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L^p well-posedness for the complex Ginzburg-Landau equation

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1 Introduction

This note is based on a joint work with Prof. Naoki Tanaka (Shizuoka University).

We consider the initial-boundary value problem for the complex Ginzburg-Landau equation:

$$
\text{(CGL)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} - (\lambda + i\mu)\Delta u + (\kappa + i\nu)|u|^{q-2}u - \gamma u = 0, & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), & x \in \partial\Omega. \end{cases}
$$

Here Ω is a general domain in $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$, $\lambda > 0$, $\kappa > 0$, $\mu, \nu, \gamma \in \mathbb{R}$, and $i = \sqrt{-1}$. The complex Ginzburg-Landau equation with $q = 4$ was derived to describe the destabilization of plane shear flow and the instability problem in nonlinear chemical kinetics. The general case $(q \geq 2)$ has been studied as a model for turbulent dynamics. For details we refer to [1].

Our aim in this note is to show the time global well-posedness for (CGL) in $L^p(\Omega)$. The weak or strong global well-posedness for (CGL) has been studied by many authors. We refer to [2, 3, 4, 6, 7, 8, 9, 10, 11] and references therein. Among them, Ginibre-Velo [4] established the global well-posedness for (CGL) in $L^p(\mathbb{R}^N)$ and locally uniform $L^p(\mathbb{R}^N)$ spaces under the assumptions that $p \geq 2$,

$$|\mu|/\lambda < 2\sqrt{p-1}/|p-2|, \tag{1.1}$$

$$2 \leq q \leq 2 + 2p/N. \tag{1.2}$$

It should be remarked that Yokota and Okazawa [11] studied (CGL) for $u_0 \in L^2(\Omega) \cap L^p(\Omega)$. They proved the unique existence of strong solution to (CGL) and the continuous dependence on its initial data in $L^2(\Omega)$ under the assumptions that $p \geq 2$,

$$|\mu|/\lambda \leq 2\sqrt{p-1}/|p-2|,$$

$$2 \leq q < 2 + 2p/N.$$

Our main result is an extension of one of the results of [4].

Main Theorem. Assume $p \in (1,\infty)$, (1) and (2). Then for each $u_0 \in L^p(\Omega)$ there exists a unique solution $u(\cdot;u_0)$ to (CGL) in the class

$$C([0,\infty);L^p(\Omega)) \cap C^1((0,\infty);L^p(\Omega)) \cap C((0,\infty);W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)).$$

Moreover, the following continuous dependence of solutions on their initial data holds: for each $\tau > 0$ and $r > 0$ there exists $M(\tau,r) > 0$ such that

$$\|u(t;u_0) - u(t;\hat{u}_0)\|_{L^p} \le M(\tau,r)\|u_0 - \hat{u}_0\|_{L^p}$$

for $t \in [0,\tau]$ and u_0, $\hat{u}_0 \in L^p(\Omega)$ with $\|u_0\|_{L^p} \le r$ and $\|\hat{u}_0\|_{L^p} \le r$.

Our approach to (CGL) is quite different from that of [4]. First we rewrite (CGL) as an abstract semilinear Cauchy problem

$$u'(t) = Au(t) + Bu(t) \quad (t > 0), \qquad u(0) = u_0$$

in a Banach space $X = L^p(\Omega)$. Then we apply a characterization theorem for semigroups of locally Lipschitz operators associated with the above Cauchy problem obtained by [7] to prove our main theorem.

2 Characterization theorem for semigroups of locally Lipschitz operators

In this section we introduce semigroups of locally Lipschitz operators and recall the characterization theorem for them.

We consider a semilinear Cauchy problem

$$u'(t) = Au(t) + Bu(t) \quad (t > 0), \qquad u(0) = u_0 \in D \qquad \text{(SP; } u_0\text{)}$$

in a Banach space $(X, \|\cdot\|)$. Here A is the infinitesimal generator of an analytic (C_0) semigroup $\{T(t) \mid t \ge 0\}$ on X satisfying the condition below:

(A) There exist constants $M \ge 1$ and $\omega_A < 0$ such that $\|T(t)\| \le Me^{\omega_A t}$ for $t \ge 0$.

Let $\alpha \in (0,1)$ and let Y be the Banach space $D((-A)^\alpha)$ equipped with the norm $\|x\|_Y = \|(-A)^\alpha x\|_X$ for $x \in D((-A)^\alpha)$. Let D be a subset of X and let φ be a proper lower semicontinuous functional from X into $[0,\infty]$ such that $D(\varphi) = D$ and for each $r > 0$, $C_r := D_r \cap Y$ is dense in D_r, where $D_r = \{x \in X; \varphi(x) \le r\}$ for $r > 0$. Let $C = D \cap Y$. The operator B from C into X is assumed to satisfy the following conditions:

(B1) For each $r > 0$ the operator B is continuous from C_r into X.

(B2) For each $r > 0$ there exists $M_B(r) > 0$ such that

$$\|Bx\|_X \le M_B(r)(1 + \|x\|_Y) \quad \text{for } x \in C_r. \tag{2.1}$$

Now we give the definition of semigroups of locally Lipschitz operators.

Definition 2.1. A one-parameter family $\{S(t); t \geq 0\}$ of locally Lipschitz operators from D into itself is called a *semigroup of locally Lipschitz operators on D with respect to the functional φ* if the following three conditions are satisfied:

(S1) $S(0)x = x$ for $x \in D$, and $S(t+s)x = S(t)S(s)x$ for s, $t \geq 0$ and $x \in D$.

(S2) For each $x \in D$, $S(\cdot)x : [0, \infty) \to X$ is continuous.

(S3) For each $\tau \geq 0$ and $r \geq 0$ there exists $M(\tau, r) > 0$ such that

$$\|S(t)x - S(t)y\| \leq M(\tau, r)\|x - y\| \quad \text{for } x,\ y \in D_r \text{ and } t \in [0, \tau].$$

The characterization theorem is stated as follows:

Theorem 2.2. ([7, Proposition 2.4 and Theorem 3.5]) *Let $a_0 \geq 0$. The following two statements* (i) *and* (ii) *are equivalent:*

(i) *There exists a semigroup $\{S(t); t \geq 0\}$ of locally Lipschitz operators on D with respect to φ such that for $x \in D$, $S(t)x \in C$ for $t > 0$, $BS(t)x \in C((0, \infty); X) \cap L^1_{loc}(0, \infty; X)$ and $S(t)x$ satisfies the integral equation*

$$S(t)x = T(t)x + \int_0^t T(t-s)BS(s)x\, ds \qquad \textit{for } t \geq 0,$$

and the growth condition

$$\varphi(S(t)x) \leq e^{a_0 t}\varphi(x) \qquad \textit{for } x \in D \textit{ and } t \geq 0.$$

(ii) *The following three conditions are satisfied:*

(ii-1) *There exist $\tau > 0$ and a family $\{V_r(\cdot, \cdot, \cdot); r > 0\}$ of nonnegative functionals on $[0, \tau] \times X \times X$ such that*

(V1) *for each $r > 0$ and x, $y \in D_r$, $V_r(\cdot, x, y) : [0, \tau] \to [0, \infty)$ is continuous,*

(V2) *for each $r > 0$ there exists $L(r) > 0$ such that*

$$|V_r(t, x, y) - V_r(t, \hat{x}, \hat{y})| \leq L(r)(\|x - \hat{x}\|_X + \|y - \hat{y}\|_X)$$

for (t, x, y), $(t, \hat{x}, \hat{y}) \in [0, \tau] \times X \times X$,

(V3) *for each $r > 0$ there exist $M(r) \geq m(r) > 0$ such that*

$$m(r)\|x - y\|_X \leq V_r(t, x, y) \leq M(r)\|x - y\|_X$$

for $(t, x, y) \in [0, \tau] \times D_r \times D_r$.

(ii-2) *For each* $r > 0$ *there exist* $R \geq r$ *and* $\omega \geq 0$ *such that*

$$\liminf_{h \downarrow 0} (V_R(t+h, J(h)x, J(h)y) - V_R(t,x,y))/h \leq \omega V_R(t,x,y)$$

for $(t,x,y) \in [0,\tau) \times C_r \times C_r$, *where*

$$J(t)w = T(t)w + \int_0^t T(s)Bw\,ds \qquad \text{for } w \in C \text{ and } t \geq 0.$$

(ii-3) *There exists* $b_0 \in (0,1)$ *such that to each* $x \in C$ *and* $\varepsilon > 0$ *there correspond* $\delta \in (0,\varepsilon]$, $x_\delta \in C$ *and* $z_\delta \in Y$ *satisfying*

$$x_\delta = J(\delta)x + z_\delta, \quad \|z_\delta\|_X \leq \varepsilon\delta, \quad \|z_\delta\|_Y \leq \varepsilon\delta^{b_0},$$
$$(\varphi(x_\delta) - \varphi(x))/\delta \leq a_0\varphi(x) + \varepsilon.$$

Moreover, for each $x \in D$, $S(t)x$ *is continuously differentiable on* $(0,\infty)$ *in* X, $AS(t)x$ *is continuous on* $(0,\infty)$ *in* X, *and* $S(t)x$ *satisfies* (SE;x) *for* $t > 0$, *if the operator* B *satisfies that for each* $\rho > 0$ *there exists* $L_B(\rho) > 0$ *such that*

$$\|Bu - Bv\|_X \leq L_B(\rho)\|u - v\|_Y \quad \text{for } u,v \in C \text{ and } \|u\|_Y \leq \rho,\ \|v\|_Y \leq \rho. \tag{2.2}$$

3 Outline of the Proof of Main Theorem

In order to prove our main theorem, we first rewrite (CGL) as an abstract Cauchy problem (SP;u_0) in $X = L^p(\Omega)$ and then apply the characterization theorem. In what follows K stands for various constants.

Let $X = L^p(\Omega)$ and $\|u\|_X = \|u\|_{L^p}$ for $u \in X$. Then, the operator A defined by

$$Au = (\lambda + i\mu)(\Delta u - u) \qquad \text{for } u \in D(A) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$$

is the generator of an analytic (C_0) semigroup $\{T(z)\}$ on X such that $T(z)$ is analytic in the sector $|\arg z| < \omega_p$ and $\|T(t)\|_X \leq e^{-\lambda t}$ for $t \geq 0$, where $\omega_p = \tan^{-1}(2\sqrt{p-1}/|p-2|)$.

In what follows we assume that $q > 2$. By (1.1) we can choose $\tilde{p}$ such that $p < \tilde{p} < p + q - 2$,

$$|\mu|/\lambda < 2\sqrt{\tilde{p}-1}/|\tilde{p}-2|, \tag{3.1}$$
$$\tilde{\theta} := (N/2)(1/p - 1/(\tilde{p}(q-1))) < 1. \tag{3.2}$$

Let $\beta = (N/2)(1/p - 1/\tilde{p})$ and $\theta = (1/\tilde{p} - 1/p + 2/N)^{-1}(1/\tilde{p} - 1/(p(q-1)))$. Then it is easily seen that β, θ, $\beta + (q-1)\theta(1-\beta) \in (0,1)$. The Gagliardo-Nirenberg inequality implies that $D(A) \subset L^{\tilde{p}}(\Omega) \cap L^{p(q-1)}(\Omega) \cap L^{\tilde{p}(q-1)}(\Omega)$. Choose $\alpha \in (0,1)$ satisfying

$$\tilde{\theta} < \alpha < 1 \qquad \text{and} \qquad \beta + (q-1)\theta(1-\beta) < \alpha < 1. \tag{3.3}$$

Let $D = L^p(\Omega) \cap L^{\tilde{p}}(\Omega)$. We define a lower semicontinuous functional $\varphi : X \to [0,\infty]$ by $\varphi(u) = \|u\|_{L^p}^p + \|u\|_{L^{\tilde{p}}}^{\tilde{p}}$ if $u \in D$ and $\varphi(u) = \infty$ otherwise. Let $Y = D((-A)^\alpha)$. We introduce a nonlinear operator B defined by

$$Bu = -(\kappa + i\nu)|u|^{q-2}u + (\lambda + i\mu + \gamma)u \qquad \text{for } u \in D(B) = C \ (= D \cap Y). \tag{3.4}$$

Then (CGL) is rewritten as a semilinear Cauchy problem (SP;u_0). The operators A and B have the properties below.

Lemma 3.1. *For $s \in \{p, \tilde{p}\}$, the following are valid.*

(i) *The operator A generates an analytic semigroup $\{T(t); t \geq 0\}$ on X such that*

$$\|T(t)v\|_{L^s} \leq e^{-\lambda t}\|v\|_{L^s} \quad \textit{for } v \in D,\ t \geq 0.$$

(ii) *The Banach space Y is continuously embedded in $D \cap L^{p(q-1)}(\Omega) \cap L^{\tilde{p}(q-1)}(\Omega)$.*

(iii) *For each $r > 0$ there exists $M_B(r) > 0$ such that*

$$\|Bv\|_{L^s} \leq M_B(r)(1 + \|v\|_Y) \quad \textit{for } v \in C_r.$$

(iv) *For each $\rho > 0$ there exists $L_B(\rho) > 0$ such that*

$$\|Bv - B\hat{v}\|_{L^s} \leq L_B(\rho)\|v - \hat{v}\|_Y \quad \textit{for } v,\ \hat{v} \in C \textit{ with } \|v\|_Y \leq \rho,\ \|\hat{v}\|_Y \leq \rho.$$

(v) *The domain of A is continuously embedded in $L^{s+q-2}(\Omega)$ and it holds that*

$$\mathrm{Re}\,\langle Av + Bv, |v|^{s-2}v\rangle + \kappa\|v\|_{L^{s+q-2}}^{s+q-2} - \gamma\|v\|_{L^s}^s \leq 0$$

for $v \in D(A) \cap W^{2,\tilde{p}}(\Omega) \cap W_0^{1,\tilde{p}}(\Omega)$, where $\langle w, z\rangle = \int_\Omega w(x)\overline{z(x)}\,dx$.

(vi) *There exist constants $a > 0$ and $b > 0$ such that*

$$\begin{aligned}&\mathrm{Re}\,\langle Au + Bu - (A\hat{u} + B\hat{u}), |u - \hat{u}|^{p-2}(u - \hat{u})\rangle \\ &\quad \leq (a + b(\|u\|_{L^{p+q-2}}^{p+q-2} + \|\hat{u}\|_{L^{p+q-2}}^{p+q-2}))\|u - \hat{u}\|_{L^p}^p \quad \textit{for } u, \hat{u} \in D(A).\end{aligned}$$

Let $\tau > 0$ and define a family $\{V_r(\cdot,\cdot,\cdot); r > 0\}$ of nonnegative functions on $[0,\tau] \times X \times X$ by

$$V_r(t,u,v) = \exp((b/\kappa p)((\|u\|_X \wedge r^{1/p})^p + (\|v\|_X \wedge r^{1/p})^p))(\|u - v\|_X \wedge (2r^{1/p}))$$

for $(t,u,v) \in [0,\tau] \times X \times X$, where b is a positive number satisfying Lemma 3.1 (vi) and $\xi \wedge \eta = \min(\xi,\eta)$ for $\xi, \eta \in \mathbb{R}$.

By applying Theorem 2.2, we obtain the following proposition.

Proposition 3.2. *There exists a semigroup $\{S(t); t \geq 0\}$ of locally Lipschitz operators on D with respect to φ satisfying the following:*

(i) *For $u_0 \in D$, $S(\cdot)u_0 \in C([0,\infty); X) \cap C^1((0,\infty); X) \cap C((0,\infty); D(A))$.*

(ii) *For $u_0 \in D$, $u(t) \equiv S(t)u_0$ gives a C^1 solution to* (SP;u_0) *satisfying*

$$\|u(t)\|^p + p\kappa \int_0^t e^{p\gamma(t-s)} \|u(s)\|_{L^{p+q-2}}^{p+q-2} \, ds \leq e^{p\gamma t}\|u_0\|^p \quad \text{for } t \geq 0. \tag{3.5}$$

(iii) *For $u_0 \in D$ and $t \geq 0$, $\|S(t)u_0\| \leq e^{\gamma t}\|u_0\|$ and $\|S(t)u_0\|_{L^{\tilde{p}}} \leq e^{\gamma t}\|u_0\|_{L^{\tilde{p}}}$.*

(iv) *For $\tau > 0$ and $r \geq 0$ there exists $M_{\tau,r} > 0$ such that*

$$\|S(t)u_0 - S(t)\hat{u}_0\| \leq M_{\tau,r}\|u_0 - \hat{u}_0\|$$

for $t \in [0,\tau]$ and u_0, $\hat{u}_0 \in D$ with $\|u_0\| \leq r$ and $\|\hat{u}_0\| \leq r$.

By Proposition 3.2 (iv), we observe that the family $\{S(t); t \geq 0\}$ can be uniquely extended to a semigroup $\{\widetilde{S}(t); t \geq 0\}$ of locally Lipschitz operators on X. We have only to show that for each $u_0 \in X = L^p(\Omega)$, $\widetilde{S}(t)u_0$ gives a C^1 solution to (SP;u_0). Let $u_0 \in X$. Since $u(t) = \widetilde{S}(t)u_0$ also satisfies the inequality (3.5), we see that $\widetilde{S}(t)u_0 \in L^p(\Omega) \cap L^{p+q-2}(\Omega)$ for a.e. $t > 0$. This fact ensures that there exists a decreasing sequence $\{t_n\}$ of positive numbers such that $\lim_{n\to\infty} t_n = 0$ and $\widetilde{S}(t_n)u_0 \in L^p(\Omega) \cap L^{\tilde{p}}(\Omega)$ for $n \geq 1$. Since $\widetilde{S}(t)u_0 = S(t-t_n)\widetilde{S}(t_n)u_0$ for $t > t_n$, $\widetilde{S}(t)u_0$ gives a global C^1 solution. □

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