A Continuum Dynamics on Vector Bundle

理化学研究所 山岡 英孝 (Hidetaka Yamaoka)[1]
Computational Cell Biomechanics Team,
VCAD System Research Program,
RIKEN

Abstract

The Cosserat theory for continua with microstructure can be geometrically interpreted as a continuum dynamics on vector bundle. In this talk, we begin with geometrical settings for the continua with microstructure and construct its dynamics as continuum dynamics on vector bundle. As an example, we deal with a Cosserat rod, which is considered as a one-dimensional continuum with microstructure. In addition, we suggest that we can describe lower microstructure of the elastic rods, by extending the base manifold of the vector bundle to a three-dimensional continuum. Such rod model well represents biomolecules involving various interacting factors, so that the model can be applied to analysis of deformation behavior of biomolecules. We expect that our geometrical method would contribute to development of the molecular biomechanics.

1 Introduction

We attempt to construct geometrical foundations and dynamical frameworks of a directed medium based on the fiber bundle theory. The directed medium is a continuum with microstructures that is described by a deformable vector, called a director. Studies on the directed medium were actively pursued in the 1960s, for example, by Ericksen [1], Toupin [2], and Eringen and Şuhubi [3] and in recent years, have been investigated from many point of views such as elast-plasticity [4, 5], advanced materials [6], and biomechanics [7, 8]. In contrast, since about the 1960s, the elastic theory has been reconstructed using differential geometry, for example, by Green and Rivilin [9], Noll [10], and Wang [11]. A modern text by Marsden and Hughes [12] helps us to consider geometrical settings of elasticity (see also an early textbook [13]). In this study, we develop the dynamics of the directed medium based on the fiber bundle theory in differential geometry.

In geometric continuum mechanics, an elastic body is viewed as a differentiable manifold, while a directed medium is viewed as a vector bundle whose fiber denotes a collection of the deformable directors. Hence, the mechanical behaviors of the directed medium should be described as the continuum dynamics on a tangent bundle of a vector bundle. Thus, we begin with a geometrical setting of the continuum dynamics on a vector bundle, and derive a weak form and equations of motion for the directed medium. For future applications, we use elasticity notations to provide a framework of continuum dynamics on the vector bundle and present some figures for better understanding. Moreover, we

---

[1]hyamaoka@riken.jp

apply our resultant equations to a Cosserat rod, as an example, and find that the derived equations of motion coincide with the balance laws of large deformable rods. It is simple to prove such a coincidence if the equations of motion are restricted to the special Cosserat rod with undeformed cross-section.

We can use our description to examine such macro–micro interactive mechanisms, if we have to consider only the geometrical structures of objects using the mechanisms, *i.e.*, the corresponding base manifolds and fiber spaces. Such geometrical considerations help us to improve our understanding of the complicated mechanical behaviors of various structures associated with the macro–micro interactive mechanisms.

2 Geometry and kinematics

In the geometric continuum mechanics, an elastic body is viewed as an m-dimen-sional Riemannian manifold $\mathscr{B}$, and deforms in an ambient space $\mathscr{S}$, an n-dimen-sional Rieman-nian manifold $(m \leq n)$ [12]. In contrast, when we consider a deformation of a continuum with microstructure, we must replace these manifolds with principal fiber bundles, de-noting by $\mathscr{P}$ and $\mathscr{Q}$, respectively, over the manifolds $\mathscr{B}$ and $\mathscr{S}$. The microstructure is often expressed by a r-dimensional vector $(r \leq n)$, called director. In this case, the space consisting of the directors, $\mathscr{V}$, is exactly the fiber of the bundle $\mathscr{P} \to \mathscr{B}$, and then $\mathscr{P}$ can be considered as the real vector bundle $\mathscr{P} \simeq \mathscr{B} \times \mathscr{V}$.

A configuration of the continuum with microstructure is given by a smooth embedding $\Phi : \mathscr{P} \to \mathscr{Q}$, then the configuration space is a space of all embeddings,

$$\mathscr{C} = \{\Phi : \mathscr{P} \to \mathscr{Q}, \text{ smooth embedding}\}. \tag{2.1}$$

Indeed, for an arbitrary point $p = (b, v) \in \mathscr{P} \simeq \mathscr{B} \times \mathscr{V}$ $(b \in \mathscr{B}$ and $v \in \mathscr{V})$, we can define the embedding Φ through embedding of the base manifold $\phi : \mathscr{B} \to \mathscr{S}$ and projection onto the fiber $\varphi : \mathscr{B} \times \mathscr{V} \to \mathbb{R}^r$;

$$\Phi(p) = (\phi(b), \varphi(b, v)). \tag{2.2}$$

It is easy to verify that this map is the embedding. We note here that the ambient bundle $\mathscr{Q}$ is the vector bundle $\mathscr{Q} \to \mathscr{S}$ with fiber $\mathbb{R}^r$, *i.e.*, $\mathscr{Q} \to \mathscr{S} \times \mathbb{R}^r$, as shown in Fig. 1.

For sake of simplicity, we take the ambient space as the n-dimensional Euclidean space; $\mathscr{S} \simeq \mathbb{R}^n$, and we embed the fiber space $\mathbb{R}^r$ into the same Euclidean space $\mathbb{R}^n$. Then we denote position vectors of points in the reference body $\mathscr{B}_0$ and current body $\mathscr{B}$ by $X = \phi_0(b_0)$ $(b_0 \in \mathscr{B}_0)$ and $x = \phi(b)$ $(b \in \mathscr{B})$, respectively. Also, we denote reference and current directors associated with their points by $\Xi = \varphi_0(b_0, v_0)$ $(v_0 \in \mathscr{V}_0)$ and $\xi = \varphi(b, v)$ $(v \in \mathscr{V})$, as shown Fig. 1. Accordingly, three deformation gradient tensors can be defined as

$$F = \frac{\partial x}{\partial X}, \quad \mathfrak{F} = \frac{\partial \xi}{\partial X}, \quad \mathcal{F} = \frac{\partial \xi}{\partial \Xi}, \tag{2.3}$$

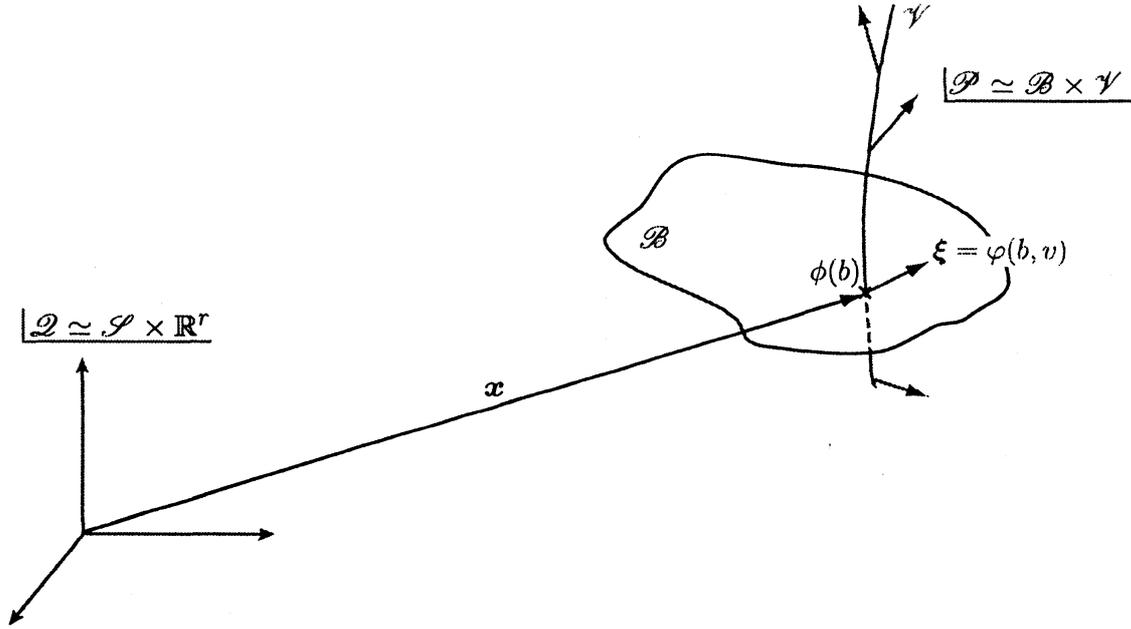called the macro, mixture, and micro deformation gradient, respectively.

Figure 1: Illustration of the continuum with microstructure, $\mathscr{P} \simeq \mathscr{B} \times \mathscr{V}$, embedded into the ambient bundle, $\mathscr{Q} \simeq \mathscr{S} \times \mathbb{R}^r$, and its local coordinates in $\mathscr{Q}$.

Since the trivialization of bundle $\mathscr{Q}$ is expressed as

$$\mathscr{Q} \simeq \mathscr{S} \times \mathbb{R}^r, \tag{2.4}$$

the flat connection is defined on the bundle $\mathscr{Q}$. Accordingly, the tangent bundle $T\mathscr{Q}$ is decomposed into the tangent bundles $T\mathscr{S}$ and $T\mathbb{R}^r$;

$$T\mathscr{Q} \simeq T\mathscr{S} \oplus T\mathbb{R}^r. \tag{2.5}$$

Also, the cotangent bundle $T^*\mathscr{Q}$ is decomposed by the flat connection, so that one-form $dq \in T_q^*\mathscr{Q}$ is put in the form

$$dq = dx + d\xi, \tag{2.6}$$

and expressed, in terms of the reference coordinates $q_0 = (X, \Xi)$, as

$$dq = (F + \mathfrak{F})dX + \mathcal{F}d\Xi. \tag{2.7}$$

From Eq. (2.7), the quadratic form dq^2 is calculated as

$$
\begin{aligned}
dq^2 =& dX^\top (F + \mathfrak{F})^\top (F + \mathfrak{F})dX \\
& + dX^\top (F + \mathfrak{F})^\top \mathcal{F}d\Xi + d\Xi^\top \mathcal{F}^\top (F + \mathfrak{F})dX \\
& + d\Xi^\top \mathcal{F}^\top \mathcal{F}d\Xi,
\end{aligned}
\tag{2.8}
$$

where $\top$ denotes the transposition of tensors. Here, we set

$$C = (F + \mathfrak{F})^\top (F + \mathfrak{F}), \tag{2.9a}$$

$$\mathcal{C} = \mathcal{F}^\top \mathcal{F}, \tag{2.9b}$$

$$\mathfrak{C} = \mathcal{F}^\top (F + \mathfrak{F}), \tag{2.9c}$$

called macro deformation, micro deformation, and (macro-micro) mixture deformation, respectively. The reference deformations put in the form

$$C_0 = (I + \mathfrak{F}_0)^\mathsf{T}(I + \mathfrak{F}_0), \tag{2.10a}$$

$$\mathcal{C}_0 = I, \tag{2.10b}$$

$$\mathfrak{C}_0 = I^\mathsf{T}(I + \mathfrak{F}_0), \tag{2.10c}$$

where I is n-th order identity tensor, and where $\mathfrak{F}_0 = \partial \Xi / \partial X$, as well. Thus, we have the difference of the current and reference quadratic forms,

$$\begin{aligned} dq^2 - dq_0^2 = &dX^\mathsf{T}(C - C_0)dX \\ &+ dX^\mathsf{T}(\mathcal{C} - \mathcal{C}_0)^\mathsf{T}d\Xi + d\Xi^\mathsf{T}(\mathcal{C} - \mathcal{C}_0)dX \\ &+ d\Xi^\mathsf{T}(\mathfrak{C} - \mathfrak{C}_0)d\Xi, \end{aligned} \tag{2.11}$$

so that we define the strains as

$$E = \frac{1}{2}(C - C_0), \tag{2.12a}$$

$$\mathcal{E} = \frac{1}{2}(\mathcal{C} - \mathcal{C}_0), \tag{2.12b}$$

$$\mathfrak{E} = \frac{1}{2}(\mathfrak{C} - \mathfrak{C}_0). \tag{2.12c}$$

Then we call them macro strain, micro strain, and (macro-micro) mixture strain, respectively. Here, we comment on the terminologies used by Eringen's textbook for the microcontinuum [14]. In the textbook, our mixture deformation $\mathfrak{C}$ is decomposed into $\mathcal{F}^\mathsf{T}F$ and $\mathcal{F}^\mathsf{T}\mathfrak{F}$, called the Cosserats deformation tensor, when the director is deformed rigidly, "micropolar continua" according to the textbook, and the wryness tensor, respectively. Additionally, the micro deformation tensor is defined in the same manner, while the macro deformation tensor is linearized to $F^\mathsf{T}F$.

3 Dynamics

Now, we consider a Lagrangian $\mathcal{L} = \mathcal{T} - \mathcal{W}$, where $\mathcal{T}$ is the kinetic energy, defined through a metric on $\mathscr{C}$, and $\mathcal{W}$ is a potential function on $\mathscr{C}$. Then the dynamics of the continuum with microstructure is described on the tangent bundle $T\mathscr{C}$ of the configuration space $\mathscr{C}$, that is, the Lagrangian $\mathcal{L}$ is defined as a function of the tangent bundle $T\mathscr{C}$ to $\mathbb{R}$,

$$\mathcal{L}(\Phi, \dot\Phi) = \mathcal{T}(\Phi, \dot\Phi) - \mathcal{W}(\Phi). \tag{3.1}$$

Usually, the strain energy $\mathcal{W}(\Phi)$ is expressed as a functional of $\mathscr{W}$, which is difined as a function of the deformation gradient tensors F, $\mathfrak{F}$, and $\mathcal{F}$:

$$\mathcal{W}(\Phi) = \int_{\mathscr{P}} \mathscr{W}(F, \mathfrak{F}, \mathcal{F})dV. \tag{3.2}$$

Here, dV is the volume form of the material bundle $\mathscr{P}$.

We also denote the Lagrangian and kinetic enegy densities by $\mathscr{L}$ and $\mathscr{T}$, respectively, i.e., $\mathcal{L} = \int \mathscr{L}dV$ and $\mathcal{T} = \int \mathscr{T}dV$, and, for simplicity, we consider those densities as functions of the local coordinates $(\boldsymbol{x}, \boldsymbol{\xi}, \dot{\boldsymbol{x}}, \dot{\boldsymbol{\xi}})$ on the tangent bundle $T\mathscr{Q}$. In this case, the Hamilton's principle for any time interval $[t_0, t_1]$ is expressed as follows:

$$\int_{t_0}^{t_1} \int_{\Phi(\mathscr{P})} \left(\frac{\partial \mathscr{L}}{\partial \boldsymbol{x}} \cdot \delta \boldsymbol{x} + \frac{\partial \mathscr{L}}{\partial \dot{\boldsymbol{x}}} \cdot \delta \dot{\boldsymbol{x}} + \frac{\partial \mathscr{L}}{\partial \boldsymbol{\xi}} \cdot \delta \boldsymbol{\xi} + \frac{\partial \mathscr{L}}{\partial \dot{\boldsymbol{\xi}}} \cdot \delta \dot{\boldsymbol{\xi}} \right.$$
$$\left. + \frac{\partial \mathscr{L}}{\partial \boldsymbol{F}} : \delta \boldsymbol{F} + \frac{\partial \mathscr{L}}{\partial \mathfrak{F}} : \delta \mathfrak{F} + \frac{\partial \mathscr{L}}{\partial \mathcal{F}} : \delta \mathcal{F} \right) \Phi(dV)dt = 0, \qquad (3.3)$$

where we have denoted, by $\cdot$ and $:$, the inner product of vectors and double contraction of tensors, respectively, or equivalently the simple-dot and double-dot products in the dyadics. Then, by performing partial integration and using the divergence theorem, we obtain the weak form for the continuum with microstructure,

$$\int_{t_0}^{t_1} \int_{\Phi(\mathscr{P})} \left[\left(\frac{\partial}{\partial t}\left(\frac{\partial \mathscr{T}}{\partial \dot{\boldsymbol{x}}}\right) - \frac{\partial \mathscr{T}}{\partial \boldsymbol{x}} - \frac{\partial}{\partial \boldsymbol{X}} \cdot \left(\frac{\partial \mathscr{W}}{\partial \boldsymbol{F}}\right) \right) \cdot \delta \boldsymbol{x} \right.$$
$$\left. + \left(\frac{\partial}{\partial t}\left(\frac{\partial \mathscr{T}}{\partial \dot{\boldsymbol{\xi}}}\right) - \frac{\partial \mathscr{T}}{\partial \boldsymbol{\xi}} - \frac{\partial}{\partial \boldsymbol{X}} \cdot \left(\frac{\partial \mathscr{W}}{\partial \mathfrak{F}}\right) - \frac{\partial}{\partial \Xi} \cdot \left(\frac{\partial \mathscr{W}}{\partial \mathcal{F}}\right) \right) \cdot \delta \boldsymbol{\xi} \right] \Phi(dV)dt$$
$$+ \int_{t_0}^{t_1} \int_{\Phi(\partial\mathscr{P})} \left[\boldsymbol{N} \cdot \frac{\partial \mathscr{W}}{\partial \boldsymbol{F}} \cdot \delta \boldsymbol{x} + \boldsymbol{N} \cdot \left(\frac{\partial \mathscr{W}}{\partial \mathfrak{F}} + \frac{\partial \mathscr{W}}{\partial \mathcal{F}}\right) \cdot \delta \boldsymbol{\xi} \right] \Phi(dA)dt = 0, \qquad (3.4)$$

where $\partial\mathscr{P}$ denotes the boundary of the material bundle $\mathscr{P}$, and $\boldsymbol{N}$ and dA are the unit normal to $\partial\mathscr{P}$ and the area form of $\partial\mathscr{P}$, respectively.

Finally, we assume that the Lagrangian density $\mathscr{L} = \mathscr{T} - \mathscr{W}$ has a compact support and that the variations are fixed at the end points, $\delta \boldsymbol{x} = \delta \boldsymbol{\xi} = 0$ $(t = t_0, t_1)$. Then we have equations of motion for the continuum with microstructure

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathscr{T}}{\partial \dot{\boldsymbol{x}}}\right) = \frac{\partial \mathscr{T}}{\partial \boldsymbol{x}} + \frac{\partial}{\partial \boldsymbol{X}} \cdot \left(\frac{\partial \mathscr{W}}{\partial \boldsymbol{F}}\right), \qquad (3.5a)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathscr{T}}{\partial \dot{\boldsymbol{\xi}}}\right) = \frac{\partial \mathscr{T}}{\partial \boldsymbol{\xi}} + \frac{\partial}{\partial \boldsymbol{X}} \cdot \left(\frac{\partial \mathscr{W}}{\partial \mathfrak{F}}\right) + \frac{\partial}{\partial \Xi} \cdot \left(\frac{\partial \mathscr{W}}{\partial \mathcal{F}}\right). \qquad (3.5b)$$

By introducing the generalized Piola-Kirchhoff stress tensors,

$$\boldsymbol{P} = \frac{\partial \mathscr{W}}{\partial \boldsymbol{F}}, \qquad \mathfrak{P} = \frac{\partial \mathscr{W}}{\partial \mathfrak{F}}, \qquad \mathcal{P} = \frac{\partial \mathscr{W}}{\partial \mathcal{F}}, \qquad (3.6)$$

we obtain the equations of motion, in terms of the generalized stress tensors,

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathscr{T}}{\partial \dot{\boldsymbol{x}}}\right) = \frac{\partial \mathscr{T}}{\partial \boldsymbol{x}} + \boldsymbol{P} \cdot \left(\frac{\partial}{\partial \boldsymbol{X}}\right), \qquad (3.7a)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathscr{T}}{\partial \dot{\boldsymbol{\xi}}}\right) = \frac{\partial \mathscr{T}}{\partial \boldsymbol{\xi}} + \mathfrak{P} \cdot \left(\frac{\partial}{\partial \boldsymbol{X}}\right) + \mathcal{P} \cdot \left(\frac{\partial}{\partial \Xi}\right). \qquad (3.7b)$$

Thus, the balance laws for the continuum with microstructure are reconstituted.

4 An example: Cosserat rods

As an example, we consider a Cossrat rod laid in the three-dimensional Euclidean space $\mathbb{R}^3$ ($n = 3$). In this case, the bundle $\mathscr{P}$ is the Cosserat rod, a one-dimensional continuum $\mathscr{B}$ expresses the center axis of rod, and the director attached to each point of $\mathscr{B}$ is the three-dimensional vector to describe points in the cross-section at each point on the axis ($m = 1, r = 3$). Then, it is enough that the vector bundle $\mathscr{Q}$ is taken as $\mathbb{R}^3 \times \mathbb{R}^3$.

We parameterize a position vector $r(s, t)$ of an arbitrary point on $\phi(\mathscr{B})$ by the arc-length parameter s and the time parameter t. We define a right-handed orthonormal basis, $\{d_1(s, t), d_2(s, t), d_3(s, t)\}$, along $\phi(\mathscr{B})$ at s with $d_1 = \partial r/\partial s$, and introduce the curvature vector $\kappa(s, t)$ in the current body through

$$\frac{\partial d_k}{\partial s} = \kappa \times d_k. \tag{4.1}$$

The component $\kappa^1 = \langle d_1, \kappa \rangle$ of κ gives the torsion of $\phi(\mathscr{B})$ in the current configuration; the two components, $\kappa^\alpha = \langle d_\alpha, \kappa \rangle$, $\alpha = 2, 3$, are components of the current curvature of $\phi(\mathscr{B})$ and are related to the geometric curvature $\tilde{\kappa}$ of the current axial curve through the formula $(\tilde{\kappa})^2 = (\kappa^2)^2 + (\kappa^3)^2$. Then the Cosserat rod is provided as

$$x(s, t) = r(s, t), \qquad \xi(s, t) = \xi^k d_k(s, t). \tag{4.2}$$

It is illustrated as Fig. 2. We use curvilinear coordinates with respect to $\{d_k\}$. Throughout this article, the summation convention is used for repeated indices, with Latin indices taking the values $\{1, 2, 3\}$ and Greek indices taking the values $\{2, 3\}$.
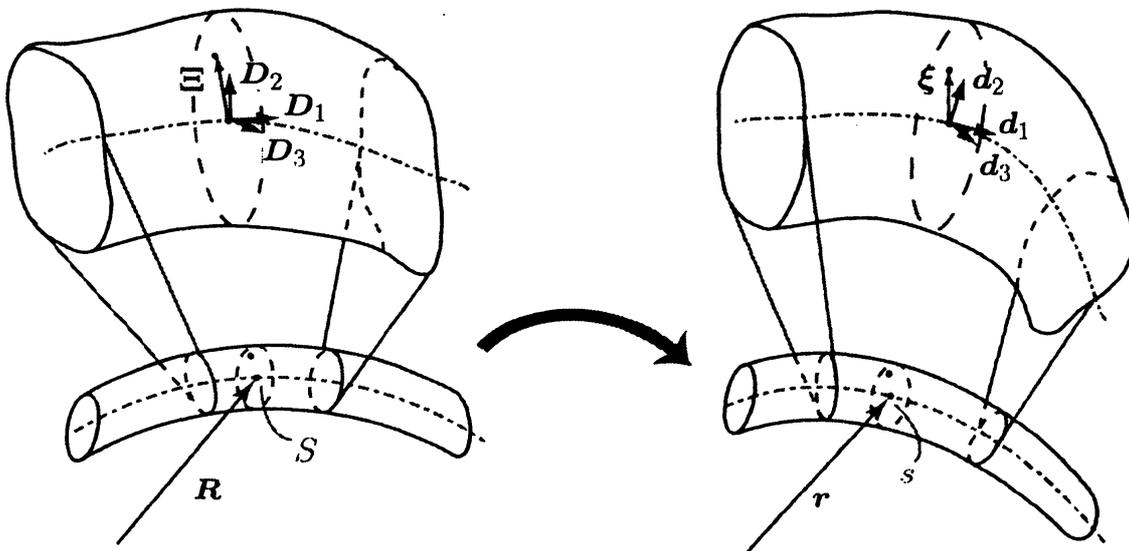

Figure 2: Illustration of the reference and current configurations of the Cosserat rod.

Additionally, we denote quantities for reference body by the capital letters of the corresponding ones for current body, that is, $R(S, 0)$ is the position vector of an arbitrary point on $\phi_0(\mathscr{B}_0)$ by the arc-length parameter S associated with $\phi_0(\mathscr{B}_0)$, and

$\{D_1(S,0), D_2(S,0), D_3(S,0)\}$ is a right-handed orthonormal basis along $\phi_0(\mathscr{B}_0)$ at S such that $D_1 = \partial R/\partial S$. Then, the reference curvature vector $\mathcal{K}_0(S,0)$ is similarly defined by

$$\frac{\partial D_k}{\partial S} = \mathcal{K}_0 \times D_k. \tag{4.3}$$

The component $\mathcal{K}_0^1 = \langle D_1, \mathcal{K}_0 \rangle$ of $\mathcal{K}_0$ gives the torsion of $\phi_0(\mathscr{B}_0)$ in the reference configuration; the two components, $\mathcal{K}_0^\alpha = \langle D_\alpha, \mathcal{K}_0 \rangle$, $\alpha = 2, 3$, are components of the reference curvature of $\phi_0(\mathscr{B}_0)$ and are related to the geometric curvature $\widetilde{\mathcal{K}}_0$ of the reference axial curve through the formula $(\widetilde{\mathcal{K}}_0)^2 = (\mathcal{K}_0^2)^2 + (\mathcal{K}_0^3)^2$. Thus, the reference configuration is provide as

$$X(S,0) = R(S,0), \qquad \Xi(S,0) = \Xi^\alpha D_\alpha(S,0). \tag{4.4}$$

It is shown as Fig. 2. Here, we consider that the reference configuration is unstressed state, and then the cross-sections of the reference filament is assumed to be normal to its axial curve. If the cross-sections of the current filament remain normal to the current axial curve, we may constrain $\xi^1 = 0$. When we suppose the special Cosserat rod, in which it is assumed that the cross-sections of the current filament remain plane, undeformed, and normal to the current axial curve, we have to append the constraints $\xi^1 = 0$ and $\xi^\alpha = \Xi^\alpha$.

Further, the extension $\varepsilon(s,t)$ of the axial curve can be defined through

$$\frac{\partial s}{\partial S} = (1 + \varepsilon). \tag{4.5}$$

Using the above the deference relation, we obtain the deformation gradients

$$F = (1 + \varepsilon)d_1 \otimes D^1, \tag{4.6a}$$

$$\mathfrak{F} = (1 + \varepsilon)\left(\frac{\partial \xi^k}{\partial s} + \xi^k R(\kappa)\right) d_k \otimes D^1, \tag{4.6b}$$

$$\mathcal{F} = \frac{\partial \xi^k}{\partial \Xi^\alpha} d_k \otimes D^\alpha, \tag{4.6c}$$

where $R(a)$ is the skew symmetric tensor associated with a polar vector a. Then the current and reference deformations are calculated as, respectively,

$$C = (1 + \varepsilon)^2 \left\| \frac{\partial}{\partial s}(r + \xi) \right\|^2 D^1 \otimes D^1, \tag{4.7a}$$

$$\mathfrak{C} = \left\langle \frac{\partial \xi}{\partial \Xi^\alpha}, (1 + \varepsilon)\frac{\partial}{\partial s}(r + \xi) \right\rangle D^\alpha \otimes D^1, \tag{4.7b}$$

$$\mathcal{C} = \left\langle \frac{\partial \xi}{\partial \Xi^\alpha}, \frac{\partial \xi}{\partial \Xi^\beta} \right\rangle D^\alpha \otimes D^\beta, \tag{4.7c}$$

and

$$C_0 = \left\| \frac{\partial}{\partial S}(R + \Xi) \right\|^2 D^1 \otimes D^1, \tag{4.8a}$$

$$\mathfrak{C}_0 = \left\langle D_\alpha, \frac{\partial}{\partial S}(R + \Xi) \right\rangle D^\alpha \otimes D^1 = \epsilon_{1\alpha\beta}\mathcal{K}_0^1 \Xi^\alpha D^\beta \otimes D^1, \tag{4.8b}$$

$$\mathcal{C}_0 = \delta_{\alpha\beta}D^\alpha \otimes D^\beta, \tag{4.8c}$$

where ϵ_{klm} is the Edinton's epsilon, and $\| \cdot \|$ denotes the standard inner product on the Euclidean spaces. Thus, we obtain the strains as follows:

$$E = \frac{1}{2} \left[(1+\varepsilon)^2 \| d_1 + \xi^\alpha R(\kappa) d_\alpha \|^2 - \| D_1 + \Xi^\alpha R(\mathcal{K}_0) D_\alpha \|^2 \right] D^1 \otimes D^1, \qquad (4.9a)$$

$$\mathfrak{E} = \frac{1}{2} \left[\left\langle \frac{\partial \xi}{\partial \Xi^\alpha}, (1+\varepsilon)\frac{\partial}{\partial s}(r+\xi) \right\rangle - \left\langle D_\alpha, \frac{\partial}{\partial S}(R+\Xi) \right\rangle \right] D^\beta \otimes D^1, \qquad (4.9b)$$

$$\mathcal{E} = \frac{1}{2} \left[\left\langle \frac{\partial \xi}{\partial \Xi^\alpha}, \frac{\partial \xi}{\partial \Xi^\beta} \right\rangle - \delta_{\alpha\beta} \right] D^\alpha \otimes D^\beta. \qquad (4.9c)$$

Because of using the moving frame, we must rewrite the variational formulation. To this end, we begin with defining the variation δk, associated with the orthonomal basis, through

$$\delta d_k = \delta k \times d_k, \qquad (4.10)$$

so that the variation of the director ξ is expressed as

$$\delta \xi = \delta k \times \xi, \qquad (4.11)$$

and the variations of the deformation gradients F, $\mathfrak{F}$, and $\mathcal{F}$ become

$$\delta F = \frac{\partial}{\partial R}(\delta r) + R(\delta k)F, \qquad (4.12a)$$

$$\delta \mathfrak{F} = \frac{\partial}{\partial R}(\delta \xi) + R(\delta k)\mathfrak{F}, \qquad (4.12b)$$

$$\delta \mathcal{F} = \frac{\partial}{\partial \Xi}(\delta \xi) + R(\delta k)\mathcal{F}. \qquad (4.12c)$$

Then the weak form is rewretten as

$$\int_{t_0}^{t_1} \int_{\Phi(\mathscr{P})} \left[\left(\frac{\partial}{\partial t}\left(\frac{\partial \mathscr{T}}{\partial \dot{r}} \right) - \frac{\partial \mathscr{T}}{\partial r} - \frac{\partial}{\partial R} \cdot \left(\frac{\partial \mathscr{W}}{\partial F} \right) \right) \cdot \delta r \right.$$
$$- \left(\frac{\partial \mathscr{W}}{\partial F} : R(\delta k)F + \frac{\partial \mathscr{W}}{\partial \mathfrak{F}} : R(\delta k)\mathfrak{F} + \frac{\partial \mathscr{W}}{\partial \mathcal{F}} : R(\delta k)\mathcal{F} \right)$$
$$\left. + \left(\xi \times \left(\frac{\partial}{\partial t}\left(\frac{\partial \mathscr{T}}{\partial \dot{\xi}} \right) - \frac{\partial \mathscr{T}}{\partial \xi} - \frac{\partial}{\partial R} \cdot \left(\frac{\partial \mathscr{W}}{\partial \mathfrak{F}} \right) - \frac{\partial}{\partial \Xi} \cdot \left(\frac{\partial \mathscr{W}}{\partial \mathcal{F}} \right) \right) \right) \cdot \delta k \right] \Phi(dV)dt$$
$$+ \int_{t_0}^{t_1} \int_{\Phi(\partial \mathscr{P})} \left[N \cdot \frac{\partial \mathscr{W}}{\partial F} \cdot \delta r + \xi \times \left(N \cdot \left(\frac{\partial \mathscr{W}}{\partial \mathfrak{F}} + \frac{\partial \mathscr{W}}{\partial \mathcal{F}} \right) \right) \cdot \delta k \right] \Phi(dA)dt = 0. \qquad (4.13)$$

Hence, under the fixed end points conditons, $\delta r = \delta k = 0$, the equations of motion for the Cosserat rod is derived as

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathscr{T}}{\partial \dot{r}} \right) = \frac{\partial \mathscr{T}}{\partial r} + \frac{\partial}{\partial R} \cdot \left(\frac{\partial \mathscr{W}}{\partial F} \right), \qquad (4.14a)$$

$$\xi \times \frac{\partial}{\partial t}\left(\frac{\partial \mathscr{T}}{\partial \dot{\xi}} \right) = \xi \times \left(\frac{\partial \mathscr{T}}{\partial \xi} + \frac{\partial}{\partial R} \cdot \left(\frac{\partial \mathscr{W}}{\partial \mathfrak{F}} \right) + \frac{\partial}{\partial \Xi} \cdot \left(\frac{\partial \mathscr{W}}{\partial \mathcal{F}} \right) \right)$$
$$+ F \stackrel{\times}{:} \frac{\partial \mathscr{W}}{\partial F} + \mathfrak{F} \stackrel{\times}{:} \frac{\partial \mathscr{W}}{\partial \mathfrak{F}} + \mathcal{F} \stackrel{\times}{:} \frac{\partial \mathscr{W}}{\partial \mathcal{F}}, \qquad (4.14b)$$

where $\overset{\times}{\cdot}$ denotes the cross-dot product in the dyadics, and it is defined as

$$(a_1 \otimes a_2) \overset{\times}{\cdot} (a_3 \otimes a_4) = (a_1 \times a_3)(a_2 \cdot a_4), \tag{4.15}$$

for any vectors $a_i \in \mathbb{R}^3$.

In terms of the local coordinates, the kinetic enegy density is expressed as

$$\mathscr{T}(r, \xi, \dot{r}, \dot{\xi}) = \frac{1}{2}\rho(r)\left(\|\dot{r}\|^2 + \|\dot{\xi}\|^2\right), \tag{4.16}$$

where $\rho(r)$ is a mass density of the body at r in the current configuration $\phi(\mathscr{B})$. Thus, we obtain the equations of motion for the special Cosserat rods expressed in terms of the generalized stress tensors,

$$\frac{\partial}{\partial t}(\rho\dot{r}) = \frac{1}{2}\frac{\partial\rho}{\partial r}\|\dot{r}\|^2 + \frac{\partial}{\partial S}\left(P \cdot D^1\right), \tag{4.17a}$$

$$\xi \times \frac{\partial}{\partial t}(\rho\dot{\xi}) = \xi \times \left(\frac{\partial}{\partial S}\left(\mathfrak{P} \cdot D^1 + \mathcal{P} \cdot \Xi^\alpha R(\mathcal{K}_0)^\mathsf{T} D^\alpha\right) + \frac{\partial}{\partial\Xi^\alpha}\left(\mathcal{P} \cdot D^\alpha\right)\right)$$
$$+ \frac{\partial r}{\partial S} \times \left(P \cdot D^1\right) + \frac{\partial\xi}{\partial S} \times \left(\mathfrak{P} \cdot D^1 + \mathcal{P} \cdot D^\alpha\right) + \frac{\partial\xi}{\partial\Xi^\alpha} \times \left(\mathcal{P} \cdot D^\alpha\right). \tag{4.17b}$$

By these expressions, it is well to reconstruct the balance laws for the Cosserat rod.

At the last in this section, we reduce the above equations to those for the special Cosserat rod, that is, we impose the constraints $\xi^1 = 0$ and $\xi^\alpha = \Xi^\alpha$. In this case, the micro deformation gradient becomes

$$\mathcal{F} = \delta^\alpha_\beta d_\alpha \otimes D^\beta, \tag{4.18}$$

so that the generalized micro stress tensor vanishes; $\mathcal{P} = 0$, because of the assumption about the undeformation of the cross-sections. Indeed, the micro strain vanishes, i.e., $\mathcal{E} = 0$. Here, we note that the linearlized macro and mixture strains become

$$E_{\text{linear}} = \varepsilon D^1 \otimes D^1, \tag{4.19a}$$

$$\mathfrak{E}_{\text{linear}} = \frac{1}{2}\epsilon_{1\alpha\beta}(\kappa^1 - \mathcal{K}^1_0)\Xi^\alpha D^\beta \otimes D^1. \tag{4.19b}$$

Then we obtain the well-known equations of motion for the special Cosserat rods expressed in terms of the generalized stress tensors,

$$\frac{\partial}{\partial t}(\rho\dot{r}) = \frac{1}{2}\frac{\partial\rho}{\partial r}\|\dot{r}\|^2 + \frac{\partial}{\partial S}\left(P \cdot D^1\right), \tag{4.20a}$$

$$\xi \times \frac{\partial}{\partial t}(\rho\dot{\xi}) = \frac{\partial}{\partial S}\left(\xi \times \left(\mathfrak{P} \cdot D^1\right)\right) + \frac{\partial r}{\partial S} \times \left(P \cdot D^1\right). \tag{4.20b}$$

We comment that $P \cdot D^1$ and $\xi \times \left(\mathfrak{P} \cdot D^1\right)$ is exactly the stress and couple-stress along the center axis, respectively.

5   Summary

In this study, we developed formulations for continuum dynamics on a tangent bundle of a vector bundle that accurately describes the mechanical behavior of a directed medium. Indeed, the dynamics of the one-dimensional continuum with a director are well expressed as one of the Cosserat rod, in which the cross-sectional structure is considered as the microstructure of the rod. For future developments, it is important to examine geometrical structures of various continua with microstructures. Especially, in the case where we consider a classification of microcontinua, it is necessary to investigate group actions on the bodies and microstructures. For example, the group structures correspond to Eringen's classification, i.e., micromorphic, microstretch, and micropolar continua [14].

Moreover, we can extend the Cosserat rod to a model describing smaller microstructure of the elastic rod. Then the expressions for the smaller microstructure to analyze deformation behavior of filaments including biopolymers. When a biopolymer expresses a certain function within a living organism, its conformation is an important factor that determes the function. Therefore, we believe to obtain a new knowledge of the interactions between the dynamical situations and the biological circumstances of biopolymers, which have been investigated recently by considering the deformation behavior of biopolymers together with their microstructures.

References

[1] J. L. Ericksen, "Conservation laws for liquid crystals," *Trans. Soc. Rheol.*, **5**, 23–34 (1961).

[2] R. A. Toupin, "Theories of elasticity with couple-stress," *Arch. Ration. Mech. Anal.*, **17**, 85–112 (1964).

[3] A. C. Eringen and E. S. Şuhubi, "Nonlinear theory of simple micro-elastic solids–I," *Int. J. Eng. Sci.*, **2**, 189–203 (1964).

[4] A. Riahi and J. H. Curran, "Full 3D finite element Cosserat formulation with application in layered structures," *Appl. Math. Model.*, **33**, 3450–3464 (2009).

[5] S. Forest and R. Sievert, "Elastoviscoplastic constitutive frameworks for generalized continua," *Acta Mechanica*, **160**, 71–111 (2003).

[6] D. A. Burton and T. Gould, "Dynamical model of Cosserat nanotubes," *J. Phys.*, **62**, 23–33 (2007).

[7] J. Rosenberg and R. Cirmrman, "Microcontinuum approach in biomechanical modeling," *Math. Comput. Simul.*, **61**, 249–260 (2003).

[8] K. A. Hohhman, R. S. Manning and J. H. Maddocks, "Link, twist, enegy and the stability of DNA minicircles," *Biopolymers*, **70**, 145–157 (2003).

[9] A. E. Green and R. S. Rivilin, "The mechanics of non-linear materials with memory," *Arch. Ration. Mech. Anal.*, **1**, 1 (1957).

[10] W. Noll, "A mathematical theory of the mechanical behavior of continuous media," *Arch. Ration. Mech. Anal.*, **2**, 197–226 (1958).

[11] C.-C. Wang, "On the geometric structures of simple bodies, a mathematical foundation for the theory of continuous distributions of dislocations," *Arch. Ration. Mech. Anal.*, **27**, 33–94 (1967).

[12] J. E. Marsden and T. J.R. Hughes, *Mathematical foundations of elasticity*, Prentice-Hall, New York, 1983 (reprinted by Dover, New York, 1994).

[13] C.-C. Wang and C. Truesdell, *introduction to rational elasticity*, Noordhoff International Publishing, Leyden, 1973.

[14] A. C. Eringen, *Microcontinuum Field Theories: Foundations and Solids*, Springer-Verlag, New York, 1999.