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Nonexistence of backward self-similar weak solutions to the Euler equations

東北大学大学院理学研究科 高田 了 (Ryo Takada)
Mathematical Institute, Tohoku University

Dedicated to Professor Kenji Nishihara on his sixtieth birthday

1 Introdunction and Main Result

Let us consider the Euler equations in $\mathbb{R}^n$ with $n \geqslant 2$, describing the motion of perfect incompressible fluids,

$$\begin{cases} \dfrac{\partial v}{\partial t} + (v\cdot\nabla)v + \nabla p = 0, & (x,t) \in \mathbb{R}^n \times (0,\infty), \\ \operatorname{div} v = 0, & (x,t) \in \mathbb{R}^n \times (0,\infty), \end{cases} \tag{E}$$

where $v = v(x,t) = (v_1(x,t), \cdots, v_n(x,t))$ and $p = p(x,t)$ denote the unknown velocity vector and the unknown pressure of the fluid at the point $(x,t) \in \mathbb{R}^n \times (0,\infty)$, respectively.

There are a number of results on local-in-time existence and uniqueness of smooth solutions to (E). Kato [10] proved that for the given initial velocity $v_0 \in [H^m(\mathbb{R}^n)]^n$ with $m > n/2+1$ satisfying $\operatorname{div} v_0 = 0$, there exist $T = T(\|v_0\|_{H^m}) > 0$ and a unique solution v of (E) with $v(x,0) = v_0(x)$ in the class $C([0,T];[H^m(\mathbb{R}^n)]^n)$. Kato and Ponce [11] extended this result to the fractional-order Sobolev space $W^{s,p}(\mathbb{R}^n) = (1-\Delta)^{-s/2}L^p(\mathbb{R}^n)$ for $s > n/p+1, p \in (1,\infty)$. Later, Chae [2] obtained a local-in-time existence result in the Triebel-Lizorkin space $F^s_{p,q}(\mathbb{R}^n)$ with $s > n/p+1, (p,q) \in (1,\infty)^2$. Moreover, a number of studies on the Euler equations in the Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ have been done by Vishik [20] [21] [22], Chae [3], Zhou [23], Pak and Park [17] and the author [18].

It is an interesting question whether the local-in-time solution $v(x,t)$ blows up at $t = T$ or can be extended to the solution in the same class beyond T. Beale, Kato and Majda [1] showed a criterion for solutions in the class $C([0,T);[H^m(\mathbb{R}^3)]^3)$ in terms of the vorticity $\omega = \operatorname{curl} v$, which states that if $\omega \in L^1(0,T;[L^\infty(\mathbb{R}^3)]^3)$, then v can be continued to the solution in the class $C([0,T');[H^m(\mathbb{R}^3)]^3)$ for some $T' > T$. Kozono and Taniuchi [12] extended this results by replacing the L^∞-norm by the BMO-norm for the vorticity, and $H^m(\mathbb{R}^n)$ by $W^{s,p}(\mathbb{R}^n)$ for the velocity, respectively. Moreover, Kozono, Ogawa and Taniuchi [13] gave a criterion which is a refinement of the above results in the sense that the BMO-norm is replaced by the Besov space $\dot{B}^0_{\infty,\infty}$-norm for the vorticity (We remark the continuous embedding properties $L^\infty(\mathbb{R}^n) \hookrightarrow \mathrm{BMO}(\mathbb{R}^n) \hookrightarrow \dot{B}^0_{\infty,\infty}(\mathbb{R}^n)$). Later, Chae [2] improved these results by replacing the $W^{s,p}(\mathbb{R}^n)$ by the Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$ for the velocity, and obtained similar results in terms of the Besov spaces [3].

The purpose of this paper is to investigate the relation between the blow-up phenomena and the backward self-similar solutions of (E). It is known that if (v,p) solves (E), then so does the pair of family $(v^{\lambda,\alpha}, p^{\lambda,\alpha})$ for all $\lambda > 0$ and all $\alpha \in \mathbb{R} \setminus \{-1\}$, where

$$v^{\lambda,\alpha}(x,t) = \lambda^{\alpha} v(\lambda x, \lambda^{\alpha+1} t), \quad p^{\lambda,\alpha}(x,t) = \lambda^{2\alpha} p(\lambda x, \lambda^{\alpha+1} t)$$

for $(x,t) \in \mathbb{R}^n \times (0,\infty)$. From the above scaling properties, the singular solution (v,p) of the self-similar type for (E) should be of the form

$$v(x,t) = \frac{1}{(T-t)^{\frac{\alpha}{\alpha+1}}} V\left(\frac{x}{(T-t)^{\frac{1}{\alpha+1}}}\right), \quad p(x,t) = \frac{1}{(T-t)^{\frac{2\alpha}{\alpha+1}}} P\left(\frac{x}{(T-t)^{\frac{1}{\alpha+1}}}\right) \tag{1.1}$$

for some $\alpha \in \mathbb{R} \setminus \{-1\}$, where (V,P) is a solution of the following system

$$\begin{cases} \dfrac{\alpha}{\alpha+1} V + \dfrac{1}{\alpha+1} (x \cdot \nabla) V + (V \cdot \nabla) V + \nabla P = 0, & x \in \mathbb{R}^n, \\ \operatorname{div} V = 0, & x \in \mathbb{R}^n. \end{cases} \tag{SE$_\alpha$}$$

Note that (SE$_\alpha$) may be regarded as the Euler version of Leray's idea for the Navier-Stokes equations introduced in [14]. If (SE$_\alpha$) possesses a non-trivial solution V, then v of the form (1.1) would be a non-trivial solution to (E) and develop a singularity at time $t = T$. Concerning the 3-dimensional Navier-Stokes equations, the question of the existence of self-similar soluiotns was originally proposed by Leray [14], and its nonexistence in the energy class was proved by Nečas, Růžička and Šverák [16] (see also Málek, Nečas, Pokorný and Schonbek [15]). Later on, Tsai [19] relaxed the hypothesis of nonexistence on the asymptotic decay properties of backward self-similar solutions. For the 3-dimensional Euler equations, similar nonexistence results have been obtained by Chae [4] [5]. In [5], he excluded any possibility of self-similar singularities assuming fast decay near infinity for the vorticity. Moreover, more refined notions of asymptotically self-similar singularity and locally self-similar blow-up were considered by Chae [5] [6] and by Hou and Li [9] for both the Euler and the Navier-Stokes equations, and they obtained the nonexistence results.

In this paper, we consider the self-similar singularities for weak solutions of (E) in the energy class and prove that the weak solutions to (E) in the form (1.1) must be trivial if the pressure satisfies some integrability and sign conditions. Moreover, we also show the nonexistence of self-similar blow-up phenomena for strong solutions to (E) under the slow dacay condition at infinity for the velocity itself provided $\alpha \neq n/2$. We remark that in terms of the asymptotic decay at space infinity, our assumption for the velocity is slightly weak in comparison with that of L^2-functions. Note that the classical solution of the Euler equation (E) conserves the energy, that is, $\|v(\cdot,t)\|_{L^2}^2$ is a constant function on $(0,T)$. Hence the energy space for the Euler equation (E) is $L^\infty(0,T;[L^2(\mathbb{R}^n)]^n)$.

Before stating our result, we introduce some definitions. A pair $(V,P) \in [L^2_{\mathrm{loc}}(\mathbb{R}^n)]^n \times L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is called a weak solution of (SE$_\alpha$) if V is divergence-free in the distribution sense, and

$$\begin{aligned} &\frac{\alpha}{\alpha+1} \int_{\mathbb{R}^n} V(x) \cdot \varphi(x) dx - \frac{1}{\alpha+1} \int_{\mathbb{R}^n} V(x) \cdot \operatorname{div}(\varphi \otimes x)(x) dx \\ &- \int_{\mathbb{R}^n} \operatorname{tr}\left[V \otimes V \nabla \varphi\right](x) dx - \int_{\mathbb{R}^n} P(x) \operatorname{div} \varphi(x) dx = 0 \end{aligned} \tag{1.2}$$

holds for all vector test functions $\varphi \in [C_0^\infty(\mathbb{R}^n)]^n$.

Definition 1.1. The function space $X^{2,\infty}(\mathbb{R}^n)$ is defined to be the set of all locally square integrable functions $f \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\limsup_{R\to\infty} \int_{R<|x|<2R} |f(x)|^2 dx < \infty.$$

It is easy to see the inclusion relation $L^2(\mathbb{R}^n) \subsetneq X^{2,\infty}(\mathbb{R}^n)$. For example, if we define the function f such that $f(x) = |x|^{-n/2}$ for $|x| > 1$, and $f(x) = 0$ for $|x| \leqslant 1$, then we have $f \in X^{2,\infty}(\mathbb{R}^n) \setminus L^2(\mathbb{R}^n)$.

Our result now reads:

Theorem 1.2. *Let $\alpha \in \mathbb{R} \setminus \{-1\}$ and let (V,P) be a weak solution of* (SE$_\alpha$)*. Suppose that $(V,P) \in [X^{2,\infty}(\mathbb{R}^n)]^n \times L^1(\mathbb{R}^n)$. Then $V \in [L^2(\mathbb{R}^n)]^n$ and*

$$\int_{\mathbb{R}^n} \{V_j(x)^2 + P(x)\} dx = 0 \tag{1.3}$$

for all $j = 1, 2, \cdots, n$. In particular, if

$$\int_{\mathbb{R}^n} P(x) dx \geqslant 0,$$

then $V(x) = 0$ and $P(x) = 0$ for almost every $x \in \mathbb{R}^n$.

Remark 1.3. This type of nonexistence results were recently obtained by Chae [7] for the original Euler and Navier-Stokes equations. He treated weak solutions of the Euler and the Navier-Stokes equations in the class $L^1(0,T;[L^2(\mathbb{R}^n)]^n)$ for the velocity, and $L^1(0,T;L^1(\mathbb{R}^n))$ for the pressure, respectively. If the solution (v,p) is of the form (1.1), then above classes for (v,p) correspond to the conditions that $V \in [L^2(\mathbb{R}^n)]^n$ and $\alpha > -1$ for the velocity, and $P \in L^1(\mathbb{R}^n)$ and $-1 < \alpha < n+1$ for the pressure, respectively. Hence, our result here could be regarded as one of the improvements of his in the sense that the assumption for the velocity $L^2(\mathbb{R}^n)$ is replaced by $X^{2,\infty}(\mathbb{R}^n)$, and there is no restriction for the range of α.

We next consider the self-similar singularities of strong solutions to (E). A function $V \in [C^1(\mathbb{R}^n)]^n$ is called a strong solution of (SE$_\alpha$) if V belongs to $[L^p(\mathbb{R}^n)]^n$ for some $p \in [1,\infty]$, satisfies the divergence-free condition, and there exists a function $P \in L^q(\mathbb{R}^n)$ with some $q \in [1,\infty)$ such that

$$\begin{aligned}
&\frac{\alpha}{\alpha+1}\int_{\mathbb{R}^n} V(x)\cdot\varphi(x)dx + \frac{1}{\alpha+1}\int_{\mathbb{R}^n} (x\cdot\nabla)V(x)\cdot\varphi(x)dx \\
&+\int_{\mathbb{R}^n} (V(x)\cdot\nabla)V(x)\cdot\varphi(x)dx - \int_{\mathbb{R}^n} P(x)\,\mathrm{div}\,\varphi(x)dx = 0
\end{aligned} \tag{1.4}$$

holds for all $\varphi \in [C_0^1(\mathbb{R}^n)]^n$.

Remark 1.4. We remark the uniqueness of the pressure for the strong solution of (SE$_\alpha$). Let V be a strong solution of (SE$_\alpha$) with $V \in [L^{2p}(\mathbb{R}^n)]^n$ for some $p \in (1,\infty)$. Then the pressure P associated with V can be chosen as $\widetilde{P}$,

$$\widetilde{P} = \sum_{j,k=1}^{n} R_j R_k (V_j V_k),$$

where $\{R_j\}_{j=1}^n$ are the n-dimensional Riesz transforms. Indeed, by the boundedness of R_j and the Fourier transforms, we have $\widetilde{P} \in L^p(\mathbb{R}^n)$ and

$$-\int_{\mathbb{R}^n} \widetilde{P}(x) \Delta\psi(x)dx = \sum_{j,k=1}^{n} \int_{\mathbb{R}^n} V_j(x)V_k(x)\partial_j\partial_k\psi(x)dx, \tag{1.5}$$

for all $\psi \in \mathscr{S}(\mathbb{R}^n)$. On the other hand, since $\operatorname{div} V = 0$, if we choose the vector test function φ in (1.4) as $\varphi = \nabla\psi$, $\psi \in C_0^\infty(\mathbb{R}^n)$, we have

$$-\int_{\mathbb{R}^n} P(x) \Delta\psi(x)dx = \sum_{j,k=1}^{n} \int_{\mathbb{R}^n} V_j(x)V_k(x)\partial_j\partial_k\psi(x)dx. \tag{1.6}$$

From (1.5), (1.6) and Weyl's lemma, we have $P - \widetilde{P} \in C^\infty(\mathbb{R}^n)$ and $\Delta(P - \widetilde{P}) = 0$. Then, it follows from the mean value property for harmonic functions that $P = \widetilde{P}$, which implies the uniqueness of the pressure.

Our result on strong solutions now reads:

Theorem 1.5. *Let $\alpha \in \mathbb{R} \setminus \{-1, n/2\}$ and let V be a strong solution of* (SE$_\alpha$) *with $V \in [(X^{2,\infty} \cap L^p)(\mathbb{R}^n)]^n$ for some finite $p \in [\frac{3n}{n-1}, \frac{4n}{n-2}]$. Then $V \equiv 0$ in $\mathbb{R}^n$.*

Remark 1.6. Let $\alpha \in \mathbb{R} \setminus \{-1, n/2\}$ and let V be a strong solution of (SE$_\alpha$) with

$$V(x) = O\big(|x|^{-n/2}\big), \quad \text{as } |x| \to \infty.$$

Then, since $V \in [(X^{2,\infty} \cap L^p)(\mathbb{R}^n)]^n$ for all $p > 2$, we have $V \equiv 0$ by Theorem 1.5. On the other hand, He [8] treated 3-dimensional case and showed the nonexistence result under the stronger condition such as

$$V(x) = O\big(|x|^{-k}\big), \quad P(x) = O\big(|x|^{-m}\big), \quad \text{as } |x| \to \infty,$$

where $k > 3/2$ and $m > 1/2$. Hence our result includes [8].

2 Proof of Theorems

Proof of Theorem 1.2. Let us first introduce the cut-off function $\sigma \in C_0^\infty(\mathbb{R}^n)$ such that

$$\sigma(x) = \widetilde{\sigma}(|x|) = \begin{cases} 1 & \text{if } |x| < 1, \\ 0 & \text{if } |x| > 2, \end{cases}$$

and $0 \leqslant \sigma(x) \leqslant 1$ for $1 \leqslant |x| \leqslant 2$. Given $R > 0$ and $j \in \{1, 2, \cdots, n\}$, we put

$$\sigma_R(x) = \sigma\left(\frac{x}{R}\right), \quad g_{R,j}(x) = \frac{x_j^2}{2}\sigma_R(x)$$

for $x = (x_1, \cdots, x_j, \cdots, x_n) \in \mathbb{R}^n$. Then, we choose the vector test function $\varphi \in [C_0^\infty(\mathbb{R}^n)]^n$ in (1.2) as

$$\varphi(x) = \nabla g_{R,j}(x) = \left(\frac{x_j^2}{2}\partial_{x_1}\sigma_R(x), \cdots, x_j\sigma_R(x) + \frac{x_j^2}{2}\partial_{x_j}\sigma_R(x), \cdots, \frac{x_j^2}{2}\partial_{x_n}\sigma_R(x)\right).$$

We remark that this type of vector test function was first introduced in [7]. Since

$$\frac{\alpha}{\alpha+1}\int_{\mathbb{R}^n} V(x)\cdot\varphi(x)dx = \frac{\alpha}{\alpha+1}\int_{\mathbb{R}^n} V(x)\cdot\nabla g_{R,j}(x)dx = 0$$

and

$$\frac{1}{\alpha+1}\int_{\mathbb{R}^n} V(x)\cdot\operatorname{div}(\varphi\otimes x)(x)dx = \frac{1}{\alpha+1}\int_{\mathbb{R}^n} V(x)\cdot\nabla\big((n-1)g_{R,j} + (x\cdot\nabla)g_{R,j}\big)(x)dx = 0$$

from the divergence-free condition for V, we have by (1.2) that

$$\begin{aligned}
0 &= \int_{\mathbb{R}^n} V_j(x)^2\sigma_R(x)dx \\
&\quad + \int_{\mathbb{R}^n} V_j(x)^2\left\{2x_j\partial_{x_j}\sigma_R(x) + \frac{x_j^2}{2}\partial_{x_j}^2\sigma_R(x)\right\}dx \\
&\quad + 2\sum_{k\neq j}\int_{\mathbb{R}^n} V_j(x)V_k(x)\left\{x_j\partial_{x_k}\sigma_R(x) + \frac{x_j^2}{2}\partial_{x_j}\partial_{x_k}\sigma_R(x)\right\}dx \\
&\quad + \frac{1}{2}\sum_{k,l\neq j}\int_{\mathbb{R}^n} V_k(x)V_l(x)x_j^2\partial_{x_k}\partial_{x_l}\sigma_R(x)dx \\
&\quad + \int_{\mathbb{R}^n} P(x)\sigma_R(x)dx \\
&\quad + \int_{\mathbb{R}^n} P(x)\left\{2x_j\partial_{x_j}\sigma_R(x) + \frac{x_j^2}{2}\Delta\sigma_R(x)\right\}dx \\
&= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned} \tag{2.1}$$

We shall derive estimates for I_2, I_3, I_4 and I_6. Let $m \in \mathbb{N}, \alpha \in (\mathbb{N}\cup\{0\})^n$ with $|\alpha| = m$ and $k, l \in \{1, 2, \cdots, n\}$. Since $\operatorname{supp}\partial_x^\alpha\sigma_R \subset \{x \in \mathbb{R}^n \,|\, R < |x| < 2R\}$, we have

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} V_k(x)V_l(x)x_j^m\partial_x^\alpha\sigma_R(x)dx\right| &\leqslant \int_{R<|x|<2R} |V_k(x)V_l(x)||x_j|^m\left|\frac{1}{R^m}\partial_x^\alpha\sigma\left(\frac{x}{R}\right)\right|dx \\
&\leqslant 2^m \sup_{1<|x|<2}|\partial_x^\alpha\sigma(x)|\int_{R<|x|<2R}|V(x)|^2dx,
\end{aligned}$$

which yields

$$|I_2|+|I_3|+|I_4| \leqslant C\int_{R<|x|<2R} |V(x)|^2 dx. \tag{2.2}$$

Similarly, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^n} P(x)x_j^m \partial_x^\alpha \sigma_R(x)dx\right| &\leqslant \int_{R<|x|<2R} |P(x)||x_j|^m \left|\frac{1}{R^m}\partial_x^\alpha \sigma\left(\frac{x}{R}\right)\right| dx \\ &\leqslant 2^m \sup_{1<|x|<2} |\partial_x^\alpha \sigma(x)| \int_{R<|x|<2R} |P(x)|dx,\end{aligned}$$

which yields

$$|I_6| \leqslant C\int_{R<|x|<2R} |P(x)|dx. \tag{2.3}$$

Since $P \in L^1(\mathbb{R}^n)$, it holds that

$$I_5 \longrightarrow \int_{\mathbb{R}^n} P(x)dx \tag{2.4}$$

as $R \longrightarrow \infty$. From (2.1), (2.2) and (2.3), we have

$$\begin{aligned}\int_{\mathbb{R}^n} V_j(x)^2 \sigma_R(x)dx \leqslant C\int_{R<|x|<2R} |V(x)|^2 dx + \left|\int_{\mathbb{R}^n} P(x)\sigma_R(x)dx\right| \\ + C\int_{R<|x|<2R} |P(x)|dx.\end{aligned} \tag{2.5}$$

Since $V \in [X^{2,\infty}(\mathbb{R}^n)]^n$ and $P \in L^1(\mathbb{R}^n)$, we obtain from (2.4) and (2.5) that

$$\limsup_{R\to\infty} \int_{\mathbb{R}^n} V_j(x)^2 \sigma_R(x)dx \leqslant C \limsup_{R\to\infty} \int_{R<|x|<2R} |V(x)|^2 dx + \|P\|_{L^1} < \infty,$$

which implies $V_j \in L^2(\mathbb{R}^n)$. Since $j \in \{1,2,\cdots,n\}$ is arbitrary, we have $V \in [L^2(\mathbb{R}^n)]^n$.

Now, we shall prove the identities (1.3). Since we have derived $V \in [L^2(\mathbb{R}^n)]^n$ and since $P \in L^1(\mathbb{R}^n)$ by the hypothesis, we have

$$\left|I_1 - \int_{\mathbb{R}^n} V_j(x)^2 dx\right| \leqslant \int_{|x|>R} V_j(x)^2 |1-\sigma_R(x)| dx \leqslant \int_{|x|>R} V_j(x)^2 dx \to 0 \tag{2.6}$$

as $R \longrightarrow \infty$. Moreover, by (2.2) and (2.3), we have

$$|I_2|+|I_3|+|I_4| \leqslant C\int_{R<|x|<2R} |V(x)|^2 dx \to 0, \tag{2.7}$$

$$|I_6| \leqslant C\int_{R<|x|<2R} |P(x)|dx \to 0 \tag{2.8}$$

as $R \longrightarrow \infty$. Hence letting $R \longrightarrow \infty$ in (2.1), from the convergences (2.6), (2.7), (2.4) and (2.8), we obtain the identity

$$\int_{\mathbb{R}^n} \{V_j(x)^2 + P(x)\}dx = 0$$

for all $j = 1,2,\cdots,n$. This completes the proof of Theorem 1.2. □

Proof of Theorem 1.5. As in the proof of Theorem 1.2, we consider the cut-off function $\sigma_R \in C_0^\infty(\mathbb{R}^n)$. Then, if we choose the test function $\varphi \in [C_0^1(\mathbb{R}^n)]^n$ in (1.4) as $\varphi(x) = \sigma_R(x)V(x)$, we obtain from integration by parts that

$$\begin{aligned}
0 &= \frac{2\alpha - n}{2(\alpha+1)} \int_{\mathbb{R}^n} \sigma_R(x)|V(x)|^2 dx - \frac{1}{2(\alpha+1)} \int_{\mathbb{R}^n} (x\cdot\nabla)\sigma_R(x)|V(x)|^2 dx \\
&\quad - \frac{1}{2}\int_{\mathbb{R}^n} (V(x)\cdot\nabla)\sigma_R(x)|V(x)|^2 dx - \int_{\mathbb{R}^n} (V(x)\cdot\nabla)\sigma_R(x)P(x)dx \\
&= J_1 + J_2 + J_3 + J_4.
\end{aligned} \tag{2.9}$$

For the estimate of J_2, we have

$$\begin{aligned}
|J_2| &\leqslant \frac{1}{2|\alpha+1|} \int_{R<|x|<2R} |x||\nabla\sigma_R(x)||V(x)|^2 dx \\
&\leqslant \frac{R}{|\alpha+1|} \int_{R<|x|<2R} \left|\frac{1}{R}\nabla\sigma\left(\frac{x}{R}\right)\right| |V(x)|^2 dx \\
&\leqslant \frac{1}{|\alpha+1|} \sup_{1<|x|<2} |\nabla\sigma(x)| \int_{R<|x|<2R} |V(x)|^2 dx.
\end{aligned} \tag{2.10}$$

Next, we derive the estimates for J_3 and J_4. Put

$$a = \frac{1 - \frac{1}{n} - \frac{3}{p}}{\frac{1}{2} - \frac{1}{p}}.$$

Note that $0 \leqslant a \leqslant 1$ for $n \geqslant 3$, $0 \leqslant a < 1$ for $n = 2$ and

$$\frac{1}{n} + \frac{2}{p} + \frac{a}{2} + \frac{1-a}{p} = 1,$$

for $n \geqslant 2$. Then, by the Hölder inequality, we have

$$\begin{aligned}
|J_3| &\leqslant \frac{1}{2}\int_{R<|x|<2R} |\nabla\sigma_R(x)||V(x)|^3 dx \\
&\leqslant \frac{1}{2}\|\nabla\sigma\|_{L^n}\|V\|_{L^p}^2\|V\chi_R\|_{L^2}^a\|V\chi_R\|_{L^p}^{1-a},
\end{aligned} \tag{2.11}$$

where χ_R is the characteristic function of the annulus $\{x \in \mathbb{R}^n \mid R < |x| < 2R\}$. As we mentioned in Remark 1.4, we have the representation of pressure $P = \sum_{j,k=1}^n R_jR_k(V_jV_k)$, which yields $\|P\|_{L^{\frac{p}{2}}} \leqslant C\|V\|_{L^p}^2$. Hence we have

$$\begin{aligned}
|J_4| &\leqslant \int_{R<|x|<2R} |\nabla\sigma_R(x)||P(x)||V(x)|dx \\
&\leqslant \|\nabla\sigma\|_{L^n}\|P\|_{L^{\frac{p}{2}}}\|V\chi_R\|_{L^2}^a\|V\chi_R\|_{L^p}^{1-a} \\
&\leqslant C\|\nabla\sigma\|_{L^n}\|V\|_{L^p}^2\|V\chi_R\|_{L^2}^a\|V\chi_R\|_{L^p}^{1-a}.
\end{aligned} \tag{2.12}$$

From (2.9), (2.10), (2.11) and (2.12), we obtain

$$\begin{aligned}
&\frac{|2\alpha-n|}{|2(\alpha+1)|}\int_{\mathbb{R}^n}\sigma_R(x)|V(x)|^2dx\\
&\quad\leqslant\frac{1}{|\alpha+1|}\sup_{1<|x|<2}|\nabla\sigma(x)|\int_{R<|x|<2R}|V(x)|^2dx\\
&\quad\quad+C\|\nabla\sigma\|_{L^n}\|V\|_{L^p}^2\|V\chi_R\|_{L^2}^a\|V\chi_R\|_{L^p}^{1-a}.
\end{aligned}\tag{2.13}$$

Since $V\in[(X^{2,\infty}\cap L^p)(\mathbb{R}^n)]^n$, we obtain from (2.13) that

$$\begin{aligned}
&\limsup_{R\to\infty}\int_{\mathbb{R}^n}\sigma_R(x)|V(x)|^2dx\\
&\leqslant\frac{2}{|2\alpha-n|}\sup_{1<|x|<2}|\nabla\sigma(x)|\limsup_{R\to\infty}\int_{R<|x|<2R}|V(x)|^2dx\\
&\quad+\frac{C|2(\alpha+1)|}{|2\alpha-n|}\|\nabla\sigma\|_{L^n}\|V\|_{L^p}^2\limsup_{R\to\infty}\|V\chi_R\|_{L^2}^a\|V\chi_R\|_{L^p}^{1-a}\\
&<\infty,
\end{aligned}$$

which implies $V\in[L^2(\mathbb{R}^n)]^n$.

Next, we will prove the convergences of J_1,J_2,J_3 and J_4. Since we have derived $V\in[L^2(\mathbb{R}^n)]^n$, we have

$$\begin{aligned}
\left|J_1-\frac{2\alpha-n}{2(\alpha+1)}\int_{\mathbb{R}^n}|V(x)|^2dx\right|&\leqslant\frac{|2\alpha-n|}{|2(\alpha+1)|}\int_{|x|>R}|V(x)|^2|1-\sigma_R(x)|dx\\
&\leqslant\frac{|2\alpha-n|}{|2(\alpha+1)|}\int_{|x|>R}|V(x)|^2dx\to 0,
\end{aligned}$$

as $R\to\infty$, and from (2.10), (2.11) and (2.12)

$$\begin{aligned}
|J_2|&\leqslant\frac{1}{|\alpha+1|}\sup_{1<|x|<2}|\nabla\sigma(x)|\int_{R<|x|<2R}|V(x)|^2dx\to 0,\\
|J_3|+|J_4|&\leqslant C\|\nabla\sigma\|_{L^n}\|V\|_{L^p}^2\|V\chi_R\|_{L^2}^a\|V\chi_R\|_{L^p}^{1-a}\to 0,
\end{aligned}$$

as $R\to\infty$. Hence letting $R\to\infty$ in (2.9), from the above convergences, we obtain

$$\frac{2\alpha-n}{2(\alpha+1)}\int_{\mathbb{R}^n}|V(x)|^2dx=0,$$

which implies $V\equiv 0$. This completes the proof of Theorem 1.5. □

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