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Approximating the path-distance-width for asteroidal triple-free graphs

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Abstract

The path-distance-width of a graph is a graph parameter to measure how close the graph is to a path. In this paper, we give an approximation algorithm with a constant approximation ratio for path-distance-width on AT-free graphs.

1 Introduction

The path-distance-width is a graph parameter to measure how close the graph is to a path [19, 18]. There are several other such graph parameters such as path-width and bandwidth. Intuitively, the classes of graphs of bounded path-distance-width, bounded bandwidth, and bounded path-width have *chain-like structures*. There are other graph classes which also have chain-like structures, such as interval graphs and AT-free graphs (see [4] for details on interval graphs and AT-free graphs). It is known that there are relationships among those graph parameters and graph classes (cf. [10]).

The study is motivated by the research on bandwidth of AT-free graphs [11, 8]. To see the motivation, let us briefly review the history of the research of bandwidth for interval graphs and AT-free graphs. Imaginably, if we restrict our input graphs to from interval graphs or AT-free graphs, then we would be able to find easily its chain-like structure (such as its interval representation or a dominating pair), and then from the chain-like structure we might be able to compute the bandwidth. It was, however, not known the computational complexity of computing the bandwidth for interval graphs [9]. But then it turned out that the decision problem can be solved in polynomial time (see [17]). Since interval graphs are AT-free graphs, it would be natural to ask whether or not the bandwidth decision problem for AT-free graphs can be solved in polynomial time. Unfortunately, it is known that the bandwidth decision problem for AT-free graphs is NP-complete (cf. [13, 11]). Fortunately, however, it is known that for AT-free graphs, the bandwidth decision problem can be approximated in polynomial time within a constant factor [11].

In a sense, bandwidth and path-distance-width have features in common. In fact, there is a similarity between the problem of computing the path-distance-width and the problem of computing the bandwidth: Both problems do not admit PTAS even for trees [1, 18]. So, it would be reasonable to ask the computational complexity of computing the path-distance-width for AT-free graphs. Unfortunately, so far, we do not know the complexity even for interval graphs. In this paper, however, we consider the problem of approximating path-distance-width for AT-free graphs and interval graphs. Although some techniques developed in the research on bandwidth can be carried over into the research on path-distance-width, the path-distance-width problem has a serious drawback which bandwidth problem does not have: Path-distance-width is not closed under the edge deletion. In many cases, this drawback makes the design and analysis of algorithms very difficult. In this study, however, it turns out that the restriction to AT-free graphs is enough to overcome the drawback for achieving a constant factor. In this paper, we give an approximation algorithm with a constant approximation ratio for path-distance-width on AT-free graphs and also a specialized approximation algorithm for interval graphs.

2 Definitions and notation

Let G be a graph. $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. We denote the maximum degree of G by $\Delta(G)$. For a subset $S \subseteq V(G)$ and a vertex $v \in V(G)$, $dist_G(S, v)$ denotes the distance between S and v in G. We denote $\max\{dist_G(S, v) \mid v \in V(G)\}$ by $e(S)$. We simply write $dist_G(u, v)$ and $e_G(u)$ instead of $dist_G(\{u\}, v)$ and $e_G(\{u\})$. The *kth power* of $G = (V, E)$, denoted by G^k, is the graph (V, E') such that $\{u, v\} \in E'$ if and only if $dist_G(u, v) \leq k$. An independent set of three vertices is called an *asteroidal triple* if every two of them are connected by a path avoiding the neighborhood of the third. A graph is *asteroidal triple-free* (AT-free for short), if it contains no asteroidal triple. $M(n)$ denotes the time complexity of multiplying two $n \times n$ matrices of integers. For the complexity of matrix multiplication, see, for example, [14, 20].

A sequence $D = (X_1, \ldots, X_t)$ of subsets of vertices is the *path-distance decomposition* (or simply *decomposition*) of a graph $G = (V, E)$ if X_i is the set of vertices of distance $i - 1$ from X_1 for each $1 \leq i \leq t$, where $t = e(X_1)$. Each X_i is called a *level* and specially X_1 is called the *initial set*. We will write the decomposition with an initial set X_1 by $D(X_1, G)$ or simply $D(X_1)$ or more simply D if it is clear from the context. For convenience, we sometimes use X_i for $i > t$ and it is treated as an empty set. The width of D, denoted by $pdw_D(G)$, is defined as $\max_{0 \leq i \leq t} |X_i|$. The path-distance-width of G, denoted by $pdw(G)$, is defined as $min_{X_1 \subseteq V} pdw_{D(X_1)}(G)$. A subset $X \subseteq V(G)$ is an *optimal initial set* of G if $pdw(G) = pdw_{D(X)}(G)$.

An *interval graph* is a graph whose vertices can be mapped to distinct intervals in the real line such that two vertices are adjacent in the graph if and only if their corresponding intervals overlap. For an interval I, we denote the left and right endpoints by $l(I)$ and $r(I)$, respectively. In this paper, we identify an interval with the corresponding vertex, for instance, we sometimes write $\{I_i, I_j\} \in E(G)$, $dist_G(I_i, I_j)$, and so on. For an interval representation I of a graph G, $l(\mathsf{I})$ and $r(\mathsf{I})$ denote $\min\{l(I) \mid I \in \mathsf{I}\}$ and $\max\{r(I) \mid I \in \mathsf{I}\}$, respectively.

3 Results

3.1 Path-distance-width of the kth power of a graph

Lemma 3.1. *Let G be a graph, d be a positive integer and X_1 be a subset of $V(G)$. And let $D(X_1, G) = (X_1, \ldots, X_t)$ and $D(X_1, G^d) = (Y_1, \ldots, Y_u)$ be the decompositions of G and G^d, respectively, with X_1 as the initial set (Note that $X_1 = Y_1$). Then,*

1. *for each $2 \le i \le u$, Y_i is contained in $\cup_k X_k$, where the union is taken over all k such that $d(i-2) < k \le d(i-1)$,*
2. *for each $1 \le i \le u$, there exists an index j such that X_i is contained in Y_j.*

Proof. (1) Let v be a vertex in Y_i. As $v \in Y_i$, we have $dist_{G^d}(X_1, v) = i-1$. Since if $dist_G(X_1, v) \le d(i-2)$ then $dist_{G^d}(X_1, v) \le i-2$, we have $d(i-2) < dist_G(X_1, v)$, Similarly, we also have $dist_G(X_1, v) \le d(i-1)$. Hence, we have $d(i-2) < dist_G(X_1, v) \le d(i-1)$, and this completes the proof of (1).

(2) Suppose that there is a level X_i which intersects Y_j and Y_k for some $1 \le j < k \le u$. Then let $x \in Y_j \cap X_i$ and $y \in Y_k \cap X_i$. Since $x \in Y_j \cap X_i$ and the above (1), $d(j-2) < i \le d(j-1)$. On the other hand, since $y \in Y_k \cap X_i$ and the above (1), $d(k-2) < i \le d(k-1)$. Thus, we have $i \le d(j-1)$ and $d(k-2) < i$. However, as $j < k$, we have a contradiction $i \le d(j-1) \le d(k-2) < i$. □

Lemma 3.2. *For a graph G, $pdw(G) \le pdw(G^d) \le d \cdot pdw(G)$.*

Proof. We first show that $pdw(G) \le pdw(G^d)$. Let X_1 be an optimal initial set of G^d. From (2) of Lemma 3.1, $pdw(G) \le pdw_{D(X_1,G)}(G) \le pdw_{D(X_1,G^d)}(G^d)$.

We now show that $pdw(G^d) \le d \cdot pdw(G)$. Let X_1 be an optimal initial set of G. From (1) of Lemma 3.1, $pdw(G^d) \le pdw_{D(X_1,G^d)}(G^d) \le d \cdot pdw_{D(X_1,G)}(G) = d \cdot pdw(G)$. □

3.2 Approximability of path-distance-width for k-cocomparability graphs

In this subsection, we will need the following definition and results.

A graph $G = (V, E)$ is a *comparability graph* if there exists a linear ordering $<$ on V such that for any three vertices $u < v < w$, $\{u, v\} \in E$ and $\{v, w\} \in E$ implies $\{u, w\} \in E$. A graph $G = (V, E)$ is a *cocomparability graph* if G is the complement of a comparability graph. It is known that G is a cocomparability graph iff it has a *cocomparability ordering*, i.e., there exists a linear order $<$ on V such that for any three vertices $u < v < w$, $\{u, w\} \in E$ implies $\{u, v\} \in E$ or $\{v, w\} \in E$. There is another characterization due to Damaschke:

Theorem 3.3 ([6]). *Let G be a connected graph. Then G is a cocomparability graph iff G has a linear ordering $<$ on $V(G)$ such that $d_G(x, y) + d_G(y, z) \le d(x, z) + 2$ for all $x < y < z$.*

Actually, any cocomparability ordering satisfies the inequality in Theorem 3.3.

In [5], Chang et al. generalized cocomparability graphs and showed the following results.

Definition 1 ([5]). Let G be a graph and k a positive integer. A *k-cocomparability ordering* (k-CCPO) of G is an ordering on $V(G)$ such that for every any three vertices $u < v < w$, $dist_G(u,w) \le k$ implies $dist_G(u,v) \le k$ or $dist_G(v,w) \le k$. A graph is called a k-cocomparability graph if it admits a k-CCPO.

Note that a 1-cocomparability ordering is just a cocomparability ordering.

Lemma 3.4 ([5]). *A graph G is a k-cocomparability graph if and only if G^k is a cocomparability graph.*

Theorem 3.5 ([5]). *AT-free graphs are 2-cocomparability graphs.*

For cocomparability graphs, from Lemma 3.2, we can show the next lemma.

Lemma 3.6. *Let G be a cocomparability graph, and s be the first vertex in a cocomparability ordering of G. Then, $pdw_{D(\{s\},G)}(G) \le 4pdw(G)$.*

Proof. Consider the largest level X_i in $D(\{s\},G)$, i.e., $pdw_{D(\{s\},G)}(G) = |X_i|$. From Theorem 3.3, we have $dist_G(x,y) \le 2$ for any vertices $x,y \in X_i$, which implies that X_i is a clique in G^2. Since any clique in G^2 cannot intersect more than two levels, we know $|X_i|/2 \le pdw(G^2) \le 2pdw(G)$. Therefore, $pdw_{D(\{s\},G)}(G) \le 4pdw(G)$. □

By combining Lemmas 3.2, 3.4, and 3.6, we have the next theorem.

Theorem 3.7. *There is an $O(M(n)\log n)$ time algorithm that finds an initial set of a path-distance decomposition of width at most $4k$ times the optimal for a given graph G with n vertices, where k is the smallest integer such that G admits a k-cocomparability ordering.*

Proof. To find the initial set, we will need G^i for each $1 \le i \le d$, where d is the diameter of G. To obtain $G^2, \dots, G^d$, we first establish the distance matrix of G (i.e., the (u,v) entry in the matrix is the distance between u and v). This can be done in $O(M(n)\log n)$ time (e.g., see [15]).

Next, we find the smallest integer k such that G^k is a cocomparability graph by using the binary search. That is, in the binary search, we check if the complement graph $\overline{G^i}$ is a comparability graph. That is, we apply an $O(n^2)$ time *orientation algorithm* in [16] to $\overline{G^i}$, then we check if the orientation of $\overline{G^i}$ is transitive by computing the transitive closure in $O(M(n))$ time. If the orientation is transitive then we can conclude G^i is a cocomparability graph, otherwise G^i is not cocomparability. This recognition test can be checked in $O(M(n))$ time, Thus, the binary search can be done in $O(M(n)\log n)$ time.

After finding the smallest integer k, we compute the initial vertex s in a cocomparability ordering of G^k. To this end, we just seek an in-degree 0 vertex in the oriented graph G^k, and take it as s. The reason why we can do so is that there is a topological sort π of the oriented graph G^k such that s is the initial vertex in π (Note that π can be considered as a cocomparability ordering of G^k).

As a result, the total time is $O(M(n)\log n)$. □

From Theorem 3.5, we have the following corollary.

Corollary 3.8. *For an AT-free graph G, the path-distance-width can be approximated within* *a factor 8 in $O(M(|V(G)|))$ time.*

3.3 Approximability of path-distance-width for interval graphs

Let I be an interval representation of an interval graph G. A sequence $(I_1, \ldots, I_n)$ of the elements in I is a *left endpoint order* of I if $i \leq j$ iff $l(I_i) \leq l(I_j)$ for $I_i, I_j \in \mathsf{I}$.

Lemma 3.9. *Let $(I_1, \ldots, I_n)$ be a left endpoint order in an interval representation of an interval graph G, d be an integer such that $1 \leq d \leq e_G(I_1) - 1$, and I_i be an interval at distance d from I_1 which has the largest right endpoint among all intervals at distance d from I_1. Then, I_i intersects with all intervals at distance $d + 1$ from I_1.*

Proof. Suppose that there is an interval I_j such that I_j is at distance $d + 1$ from I_1 and I_j does not intersect with I_i. Then, we have the following two cases, and in each case we have a contradiction. Recall that we identify an interval with the corresponding vertex.

Case 1: I_j lies to the left of I_i (i.e., $r(I_j) < l(I_i)$). Consider a shortest path from I_1 to I_i. Clearly, I_j intersects an interval in the shortest path. This means that the distance I_j and I_1 is at most d, a contradiciton.

Case 2: I_j lies to the right of I_i (i.e., $r(I_i) < l(I_j)$). Clearly, I_j intersects an interval I_k at distance d from I_1. However, from the definition of I_i, we have $r(I_k) \leq r(I_i) < l(I_j)$, which is a contradiction. □

From Lemma 3.9, we have the following corollary.

Corollary 3.10. *Let $(I_1, \ldots, I_n)$ be a left endpoint order in an interval graph G and let $(X_1, \ldots, X_t)$ be the decomposition $D(\{I_1\}, G)$. Then, for each $1 \leq d \leq t - 1$ there is a vertex $u \in X_d$ which is adjacent to all vertices in X_{d+1}.*

From Corollary 3.10 and the fact that $\Delta(G)/3 \leq pdw(G)$, we have the following lemma.

Lemma 3.11. *Let $(I_1, \ldots, I_n)$ be a left endpoint order in an interval graph G. Then, $pdw_{D(\{I_1\},G)}(G) \leq \Delta(G)$. Thus, $pdw_{D(\{I_1\},G)}(G) \leq 3pdw(G)$.*

From Lemma 3.11, we have the next theorem.

Theorem 3.12. *For an interval graph G, the path-distance-width can be approximated within a factor 3 in $O(|V(G)| + |E(G)|)$ time.*

4 Conclusion

In this paper, we give approximation algorithms with constant approximation ratios for path-distance-width on AT-free graphs and interval graphs. Unfortunately, however, we do not know the computational complexity of computing the path-distance-width for AT-free graphs, indeed even for interval graphs. Also it is not elucidated the tightness of the ratios of our proposed algorithms.

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