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Finite dimensional reduction for a reaction diffusion problem with obstacle potential

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Abstract

A reaction-diffusion problem with an obstacle potential is considered in a bounded domain of $\mathbb{R}^N$. Under the assumption that the obstacle $\mathscr{K}$ is a closed convex and bounded subset of $\mathbb{R}^n$ with smooth boundary or it is a closed n-dimensional simplex, we prove that the long-time behavior of the solution semigroup associated with this problem can be described in terms of an exponential attractor. In particular, the latter means that the fractal dimension of the associated global attractor is also finite.

1 Introduction

Let us consider the following reaction-diffusion system with an obstacle potential in a bounded and regular domain $\Omega \subset \mathbb{R}^N$

$$\begin{cases} \partial_t u - \Delta_x u + \partial I_{\mathscr{K}}(u) - \lambda u \ni 0, \\ u\big|_{t=0} = u_0, \quad u\big|_{\partial\Omega} = 0. \end{cases} \tag{1.1}$$

Here $u = (u_1(t,x), \cdots, u_n(t,x))$ is an unknown vector-valued function, Δ_x is a Laplacian with respect to the variable x, $\lambda > 0$ is a given constant, $\mathscr{K}$ is a given bounded closed convex set in $\mathbb{R}^n$ containing zero and $\partial I_{\mathscr{K}}$ stands for the subdifferential of its indicator function $I_{\mathscr{K}}$, that is

$$I_{\mathscr{K}}(u) := \begin{cases} 0, & u \in \mathscr{K}, \\ \infty, & u \notin \mathscr{K}. \end{cases} \tag{1.2}$$

Equations and systems of the type (1.1) appear quite often in the mathematical analysis of phase transitions models or in reaction diffusion processes with constraints. In the first physical situation, the unknown u denotes usually the order parameter which, in the case of multicomponent systems, is an n-dimensional vector that attains values only in a bounded (convex) subset of $\mathbb{R}^n$, usually an n-dimensional simplex

$$\mathscr{K} := \left\{ p = (p_1, \ldots, p_n) \in \mathbb{R}^n \text{ such that } \sum_{i=1}^{n} p_1 \leq 1, \quad p_i \geq 0, \ i = 1, \cdots, n \right\}. \tag{1.3}$$

This is motivated by the requirement that no void nor overlapping should appear between the phases. In particular, equation (1.1) with $n = 2$ and $\lambda = 0$ appears in the Frémond models of shape memory alloys; equation (1.1) when $n = 1$ and $\lambda > 0$ is usually called Allen-Cahn equation with constraint and appears for instance in some Penrose-Fife type models for phase transition (see [20]). We refer to [10] for other models of phase change showing the ubiquity of subdifferential operators in this framework.

Equation (1.1) is one of the simplest example, albeit rich of interesting features, of a nonlinear (parabolic) evolution equation associated to a multivalued maximal monotone operator, in this case the subdifferential $\partial I_{\mathcal{K}}$. The mathematical analysis of these kind of evolutions has attracted the attention of researchers for many years. In the particular case of equation (1.1), results concerning existence, approximation and long time behavior of solutions (e.g., in terms of global attractors) are known and by now classic (without any sake of completeness and referring only to results in the Hilbert space framework, we quote [3], [4], [2], [22], [19]).

However, to the best of our knowledge, the finite/infinite dimensionality of the global attractor associated with the obstacle problems has not been yet understood (even in the simplest case of Allen-Cahn equation with double obstacle). Let us stress that this is not a purely academic question. In fact, being a subset of the phase space, the global attractor is a priori of infinite dimension. On the other hand, a deep Theorem proved by Mané (see [11]) asserts that once the global attractor is shown to have finite fractal dimension, then it is possible to construct a reduced dynamical system via projection on finite dimensional spaces which is finite dimensional (namely it lies in $\mathbb{R}^{2m+1}$, with m properly related to the fractal dimension of the attractor) and which is Hölder continuous with respect to the initial conditions. The classical machinery for proving the finite-dimensionality (in terms of fractal or/and Hausdorff dimension) of the global attractor (which perfectly works in many cases of dissipative systems generated by non-linear PDEs with *regular* non-linearities and often gives sharp estimates on the dimension, see [1, 22] and references therein) is based on the so-called volume contraction arguments. Roughly speaking, this method consists in studying the evolution of infinitesimal volumes on the attractor. To successfully implement this method, some differentiability is needed for the semigroup $S(t)$. Here differentiability means that the associated solution semigroup is (uniformly quasi-) *differentiable* with respect to the initial data at least on the attractor (see [22]). Unfortunately, this differentiability condition is usually *violated* if the underlying PDE has singularities or/and degenerations. In particular, it is clearly violated for the obstacle problems like (1.1). Thus, the classical scheme is not applicable here and this makes the problem much more difficult and interesting since it may happen that the singular/degenerate character of the equation forces the attractor to be infinite dimensional. This is indeed the case of the degenerate analogue of the real Ginzburg-Landau equation

$$\begin{cases} \partial_t u - \Delta_x(u^3) + u^3 - u = 0, & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{1.4}$$

In [8] it has been recently shown that the global attractor of the dynamical system generated by (1.4) has infinite dimension even if Ω is bounded. On the other hand, in recent years the finite-dimensionality of the global attractor has been established for many important classes of degenerate/singular dissipative systems including Cahn-Hilliard equations with logarithmic potentials (see [15]), porous media equations (under some natural restrictions which exclude the example of (1.4), see [8]), doubly non-linear parabolic equations of different types (see [17, 18, 9]), etc. In these papers, the finite dimensionality of the global attractor is typically a consequence of the existence of a more refined object called *exponential attractor*, whose existence proof is often based on proper forms of the so called squeezing/smoothing property for the differences of solutions.

We remind that the concept of exponential attractor has been introduced in [5] in order to overcome two major drawbacks of the global attractors: the slow (uncontrollable) rate of attraction and the sensitivity to perturbations. Roughly speaking, an exponential attractor (which always contains the global one) is a compact *finite-dimensional* set in the phase space which attracts *exponentially* fast the images of all bounded sets as time tends to infinity (see Section 3 for the rigorous definition). Thus it turns out that, in contrast to global attractors, the exponential attractors are much more robust to perturbations (usually, Hölder continuous with respect to the perturbation parameter). Moreover, the rate of convergence to the exponential attractor can be controlled in term of physical parameters of the system considered, see [5] and the more recent survey [16] (and also references therein) for more details. Finally, the finite dimensionality of the global attractor immediately follows from the finite dimensionality of the exponential attractor.

In this note, we report some results contained in the recent [21] about the extension of the exponential attractors theory to some classes of reaction-diffusion problems with *obstacle* potentials. In the remaining part of this introduction, we informally present the strategy adopted in [21] to prove the existence of an exponential attractor and thus obtain the finite dimensionality of the global one. First of all, we recall a general abstract result, competitive with the volume contraction method, which serves as a ground base both for proving the finiteness of the fractal dimension and both for proving, when combined with some other estimates, the existence of an exponential attractor.

Theorem 1.1 ([14],[16], [23]). *Let X be a compact subset of the Banach space E. If there exists another Banach space E_1 compactly embedded in E and a map $L : X \longrightarrow X$ such that $L(X) = X$ and, for a positive C_L,*

$$\|L(v_1) - L(v_2)\|_{E_1} \le C_L \|v_1 - v_2\|_E, \ \forall v_1, v_2 \in X, \tag{1.5}$$

then the fractal dimension of X is finite and the following estimate holds

$$dim_F(X) \le \mathcal{H}_{1/4C_L}(B_{E_1}), \tag{1.6}$$

where $\mathcal{H}_{1/4C_L}(B_{E_1})$ is a quantity, named Kolmogorov entropy, related to the (minimal) number $N_{1/4C_L}(B_{E_1})$ of balls of radius $1/4C_L$ needed to cover B_{E_1} via the formula $\mathcal{H}_{1/4C_L}(B_{E_1}) := \log_2 N_{1/4C_L}(B_{E_1})$.

In order to use this Theorem to investigate the finiteness of the fractal dimension of the global attractor, one has to properly define the map L and the space E_1. The usual choices when one has to deal with parabolic problems are to define L to be $L : u_0 \mapsto S(1)u_0$, where u_0 lies in the attractor and the space E_1 in such a way that $S(1)u_0 \in E_1$ for all u_0 on the attractor. Then, the parabolic character of the problem would guarantee some regularity for $S(1)u_0$ and thus the required compactness of $E_1 \subset E$. Of course, the choice of E_1 should be also calibrated with the requirement that also (1.5), which thus appears as a smoothing estimate for the difference of two solutions, holds. Now, turning back to our obstacle problem, the natural choice $L = S(1)$ and $E_1 = (H_0^1)^N$ turns out to be the wrong one: in fact, proving a point wise (in time) estimate for the difference of two solutions in the norm of $(H_0^1)^N$ reveals to be hopeless due to the presence of the subdifferential $\partial I_{\mathcal{K}}$. On the contrary, the correct choice here is to (at first) apply Theorem 1.1 in a space of trajectories and then try to recover the results in the physical phase space. In our situation, the correct choices for E and E_1 turn out to be $E := L^2(0,1;(L^2)^n)$ a $E_1 := \{u \in L^2(0,1;(H_0^1)^n) \ : \partial_t u \in L^1(0,1;(H^{-s})^n)\}$ with $s > N/2$. Correspondingly, the map L will be $L = \mathbb{S}(1)$ with $\mathbb{S}(t)$ being the lifted (semigroup) of the semigroup $S(t)$ (see [21] for the correct definition). In other words, we are applying some

modification of the method of ℓ-trajectories which was originally introduced by Málek and Nečas in [13] and later extended in [14]. In this last paper, it is also shown how to canonically recover, e.g., the finiteness of the fractal dimension of the attractor in the usual phase space once the analogous result is proven in the extended trajectory space.
However, it should be noted that the application of this theory to our obstacle problem is far from being standard and immediate. In fact, in order to obtain the estimate (1.6) when E_1 is as above, we have to control the difference $(\partial_t u_1 - \partial_t u_2)$ in $L^1(0,1;(H^{-s})^n)$. As a consequence, it is evident that we have to produce estimates for the difference between the Lagrange multipliers (namely the selection of the subdifferential $\partial I_{\mathscr{K}}(u)$ which turns the differential inclusion (1.1) into an equation) associated with the solutions u_1 and u_2. These kind of estimates, roughly speaking, look as follows

$$\int_0^1 \|\partial I_{\mathscr{K}}(u_1(t)) - \partial I_{\mathscr{K}}(u_2(t))\|_{L^1(\Omega)}\, dt \leq C\|u_1(0) - u_2(0)\|_{L^2} \tag{1.7}$$

where u_1 and u_2 denote two solutions starting from the proper absorbing set and, with a little abuse of notation, we refer to $\partial I_{\mathscr{K}}$ as it were single valued. Such kind of estimates are the core of the problem and, to the best of our knowledge, do not seem to be already known. The proof of (1.7) is contained in the paper [21], in the case in which $\mathscr{K}$ is a smooth and bounded convex subset or is the simplex (1.3). With (1.7) in hand, the existence of the exponential attractor (even in the physical phase space) and thus the finite dimensionality of the global one, can be obtained by standardly applying the method of ℓ-trajectories.

2 Well-posedness and regularity

In this section we recall some known facts about the solutions of reaction-diffusion problem (1.1) with obstacle potential and to formulate some additional estimates which will be crucial for what follows. Before entering into the details, we advise the reader that, although the unknown function u is actually a vector valued function, for the sake of simplicity, will be denoted as a scalar valued function. Consequently, also the functional spaces we will use in the course of the paper we will have a "scalar" notation. This means that a notation like, e.g., L^2 will be preferred to a (more precise) notation like $(L^2(\Omega))^n$. The same applies to dualities $(\langle\cdot,\cdot\rangle)$ and scalar products $((\cdot,\cdot))$. Moreover, we will indicate with same symbols $\mathscr{K}$ and $I_{\mathscr{K}}$ the convex in $\mathbb{R}^n$ and its indicator function and their realization in L^2. Thus, the definition of (weak) solutions of our problem is.

Definition 2.1. A function $u = u(t,x)$ is a solution of the obstacle problem (1.1) if $u(t,x) \in \mathscr{K}$ for almost all $(t,x) \in [0,T] \times \Omega$,

$$u \in C([0,T];L^2) \cap L^2(0,T;H_0^1), \quad \partial_t u \in L^2(0,T;H^{-1}), \tag{2.1}$$

and the following variational inequality holds for almost every $t \in (0,T]$

$$\langle \partial_t u(t), u(t) - z\rangle + (\nabla u, \nabla(u-z)) \leq \lambda(u, u-z), \text{ for any } z \in H_0^1 \cap \mathscr{K}. \tag{2.2}$$

The next theorem is a standard result in the theory of the evolution equations associated with maximal monotone operators (see the seminal references [3], [4] and [2]).

Theorem 2.2. *[Well posedness] When $\mathscr{K}$ is a closed and bounded convex set containing the origin in $\mathbb{R}^n$ there exists a unique global weak solution for any given measurable u_0 such that $u_0(x) \in \mathscr{K}$ for almost all $x \in \Omega$.*

Let us introduce the function (named Lagrange multiplier in what follows)

$$h_u(t) := -\partial_t u(t) + \Delta_x u(t) + \lambda u, \quad h_u(t) \in \partial I_{\mathscr{K}}(u). \tag{2.3}$$

The next proposition, which has an independent interest and turns out to be crucial for the rest of our investigations, shows that the function h_u is, in fact, globally bounded in the L^∞-norm.

Proposition 2.3 ([21]). *Let the assumptions of Theorem 2.2 hold and let $h_u(t)$ be the Lagrange multiplier associated with the solution $u(t)$ of problem* (1.1). *Then, $h_u \in L^\infty(\mathbb{R}_+ \times \Omega)$ and*

$$\|h_u(t)\|_{L^\infty} \le C, \tag{2.4}$$

where the constant C depends only on $\mathscr{K}$ and λ (and is independent of u and $t \ge 0$).

We refer to [21] for all the details of the proof. Here it is worthwhile to comment a bit on the ingredients of the proof. To prove (2.4), we use an approximation argument combined with the maximum principle. The key point here is the use of an *ad hoc* approximation scheme different from the Yosida approximation usually used in these contexts. This kind of approximation consists in approximating the indicator function $I_{\mathscr{K}}$ as follows. First, we let $M(u)$ to be the distance from the point $u \in \mathbb{R}^n$ to the convex set $\mathscr{K}$, namely the real valued function M

$$M(u) := \mathrm{dist}(u, \mathscr{K}). \tag{2.5}$$

Then, M is convex, globally Lipschitz continuous and smooth outside $\mathscr{K}$. In addition there holds that

$$M(u) \ge 0, \ M(u) = 0, \ \text{if } u \in \mathscr{K}, \tag{2.6a}$$

$$|\nabla M(u)| = 1, \ \text{if } u \notin \mathscr{K}. \tag{2.6b}$$

Then, for any $\varepsilon > 0$, we introduce the real function

$$f_\varepsilon(z) := \begin{cases} 0, & z \le \varepsilon, \\ \varepsilon^{-1}(z-\varepsilon)^2, & z \ge \varepsilon. \end{cases} \tag{2.7}$$

Finally, the desired approximation is defined a

$$F_\varepsilon(u) := f_\varepsilon(M(u)). \tag{2.8}$$

Remark 2.4. Contrary to Theorem 2.2 which is based only on the energy type estimates which are valid for much more general equations, e.g., with non-scalar diffusion matrix, etc., the L^∞-estimate obtained in Proposition 2.3 is based on the maximum/comparison principle and requires the diffusion matrix to be scalar. In particular, we do not know whether or not this estimate remains true even for the case of diagonal, but non-scalar diffusion matrix.

As direct consequence of the previous Proposition, we have that (1.1) could be understood as the heat equation

$$\partial_t u - \Delta_x u = \lambda u - h_u$$

with the external forces belonging to L^∞. Thus, the parabolic interior regularity estimates give (see. e.g., [12])

$$\|u(t)\|_{C^{2-\nu}} \le C_\nu \frac{1+t^\alpha}{t^\alpha}, \quad \text{for any } \nu >, 0. \tag{2.9}$$

3 Global and exponential attractors

In this section we present the long-time behavior results for the solutions of problem (1.1) in terms of global and exponential attractors. We first recall that, due to Theorem 2.2, equation (1.1) generate a (dissipative) and Lipshitz continuous in the L^2-metric semigroup $\{S(t),\ t \geq 0\}$ in the phase space

$$\Phi = \Phi_{\mathscr{K}} := \{u \in L^\infty : \ u(x) \in \mathscr{K} \ \text{ for almost all } x \in \Omega\}, \tag{3.1}$$

i.e.,

$$S(t) : \Phi \to \Phi, \ \ S(t)u_0 := u(t), \tag{3.2}$$

where $u(t)$ is the solution to (1.1) at time t. Moreover, it can be standardly proved the following

Theorem 3.1. *[Global Attractor] Under the assumptions of Theorem 2.2, the semigroup $S(t)$ associated with the obstacle equation* (1.1) *possesses the global attractor $\mathcal{A}$ in Φ which is bounded in $C^{2-\nu}(\Omega)$ for every $\nu > 0$. This attractor is generated by all the trajectories of the semigroup $S(t)$ defined for all $t \in \mathbb{R}$:*

$$\mathcal{A} = \mathcal{K}\big|_{t=0}, \tag{3.3}$$

where $\mathcal{K} \subset L^\infty(\mathbb{R}, \Phi)$ is a set of all solutions of (1.1) *defined for all $t \in \mathbb{R}$.*

Now, we turn our attention to the construction of the exponential attractor and to the study of the finite dimensionality of the global attractor $\mathcal{A}$. First of all, recall that the semigroup $S(t)$ associated with the obstacle problem (1.1) possesses a global Lyapunov function of the form

$$\mathcal{L}(u) := \|\nabla_x u\|^2_{L^2} - \lambda \|u\|^2_{L^2}. \tag{3.4}$$

Indeed, using the test function $z = u(t-h)$ in the variational inequality (2.2), dividing it by $h > 0$ and passing to the limit $h \to 0$, we arrive at

$$\|\partial_t u(t)\|^2_{L^2} + \frac{d}{dt}\mathcal{L}(u(t)) \leq 0.$$

Therefore, according to the general theory (see e.g., [1]), every trajectory $u(t) = S(t)u_0$ tends as $t \to \infty$ to the set $\mathcal{R}$ of all equilibria of problem (1.1)

$$\operatorname{dist}(S(t)u_0, \mathcal{R}) \to 0, \quad \text{as } t \to \infty.$$

However, in contrast to the case of regular systems, the equilibria set $\mathcal{R}$ is generically *not discrete* for the obstacle type singular problems. Thus, in our situation, the existence of a Lyapunov function does not allow to obtain the stabilization of every trajectory to a single equilibrium even in "generic" situation. In addition, the semigroup $S(t)$ is not differentiable with respect to the initial data (it is in fact only globally Lipschitz continuous), so we are not able to construct the stable/unstable manifolds associated with an equilibrium. Thus, the so-called theory of *regular* attractors is not applicable to equations of the type (1.1). Moreover, due to the above mentioned non-differentiability, the standard way of proving the finite-dimensionality of the global attractor based on the volume contraction method does not work here. So, the existence of the finite-dimensional reduction for the associated long time dynamics becomes a non-trivial problem which, to the best of our knowledge, it has not been yet tackled. In this paper we will prove that the global attractor for (1.1) has finite fractal dimension by using the concept of the so-called *exponential* attractor and the estimation of the dimension based on the proper chosen squeezing/smoothing property for the difference between two solutions. This method has the advantage that it does not require the differentiability with respect to the initial data The existence of an exponential attractor is interest in itself. In fact, we recall once more that the global attractor represents the first (although extremely important) step in the understanding of the long-time dynamics of a given evolutive process. However, it may also present some severe drawbacks. Indeed, as simple examples show, the rate of convergence to the global attractor may be arbitrarily slow. This fact makes the global attractor very sensitive to perturbations and to numerical approximation. In addition, it is usually extremely difficult to estimate the rate of convergence to the global attractor and to express it in terms of the physical parameters of the system. In particular, it may even be reduced to a single point, thus failing in capturing the very rich and most interesting transient behavior of the system considered. The simplest example of such a system is the following 1D real Ginzburg-Landau equation

$$\partial_t u = \varepsilon \partial_x^2 u + u - u^3, \quad x \in [0,1], \quad u\big|_{x=0} = u\big|_{x=1} = -1.$$

In that case, the global attractor $\mathcal{A} = \{-1\}$ is trivial for all $\varepsilon > 0$. However, this attractor is, factually, invisible and unreachable if ε is small enough since the transient structures (which are very far from the attractor) have an extremely large lifetime $T \sim e^{1/\sqrt{\varepsilon}}$.

In order to overcome these drawbacks, the concept of *exponential attractor* has then been proposed in [5]) to possibly overcome this difficulty. We recall below the definition of exponential attractor adopted for our case, see e.g., [5] and [16] for more detailed exposition.

Definition 3.2. A compact subset $\mathcal{M}$ of the phase space Φ is called an *exponential attractor* for the semigroup $S(t)$ if the following conditions are satisfied:

(E1) The set $\mathcal{M}$ is *positively* invariant, i.e., $S(t)\mathcal{M} \subset \mathcal{M}$ for all $t \geq 0$;

(E2) The fractal dimension (see, e.g., [22]) of $\mathcal{M}$ in Φ is finite;

(E3) The set $\mathcal{M}$ attracts exponentially fast the image the phase space Φ. Namely, there exist $C, \beta > 0$ such that

$$\mathrm{dist}_{L^\infty}(S(t)\Phi, \mathcal{M}) \leq Ce^{-\beta t}, \quad \forall t \geq 0. \tag{3.5}$$

Thanks to the control of the convergence rate $(E3)$ it follows that, compared to the global attractor, an exponential attractor is much more robust to perturbation (usually it is Hölder continuous with respect to the perturbation parameter). However, since the the exponential attractor $\mathcal{M}$ is only *positively* invariant (see (E1)), it is obviously not unique. Thus, the concrete choice of an exponential attractor and its explicit construction becomes essential. We recall also that, in the original paper [5] the construction was extremely implicit (involving the Zorn lemma) and this fact did not allow to develop a reasonable perturbation theory. This drawback has been overcome later in [6] and [7] where an alternative and relatively simple and explicit construction for the exponentially attractor has been suggested. Note also that the construction of [7] gives an exponential attractor which is automatically Hölder continuous with respect to the reasonable perturbations of the semigroup considered and this somehow resolves the non-uniqueness problem. We refer the reader to the recent survey [16] for the detailed informations on the exponential attractors theory.
We turn back to our equation (1.1). In [21], we proved the following two Theorems.

Theorem 3.3 ([21]). *Let the assumptions of Theorem 2.2 hold and let, in addition, the boundary $S = \partial\mathscr{K}$ be smooth enough (at least, $C^{2,1}$). Then, the solution semigroup $S(t)$ associated with the obstacle problem* (1.1) *possesses an exponential attractor. As a consequence, the global attractor has finite fractal dimension.*

Theorem 3.4 ([21]). *Let the assumptions of Theorem 2.2 hold and let the set $\mathscr{K}$ be an n-dimensional simplex* (1.3). *Then, the solution semigroup $S(t)$ associated with the obstacle problem* (1.1) *possesses an exponential attractor. As a consequence, the global attractor $\mathcal{A}$ has finite fractal dimension.*

The proofs of both results are based on an argument that combines a proper choice of an approximation scheme and the maximum principle similar to the one devised to prove Proposition 2.3. In particular, let us note that the non trivial part (and also the main novelty of paper [21]) consists in showing the validity of estimate (1.7) which, at the moment, we are able to prove only for the two different kind of convex sets introduced in Theorem 3.3 and 3.4. Proving the existence of an exponential attractor in the case of a general closed and bounded convex set is still an open an interesting problem.

3.1 Some generalizations

We now discuss the applications of our method to more general problems. We start with the obvious observation that all the above results remain valid if we replace the term λu in the left-hand side of equation (1.1) by any sufficiently regular interaction function $g(u,x)$. Namely, consider the problem

$$\partial_t u - \Delta_x u + \partial I_K(u) + g(x,u) \ni 0, \tag{3.6}$$

where $g \in C(\Omega, C^1(\mathbb{R}^n, \mathbb{R}^n))$ is an arbitrary interaction function. Then, the following result holds.

Theorem 3.5. *Let $\mathscr{K}$ be a convex bounded set of $\mathbb{R}^n$ containing zero with a smooth boundary (or let K be an n-dimensional simplex* (1.3)*) and let $g \in C(\Omega, C^1(\mathbb{R}^n, \mathbb{R}^n))$ be an arbitrary (not necessarily a gradient!) non-linear interaction function. Then, the solution semigroup $S(t)$ associated with equation* (3.6) *possesses an exponential attractor $\mathcal{M}$ in the sense of Definition 3.2. Moreover, the fractal dimension of the global attractor is finite.*

Indeed, since $\mathscr{K}$ is bounded, the solution u is also automatically bounded in L^∞ (and the same will be true for the solutions u_ε of any reasonable approximate problel if $\varepsilon > 0$ is small enough no matter how the *regular* interaction function g looks like). So, the term $g(u,x)$ can be treated as a perturbation and the proof of Theorem 3.5 repeats word by word the given proof for the particular case $g(x,u) := -\lambda u$. The function $g(x,u)$ may even depend explicitly on the gradient $\nabla_x u$: $g = g(x,u,\nabla_x u)$. However, in that case we already need to impose some restrictions on the growth of g with respect to $\nabla_x u$ (since the obstacle potential controls only the L^∞-norm of a solution and the control of its $W^{1,\infty}$-norm should be then additionally obtained). In particular, if g does not grow with respect to $\nabla_x u$, i.e.,

$$|g(x,u,\nabla_x u)| \le Q(|u|)$$

for some monotone function Q independent of x and $\nabla_x u$, the proof of Theorem 3.5 still repeats word by word the case of $g(u) = -\lambda u$. However, our **conjecture** here is that the result remains true under the standard sub-quadratic growth restriction

$$|g(x,u,\nabla_x u)| \le Q(|u|)(1 + |\nabla_x u|^q)$$

with $q < 2$. As we have already pointed out, our method of estimating the Lagrange multipliers is strongly based on the maximum principle for the leading linear part of equation (1.1). For this reason, we are unable in general to extend it to the case of non-scalar diffusion matrix. However, we point out that for some particular convexes the estimate (1.7) can be verified also for diffusion matrices (say diagonal). This is the case of the convex

$$\mathscr{K} := [0, L]^n, \quad L > 0.$$

In this case, estimate (1.7) can be easily verified just multiplying the equation for the kth component of (1.1) (actually of its approximation) by $\operatorname{sgn}(u_k^1 - u_k^2)$ even in the case of diagonal diffusion matrix

$$\partial_t u - a\Delta_x u + \partial I_K(u) - \lambda u \ni 0, \tag{3.7}$$

with $a = \operatorname{diag}(a_1, \cdots, a_k)$, $a_i > 0$ for $i = 1, \cdots, n$.

One more generalization can be obtained replacing the Laplacian $\Delta_x u$ in equations (1.1) by the quasi-linear second order differential operator

$$A(u) := \operatorname{div}(a(|u|^2)\nabla_x u), \quad u = (u_1, \cdots, u_n), \quad |u|^2 := u_1^2 + \cdots + u_n^2$$

with some natural assumptions on the scalar diffusion function a. Then, it is not difficult to verify that the proofs of (1.7) given above remain true and the associated solution semigroup possesses an exponential attractor under the assumptions of Theorem 3.5.

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