quad \text{and} \quad \rho_b u_b = \rho_+ u_+. \tag{2.8}$$

Next we integrate (2.1b) and (2.1c) over (x, ∞) with using (2.6) to obtain a system of equations for $(\tilde{u}, \tilde{\theta})$ as

$$\frac{d}{dx}\begin{pmatrix}\tilde{u}\\ \tilde{\theta}\end{pmatrix} = J\begin{pmatrix}\tilde{u}-u_+\\ \tilde{\theta}-\theta_+\end{pmatrix} + \begin{pmatrix} f(\tilde{u},\tilde{\theta})\\ g(\tilde{u},\tilde{\theta})\end{pmatrix}, \tag{2.9}$$

where the matrix J and nonlinear terms $f(\tilde{u},\tilde{\theta})$ and $g(\tilde{u},\tilde{\theta})$ are defined by

$$J := \begin{pmatrix} (\rho_+ u_+^2 - R\rho_+\theta_+)/(\mu u_+) & R\rho_+/\mu \\ R\rho_+\theta_+/\kappa & c_v\rho_+u_+/\kappa \end{pmatrix},$$

$$f(\tilde{u},\tilde{\theta}) := \frac{R\rho_+\theta_+}{\mu u_+ \tilde{u}}(\tilde{u}-u_+)^2 - \frac{R\rho_+}{\mu\tilde{u}}(\tilde{u}-u_+)(\tilde{\theta}-\theta_+),$$

$$g(\tilde{u},\tilde{\theta}) := -\frac{\rho_+ u_+}{2\kappa}(\tilde{u}-u_+)^2.$$

The boundary conditions for (2.9) are prescribed as

$$(\tilde{u},\tilde{\theta})(0) = (u_b,\theta_b), \quad \lim_{x\to\infty}(\tilde{u}(x),\tilde{\theta}(x)) = (u_+,\theta_+). \tag{2.10}$$

We first summarize the existence result of the problem (2.9) and (2.10) for the outflow problem which yields the solvability of the problem (2.1), (2.2) and (2.4) considered in [3].

Proposition 2.1 ([3]). *The necessary condition for the existence of the stationary solution to the problem* (2.1), (2.2) *with the outflow boundary condition* (2.4) *is* (2.7). *Suppose that the boundary data* (u_b,θ_b) *satisfies*

$$\delta < \varepsilon_0 \tag{2.11}$$

for a certain positive constant ε_0.

(i) *For the supersonic case* $M_+ > 1$, *the problem* (2.1), (2.2) *and* (2.4) *has a unique smooth solution* $(\tilde{\rho},\tilde{u},\tilde{\theta})$ *satisfying*

$$|\partial_x^k(\tilde{\rho}(x)-\rho_+, \tilde{u}(x)-u_+, \tilde{\theta}(x)-\theta_+)| \le C\delta e^{-cx} \quad (k=0,1,\dots). \tag{2.12}$$

(ii) *For the transonic case* $M_+ = 1$, *there exist a local stable manifold* $\theta = \tilde{h}^s(u)$ *and a local center manifold* $\theta = \tilde{h}^c(u)$ *around the equilibrium point* (u_+,θ_+) *in the state space* (u,θ) *(see Figure 1). Then, if the boundary data* (u_b,θ_b) *satisfies* $\theta_b \le \tilde{h}^s(u_b)$, *then the problem* (2.1), (2.2) *and* (2.4) *has a unique smooth solution* $(\tilde{\rho},\tilde{u},\tilde{\theta})$ *satisfying*

$$|\partial_x^k(\tilde{\rho}(x)-\rho_+, \tilde{u}(x)-u_+, \tilde{\theta}(x)-\theta_+)| \le C\frac{\delta^{k+1}}{(1+\delta x)^{k+1}} + C\delta e^{-cx} \quad (k=0,1,\dots).$$

(iii) *For the subsonic case* $M_+ < 1$, *there exist a local stable manifold* $\theta = \tilde{h}^s(u)$ *and a local unstable manifold* $\theta = \tilde{h}^u(u)$. *Then, if the boundary data* (u_b,θ_b) *satisfies* $\theta_b = \tilde{h}^s(u_b)$, *then the problem* (2.1), (2.2) *and* (2.4) *has a unique smooth solution* $(\tilde{\rho},\tilde{u},\tilde{\theta})$ *satisfying* (2.12).

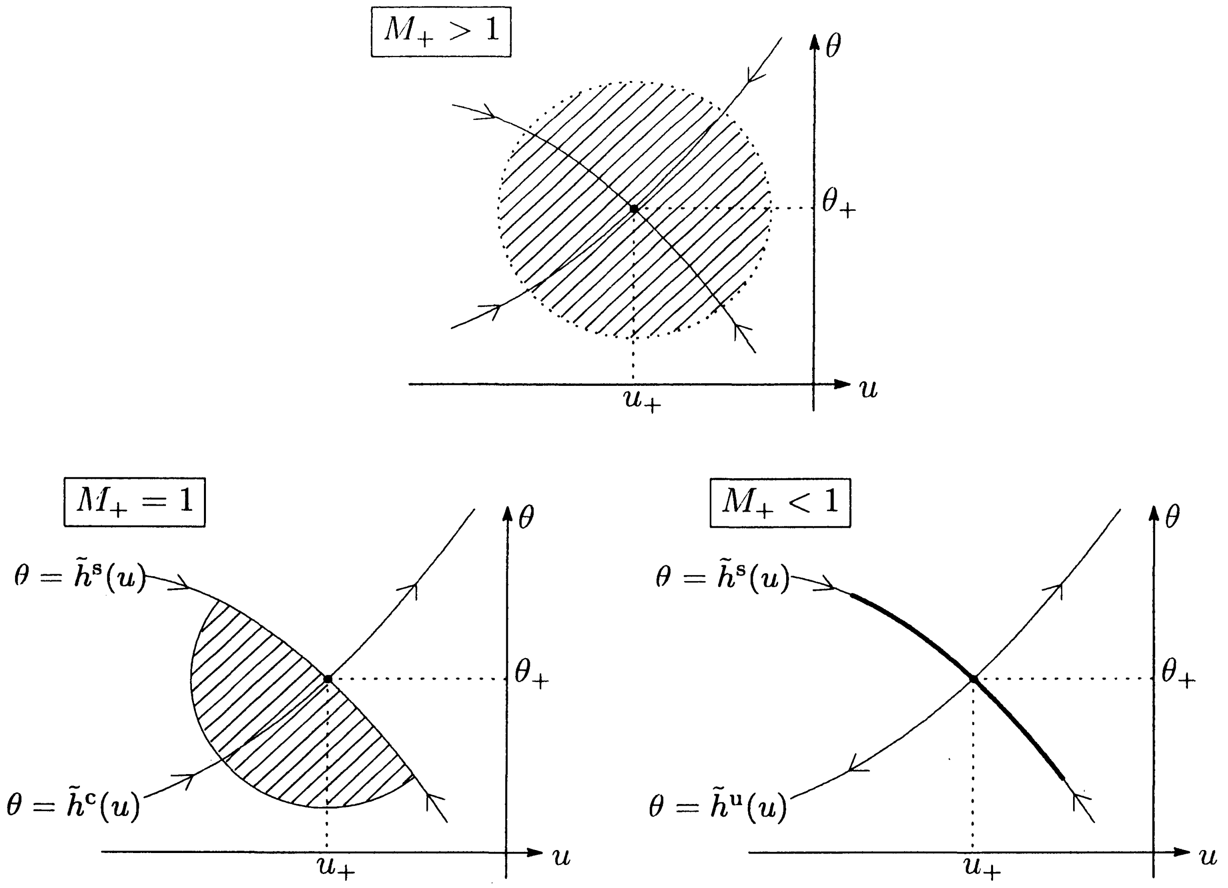

Figure 1: State space for the problem (2.9) and (2.10) with the outflow boundary condition.

We next summarize the existence result considered in [8] for the problem (2.1) and (2.2) with the inflow boundary condition (2.5).

Proposition 2.2 ([8]). *The necessary condition for the existence of the stationary solution to the problem* (2.1), (2.2) *with the inflow boundary condition* (2.5) *is* (2.8). *Suppose that the boundary data* $(u_{\mathrm{b}}, \theta_{\mathrm{b}})$ *satisfies* (2.11).

(i) *For the supersonic case* $M_+ > 1$, *there does not exist a solution to the problem* (2.1), (2.2) *and* (2.5).

(ii) *For the transonic case* $M_+ = 1$, *there exist a local center manifold* $\theta = \tilde{h}^{\mathrm{c}}(u)$ *and a local unstable manifold* $\theta = \tilde{h}^{\mathrm{u}}(u)$ *(see Figure* 2*). Then, if the boundary data* $(u_{\mathrm{b}}, \theta_{\mathrm{b}})$ *satisfies* $\theta_{\mathrm{b}} = \tilde{h}^{\mathrm{c}}(u_{\mathrm{b}})$ *and* $\theta_{\mathrm{b}} \geq \tilde{h}^{\mathrm{u}}(u_{\mathrm{b}})$, *then the problem* (2.1), (2.2) *and* (2.5) *has a unique smooth solution* $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$ *satisfying*

$$|\partial_x^k(\tilde{\rho}(x) - \rho_+, \tilde{u}(x) - u_+, \tilde{\theta}(x) - \theta_+)| \leq C\frac{\delta^{k+1}}{(1+\delta x)^{k+1}} \quad (k = 0, 1, \dots). \quad (2.13)$$

(iii) *For the subsonic case* $M_+ < 1$, *we have the same conclusion as in Proposition* 2.1-(iii).

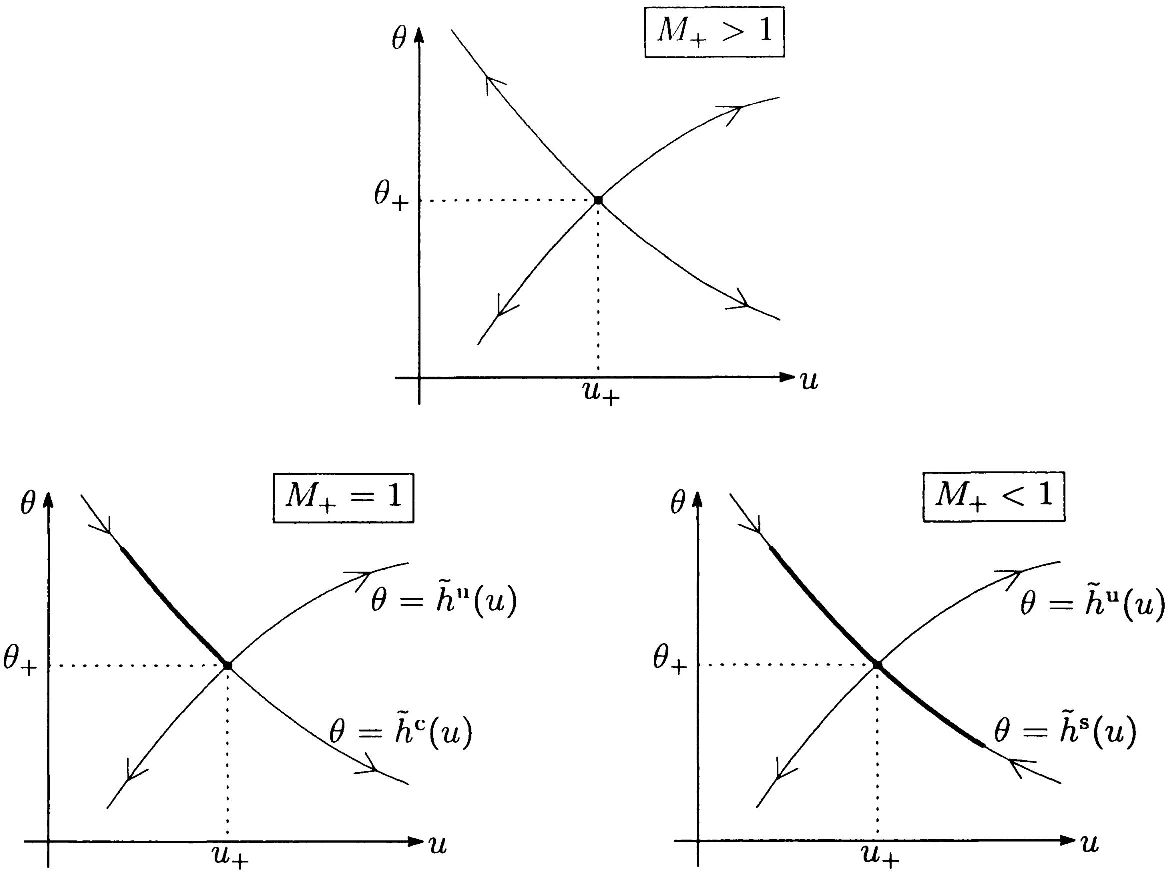

Figure 2: State space for the problem (2.9) and (2.10) with the inflow boundary condition.

2.2 Local structure of invariant manifolds

In order to verify the conditions in Proposition 2.1 and 2.2, which ensure the existence of the stationary solution, it is important to make clear the local shapes of the invariant manifolds. In the present section, we focus ourselves on the transonic case $M_+ = 1$ for the outflow problem (see Figure 1) and show that the geometric properties of the invariant manifolds $\tilde{h}^{c}$ and $\tilde{h}^{s}$ are characterized by the Prandtl number P_{r} defined by

$$P_{\mathrm{r}} := \frac{\mu}{\kappa} c_{\mathrm{p}}, \quad c_{\mathrm{p}} := \frac{\gamma}{\gamma - 1} R,$$

where c_{p} denotes a specific heat at constant pressure. Precisely we approximate $\tilde{h}^{c}$ and $\tilde{h}^{s}$ by polynomial functions around the equilibrium point as

$$\tilde{h}^{c}(u) = \sum_{k=0}^{3} c_k (u - u_+)^k + O((u - u_+)^4), \tag{2.14}$$

$$\tilde{h}^{s}(u) = \sum_{k=0}^{3} s_k (u - u_+)^k + O((u - u_+)^4). \tag{2.15}$$

Computing the eigen-vectors of the matrix J, we obtain

$$c_0 = s_0 = \theta_+, \quad c_1 = \frac{(1-\gamma)\theta_+}{u_+} > 0, \quad s_1 = \frac{\mu u_+}{\kappa(\gamma - 1)} < 0.$$

Moreover, following an idea in [1], we obtain the coefficients c_k, s_k $(k = 2, 3)$ and see that the convexity of the local invariant manifolds depends on the Prandtl number. Namely we have

Lemma 2.3 ([3]).

(i) *The local center manifold* (2.14) *satisfies* $c_2 \gtreqless 0$ *if and only if* $P_{\mathrm{r}} \gtreqless 2$. *Especially, if* $P_{\mathrm{r}} = 2$, *i.e.,* $c_2 = 0$, *the coefficient* c_3 *is negative.*

(ii) *The local stable manifold* (2.15) *satisfies* $s_2 \gtreqless 0$ *if and only if* $P_{\mathrm{r}} \gtreqless \gamma_* := (\gamma^2 - \gamma + 2)/2$. *Especially, if* $P_{\mathrm{r}} = \gamma_*$, *i.e.,* $s_2 = 0$, *the coefficient* s_3 *is positive.*

3 Asymptotic stability of stationary waves

In this section, we introduce the results in [3, 8] on the asymptotic stability of the stationary solution $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$. The next theorem shows the stability of $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$ for the outflow problem.

Theorem 3.1 ([3]). *Suppose that the same conditions as in Proposition* 2.1 *hold. In addition, the initial data* (ρ_0, u_0, θ_0) *is supposed to satisfy*

$$\begin{gathered} \rho_0 \in \mathcal{B}^{1+\sigma}(\mathbb{R}_+), \quad (u_0, \theta_0) \in \mathcal{B}^{2+\sigma}(\mathbb{R}_+), \\ (\rho_0, u_0, \theta_0) - (\tilde{\rho}, \tilde{u}, \tilde{\theta}) \in H^1(\mathbb{R}_+) \end{gathered} \tag{3.1}$$

for a certain constant $\sigma \in (0, 1)$. *Then there exists a positive constant* ε_1 *such that if*

$$\|(\rho_0, u_0, \theta_0) - (\tilde{\rho}, \tilde{u}, \tilde{\theta})\|_{H^1} + \delta \le \varepsilon_1, \tag{3.2}$$

then the initial boundary value problem (1.1), (1.2) *with the outflow boundary condition* (1.3) *has a unique solution globally in time satisfying*

$$\begin{gathered} \rho \in \mathcal{B}_T^{1+\sigma/2, 1+\sigma}, \quad (u, \theta) \in \mathcal{B}_T^{1+\sigma/2, 2+\sigma}, \\ (\rho, u, \theta) - (\tilde{\rho}, \tilde{u}, \tilde{\theta}) \in C([0, \infty); H^1(\mathbb{R}_+)) \end{gathered} \tag{3.3}$$

for an arbitrary $T > 0$. *Moreover the solution* (ρ, u, θ) *converges to the stationary solution* $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$ *uniformly as time tends to infinity:*

$$\lim_{t \to \infty} \|(\rho, u, \theta)(t) - (\tilde{\rho}, \tilde{u}, \tilde{\theta})\|_{L^\infty} = 0. \tag{3.4}$$

In the paper [8], the asymptotic stability of the stationary solution is also proved for the inflow problem.

Theorem 3.2 ([8]). *Suppose that the same condition as in Proposition* 2.2 *hold. In addition, if the initial data and the boundary data satisfy the conditions* (3.1) *and* (3.2), *then the initial boundary value problem* (1.1), (1.2) *with the inflow boundary condition* (1.4) *has a unique solution in the space* (3.3). *Moreover, the stationary solution is asymptotically stable in the sence of* (3.4).

In this paper, we focus on the outflow problem and give the outline of the proof of Theorem 3.1. Theorem 3.2 is proved in the similar computations. The crucial point is a derivation of uniform *a priori* estimates for a perturbation

$$(\varphi, \psi, \chi)(t, x) := (\rho, u, \theta)(t, x) - (\tilde{\rho}, \tilde{u}, \tilde{\theta})(x)$$

from the stationary solution in the Sobolev space H^1. Using (1.1) and (2.1), we have the system of equations for (φ, ψ, χ) as

$$\varphi_t + u\varphi_x + \rho\psi_x = -(\tilde{u}_x\varphi + \tilde{\rho}_x\psi), \tag{3.5a}$$

$$\rho(\psi_t + u\psi_x) + \frac{1}{M_+^2}(p - \tilde{p})_x = \mu\psi_{xx} - (\rho u - \tilde{\rho}\tilde{u})\tilde{u}_x, \tag{3.5b}$$

$$\frac{c_v}{M_+^2}\rho\chi_t + \frac{c_v}{M_+^2}(\rho u\theta_x - \tilde{\rho}\tilde{u}\tilde{\theta}_x) = \kappa\chi_{xx} + \mu(u_x^2 - \tilde{u}_x^2) - \frac{1}{M_+^2}(pu_x - \tilde{p}\tilde{u}_x). \tag{3.5c}$$

The initial condition for (φ, ψ, χ) follows from (1.2) as

$$(\varphi, \psi, \chi)(0, x) = (\varphi_0, \psi_0, \chi_0)(x) := (\rho_0, u_0, \theta_0)(x) - (\tilde{\rho}, \tilde{u}, \tilde{\theta})(x). \tag{3.6}$$

The boundary condition for the outflow problem is prescribed as

$$(\psi, \chi)(t, 0) = (0, 0). \tag{3.7}$$

Hereafter, for simplicity, we often use the notations $\Phi := (\varphi, \psi, \chi)$ and $\Phi_0 := (\varphi_0, \psi_0, \chi_0)$. To summarize *a priori* estimates for Φ, we define a function space $X(0, T)$, for $T > 0$, by

$$\begin{aligned} X(0,T) := \{(\varphi, \psi, \chi)\ ;\ & \varphi \in \mathcal{B}_T^{1+\sigma/2, 1+\sigma},\ (\psi, \chi) \in \mathcal{B}_T^{1+\sigma/2, 2+\sigma}, \\ & (\varphi, \psi, \chi) \in C([0,T]; H^1(\mathbb{R}_+)),\ \varphi_x \in L^2(0,T; L^2(\mathbb{R}_+)), \\ & (\psi_x, \chi_x) \in L^2(0,T; H^1(\mathbb{R}_+))\}, \end{aligned}$$

where $\sigma \in (0, 1)$ is a constant. We also employ non-negative functions $N(t)$ and $D(t)$ by

$$N(t) := \sup_{0 \le \tau \le t} \|\Phi(\tau)\|_{H^1},$$

$$D(t)^2 := |(\varphi, \varphi_x)(t, 0)|^2 + \|\varphi_x(t)\|_{L^2}^2 + \|(\psi_x, \chi_x)(t)\|_{H^1}^2.$$

Proposition 3.3. *Let $\Phi = (\varphi, \psi, \chi) \in X(0, T)$ be a solution to* (3.5), (3.6) *and* (3.7) *for a certain constant $T > 0$. Then there exist positive constants ε_2 and C independent of T such that if $N(T) + \delta \le \varepsilon_2$, then the solution Φ satisfies the estimate*

$$\|\Phi(t)\|_{H^1}^2 + \int_0^t D(\tau)^2\, d\tau \le C\|\Phi_0\|_{H^1}^2. \tag{3.8}$$

The *a priori* estimate (3.8) is obtained by using an energy method. Once (3.8) is shown, we prove Theorem 3.1 by a standard continuation argument together with a local existence of the solution. For details, see [3].

To obtain (3.8), we firstly derive a basic L^2 estimate by employing an energy form $\mathcal{E}$ defined by

$$\mathcal{E} := \frac{1}{M_+^2\gamma}\tilde{\theta}\omega\Big(\frac{\tilde{\rho}}{\rho}\Big) + \frac{1}{2}\psi^2 + \frac{c_v}{M_+^2}\tilde{\theta}\omega\Big(\frac{\theta}{\tilde{\theta}}\Big), \quad \omega(s) := s - 1 - \log s.$$

Owing to a smallness assumption on $N(T)$, a quantity $\|\Phi\|_{L^\infty}$ is also sufficiently small. Hence we see that the energy form is equivalent to $|\Phi|^2$:

$$c\varphi^2 \le \omega\Big(\frac{\tilde\rho}{\rho}\Big) \le C\varphi^2, \quad c\chi^2 \le \omega\Big(\frac{\theta}{\tilde\theta}\Big) \le C\chi^2, \quad c|\Phi|^2 \le \mathcal{E} \le C|\Phi|^2. \tag{3.9}$$

The solution, moreover, satisfies a uniform estimate

$$0 < c \le \rho(t,x),\ \theta(t,x) \le C, \quad -C \le u(t,x) \le -c < 0 \tag{3.10}$$

for $(t,x) \in [0,T] \times \mathbb{R}_+$. Hereafter we only show the key estimates summarized in Lemma 3.4 and 3.5 and omit detailed computations.

Lemma 3.4. *Suppose that the same conditions as in Proposition* 3.3 *hold. Then we have*

$$\begin{aligned}\|\Phi(t)\|_{L^2}^2 &+ \int_0^t \left(\varphi(\tau,0)^2 + \|(\psi_x,\chi_x)(\tau)\|_{L^2}^2\right) d\tau \\ &\le C\|\Phi_0\|_{L^2}^2 + C\delta \int_0^t \|\varphi_x(\tau)\|_{L^2}^2\, d\tau.\end{aligned} \tag{3.11}$$

We next obtain the estimate for the first order derivative $(\varphi_x, \psi_x, \chi_x)$.

Lemma 3.5. *Suppose that the same conditions as in Proposition* 3.3 *hold. Then we have*

$$\begin{aligned}\|\Phi_x(t)\|_{L^2}^2 &+ \int_0^t \left(\varphi_x(\tau,0)^2 + \|(\varphi_x,\psi_{xx},\chi_{xx})(\tau)\|_{L^2}^2\right) d\tau \\ &\le C\|\Phi_0\|_{H^1}^2 + C(N(T)+\delta) \int_0^t D(\tau)^2\, d\tau.\end{aligned} \tag{3.12}$$

Summing up the estimates (3.11) and (3.12) and letting $N(T)+\delta$ suitably small, we prove the desired *a priori* estimate (3.8).

4 Convergence rate for outflow problem

For the outflow problem, the convergence rate toward the stationary solution $(\tilde\rho, \tilde u, \tilde\theta)$ is obtained in [3] for the supersonic and transonic cases.

Theorem 4.1 ([3]). *Suppose that the same conditions as in Theorem* 3.1 *hold.*

(i) *For the supersonic case $M_+ > 1$, if the initial perturbation satisfies*

$$(\rho_0, u_0, \theta_0) - (\tilde\rho, \tilde u, \tilde\theta) \in L^2_\alpha(\mathbb{R}_+)$$

for a certain positive constant α, then the solution (ρ, u, θ) to (1.1), (1.2) *with the outflow boundary condition* (1.3) *satisfies the decay estimate*

$$\|(\rho,u,\theta)(t) - (\tilde\rho,\tilde u,\tilde\theta)\|_{L^\infty} \le C(1+t)^{-\alpha/2}. \tag{4.1}$$

(ii) *For the transonic case $M_+ = 1$, let $\alpha \in [1, 2(1+\sqrt{2}))$. There exists a positive constant ε_3 such that if*

$$\delta^{-1/2}\|(\rho_0,u_0,\theta_0) - (\tilde\rho,\tilde u,\tilde\theta)\|_{H^1_\alpha} \le \varepsilon_3,$$

then the solution (ρ, u, θ) satisfies the decay estimate

$$\|(\rho,u,\theta)(t) - (\tilde\rho,\tilde u,\tilde\theta)\|_{L^\infty} \le C(1+t)^{-\alpha/4}. \tag{4.2}$$

4.1 Estimate for supersonic flow

In this section we introduce the weighted energy estimates which yield the convergence rate (4.1) for the case $M_+ > 1$. To this end, we define weighted norm $E_\alpha(t)$ and $D_\alpha(t)$ by

$$E_\alpha(t)^2 := \|\Phi(t)\|_{H^1}^2 + \|\Phi(t)\|_{L^2_\alpha}^2,$$
$$D_\alpha(t)^2 := D(t)^2 + \alpha\|\Phi(t)\|_{L^2_{\alpha-1}}^2 + \|(\psi_x, \chi_x)(t)\|_{L^2_\alpha}^2.$$

Proposition 4.2. *We assume that $M_+ > 1$ and (2.11) hold. Let $\Phi = (\varphi, \psi, \chi) \in X(0,T)$ be a solution to (3.5), (3.6) and (3.7) satisfying $\Phi \in C([0,T]; L^2_\alpha(\mathbb{R}_+))$ for certain constants $\alpha > 0$ and $T > 0$. Then there exist positive constant ε_4 and C independent of T such that if $N(T) + \delta \leq \varepsilon_4$, then the solution Φ satisfies the following estimates*

$$(1+t)^j E_{\alpha-j}(t)^2 + \int_0^t (1+\tau)^j D_{\alpha-j}(\tau)^2 \, d\tau \leq C E_\alpha(0)^2, \tag{4.3}$$

for an arbitrary integer $j = 0, \ldots, [\alpha]$ and

$$(1+t)^\xi E_0(t)^2 + \int_0^t (1+\tau)^\xi D_0(\tau)^2 \, d\tau \leq C E_\alpha(0)^2 (1+t)^{\xi-\alpha} \tag{4.4}$$

for an arbitrary $\xi > \alpha$.

The convergence rate (4.1) is immediately follows from (4.4) and the Sobolev inequality. To obtain (4.3) and (4.4), we show the time and space weighted estimate of Φ in $L^2(\mathbb{R}_+)$.

Lemma 4.3. *Suppose that the same conditions as in Proposition 4.2 hold. Then we have*

$$\begin{aligned}
&(1+t)^\xi \|\Phi(t)\|_{L^2_\beta}^2 + \int_0^t (1+\tau)^\xi \left(\varphi(\tau,0)^2 + \beta\|\Phi(\tau)\|_{L^2_{\beta-1}}^2 + \|(\psi_x,\chi_x)(\tau)\|_{L^2_\beta}^2\right) d\tau \\
&\quad \leq C\|\Phi_0\|_{L^2_\beta}^2 + C\xi \int_0^t (1+\tau)^{\xi-1}\|\Phi(\tau)\|_{L^2_\beta}^2 \, d\tau + C\delta \int_0^t (1+\tau)^\xi \|\varphi_x(\tau)\|_{L^2}^2 \, d\tau
\end{aligned} \tag{4.5}$$

for arbitrary $\beta \in [0, \alpha]$ and $\xi \geq 0$.

Combining (4.5) with the uniform H^1 estimate (3.8), we have

$$\begin{aligned}
&(1+t)^\xi E_\beta(t)^2 + \int_0^t (1+\tau)^\xi (\beta\|\Phi(\tau)\|_{L^2_{\beta-1}}^2 + D_\beta(\tau)^2) \, d\tau \\
&\quad \leq C E_\beta(0)^2 + C\xi \int_0^t (1+\tau)^{\xi-1} (\|\Phi(\tau)\|_{L^2_\beta}^2 + D_\beta(\tau)^2) \, d\tau,
\end{aligned}$$

which yield the desired estimates (4.3) and (4.4) by applying an induction with respect to β and ξ studied by [2] and [11].

4.2 Estimate for transonic flow

In order to obtain the convergence rate (4.2) for the case $M_+ = 1$, we have to show the weighted estimate in H^1 norm. To do this, we define weighted norms by

$$\tilde{N}_\alpha(t) := \sup_{0\le\tau\le t} \tilde{E}_\alpha(\tau), \quad \tilde{E}_\alpha(t) := \|[\Phi(t)]\|_{1,\alpha},$$
$$\tilde{D}_\alpha(t)^2 := |(\varphi,\varphi_x)(t,0)|^2 + \delta^2[\Phi(t)]_{\alpha-2}^2 + [\varphi_x(t)]_\alpha^2 + \|[(\psi_x,\chi_x)(t)]\|_{1,\alpha}^2,$$

where $\|[\cdot]\|_{s,\alpha}$ and $[\cdot]_\alpha$ are algebraically weighted norms:

$$\|[u]\|_{s,\alpha} := \Big(\sum_{k=0}^{s}[\partial_x^k u]_\alpha^2\Big)^{1/2}, \quad [u]_\alpha := \Big(\int_{\mathbb{R}_+}(1+\delta x)^\alpha |u(x)|^2\,dx\Big)^{1/2}.$$

Proposition 4.4. *We assume that $M_+ = 1$ and (2.11) hold. Let $\Phi \in X(0,T)$ be a solution to (3.5), (3.6) and (3.7) satisfying $\Phi \in C([0,T];H^1_\alpha(\mathbb{R}_+))$ for certain constants $\alpha \in [1, 2(1+\sqrt{2}))$ and $T > 0$. Then there exist positive constants ε_4 and C independent of T such that if $\delta^{-1/2}\tilde{N}_\alpha(T) + \delta \le \varepsilon_4$, then the solution Φ satisfies the following estimates for $t \in [0,T]$:*

$$(1+t)^j\tilde{E}_{\alpha-2j}(t)^2 + \int_0^t (1+\tau)^j \tilde{D}_{\alpha-2j}(\tau)^2\,d\tau \le C\delta^{-2j}\tilde{E}_\alpha(0)^2 \tag{4.6}$$

for an arbitrary integer $j = 0, \ldots, [\alpha/2]$ and

$$(1+t)^\xi\tilde{E}_0(t)^2 + \int_0^t (1+\tau)^\xi \tilde{D}_0(\tau)^2\,d\tau \le C\delta^{-\alpha}\tilde{E}_\alpha(0)^2(1+t)^{\xi-\alpha/2} \tag{4.7}$$

for an arbitrary $\xi > \alpha/2$.

In order to prove Proposition 4.4, we have to derive time and space weighted estimates not only for Φ in L^2 but also for the first order derivative Φ_x. In deriving the time and space weighted L^2 estimate, we have to assume that the weight exponent α is less than $2(1+\sqrt{2})$ in order to obtain the dissipative term $\delta^2[\Phi]_{\beta-2}^2$. Moreover, to control nonlinear terms, we have to assume the smallness of $[\Phi]_1$. Hence we need a condition $\alpha \ge 1$, too.

Lemma 4.5. *Suppose that the same conditions as in Proposition 4.4 hold. Then we have*

$$\begin{aligned}(1+t)^\xi[\Phi(t)]_\beta^2 &+ \int_0^t (1+\tau)^\xi\big(\varphi(\tau,0)^2 + \delta^2[\Phi(\tau)]_{\beta-2}^2 + [(\psi_x,\chi_x)(\tau)]_\beta^2\big)\,d\tau \\ &\le C[\Phi_0]_\beta^2 + C\xi\int_0^t (1+\tau)^{\xi-1}[\Phi(\tau)]_\beta^2\,d\tau + C(\delta^{-1/2}\tilde{N}_\beta(t)+\delta)\int_0^t(1+\tau)^\xi\tilde{D}_\beta(\tau)^2\,d\tau\end{aligned} \tag{4.8}$$

for arbitrary constants $\beta \in [1,\alpha]$ and $\xi \ge 0$.

Next we show the estimate for the first order derivative Φ_x. Owing to the degenerate property of the transonic flow, we have to employ the spatially weighted energy method for the estimate for Φ_x.

Lemma 4.6. *Suppose that the same conditions as in Proposition* 4.4 *hold. Then we have*

$$
\begin{aligned}
&(1+t)^{\xi}[\Phi_x(t)]_\beta^2 + \int_0^t (1+\tau)^{\xi}\big(\varphi_x(\tau,0)^2 + [\varphi_x(\tau)]_\beta^2 + [(\psi_{xx},\chi_{xx})(\tau)]_\beta^2\big)\,d\tau \\
&\quad \le C\|[\Phi_0]\|_{1,\beta}^2 + C\xi\int_0^t (1+\tau)^{\xi-1}\|[\Phi(\tau)]\|_{1,\beta}^2\,d\tau \\
&\qquad + C(\delta^{-1/2}\tilde{N}_\beta(t)+\delta)\int_0^t (1+\tau)^{\xi}\tilde{D}_\beta(\tau)^2\,d\tau \qquad (4.9)
\end{aligned}
$$

for arbitrary constants $\beta \in [1,\alpha]$ *and* $\xi \ge 0$.

Sum up (4.8) and (4.9), let $\delta^{-1/2}\tilde{N}_\beta(t)+\delta$ suitably small and then apply an induction with respect to β and ξ. These computations yield the desired estimates (4.6) and (4.7). The convergence rate (4.2) follows from the estimate (4.7).

References

[1] J. Carr, *Applications of centre manifold theory*, Springer Verlag, 1981.

[2] S. Kawashima and A. Matsumura, *Asymptotic stability of traveling wave solutions of systems for one-dimensional gas motion*, Comm. Math. Phys., 101 (1985), pp. 97–127.

[3] S. Kawashima, T. Nakamura, S. Nishibata, and P. Zhu, *Stationary waves to viscous heat-conductive gases in half space: existence, stability and convergence rate*, preprint.

[4] S. Kawashima, S. Nishibata, and P. Zhu, *Asymptotic stability of the stationary solution to the compressible Navier-Stokes equations in the half space*, Comm. Math. Phys., 240 (2003), pp. 483–500.

[5] A. Matsumura, *Inflow and outflow problems in the half space for a one-dimensional isentropic model system of compressible viscous gas*, Methods Appl. Anal., 8 (2001), pp. 645–666. IMS Conference on Differential Equations from Mechanics (Hong Kong, 1999).

[6] A. Matsumura and K. Nishihara, *Asymptotic stability of traveling waves for scalar viscous conservation laws with non-convex nonlinearity*, Comm. Math. Phys., 165 (1994), pp. 83–96.

[7] A. Matsumura and K. Nishihara, *Large-time behaviors of solutions to an inflow problem in the half space for a one-dimensional system of compressible viscous gas*, Comm. Math. Phys., 222 (2001), pp. 449–474.

[8] T. Nakamura and S. Nishibata, *Stationary waves to viscous heat-conductive gas in half space with inflow boundary condition*, preprint.

[9] T. Nakamura, S. Nishibata, and T. Yuge, *Convergence rate of solutions toward stationary solutions to the compressible Navier-Stokes equation in a half line*, J. Differential Equations, 241 (2007), pp. 94–111.

[10] K. NISHIHARA, *A note on the stability of travelling wave solutions of Burgers' equation*, Japan J. Appl. Math., 2 (1985), pp. 27–35.

[11] M. NISHIKAWA, *Convergence rate to the traveling wave for viscous conservation laws*, Funkcial. Ekvac., 41 (1998), pp. 107–132.

[12] Y. UEDA, T. NAKAMURA, AND S. KAWASHIMA, *Convergence rate toward degenerate stationary wave for compressible viscous gases*, preprint.