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Local and global well-posedness for the KdV equation at the critical regularity

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1. Introduction

In this note, we study the Cauchy problem of the *KdV equation*:

$$\begin{cases} \partial_t u + \partial_x^3 u = \partial_x(u^2), & u : [-T,T] \times \mathbb{R} \to \mathbb{R} \text{ or } \mathbb{C}, \\ u(0,x) = u_0(x). \end{cases} \tag{1}$$

This is a survey of the author's papers [13, 14], and we refer to them for detailed discussion.

The KdV equation was originally derived by Korteweg and de Vries [15] as a model for the propagation of shallow water waves along a canal. It appears in various phases of mathematical physics; see [7] for a number of applications. It is also well-known as one of the simplest PDEs that have complete integrability.

We shall consider local and global *well-posedness* of (1) with initial data given in Sobolev spaces $H^s(\mathbb{R})$ defined via the norm

$$\|\phi\|_{H^s(\mathbb{R})} := \|\langle\cdot\rangle^s \widehat{\phi}\|_{L^2(\mathbb{R})},$$

where $\widehat{\cdot}$ denotes the Fourier transform and $\langle\cdot\rangle := (1+|\cdot|^2)^{1/2}$. We say the Cauchy problem is locally well-posed in H^s if for any initial datum $u_0 \in H^s$, there exists a time of local existence $T = T(\|u_0\|_{H^s}) > 0$ and the solution in $C([-T,T];H^s)$ which is unique in some sense and depends continuously on the datum. If the above T can be chosen arbitrarily large, we say the Cauchy problem is globally well-posed in H^s. Note that it does not make any differences whether we take $[-T,T]$ or $[0,T]$ as the interval of local existence, because the KdV equation has time reversal symmetry.

Our main results are the local well-posedness of (1) and the global well-posedness of real-valued (1) in $H^{-3/4}(\mathbb{R})$.

We now review the *iteration method* for proving the local well-posedness and clarify the meaning of a 'solution' to the Cauchy problem.

First, we replace the Cauchy problem with the corresponding integral equation via the Duhamel formula,

$$u(t) = e^{-t\partial_x^3}u_0 + \int_0^t e^{-(t-t')\partial_x^3}\partial_x(u(t')^2)\,dt', \quad t \in [-T,T], \tag{2}$$

where $\{e^{-t\partial_x^3}\}_{t\in\mathbb{R}}$ denotes the linear propagator defined by $\widehat{e^{-t\partial_x^3}\phi}(\xi) := e^{it\xi^3}\widehat{\phi}(\xi)$.

We then put the right hand side of (2) $\Phi_{u_0}(u)(t)$ and try to show that Φ_{u_0} is a contraction map on a certain Banach space $\mathcal{S}_T^s$ embedded in $C([-T,T];H^s)$. Note that the operator Φ_{u_0} is for now defined only on smooth functions (with enough decay at spatial infinity). For instance, if we consider negative regularities $s<0$, we will fail to define the quadratic nonlinear term for all $u \in C([-T,T];H^s)$. Thus, it is important to find a function space $\mathcal{S}_T^s$ so that the domain of Φ_{u_0} can be extended appropriately to all functions in this space.

For the contractiveness of the operator Φ_{u_0}, the following linear and bilinear estimates are basically needed:

$$\|e^{-t\partial_x^3}u_0\|_{\mathcal{S}_T^s} \le C\|u_0\|_{H^s}, \quad \Big\|\int_0^t e^{-(t-t')\partial_x^3}\partial_x(u(t')v(t'))\,dt'\Big\|_{\mathcal{S}_T^s} \le C\|u\|_{\mathcal{S}_T^s}\|v\|_{\mathcal{S}_T^s}.$$

Once these estimates are established with a Banach space $\mathcal{S}_T^s$ in which smooth functions are dense, definition of the Duhamel term in Φ_{u_0} will be extended to the whole $\mathcal{S}_T^s \times \mathcal{S}_T^s$ in the unique continuous sense. Then, we consider a function u as a solution to the Cauchy problem if u satisfies the equation $u = \Phi_{u_0}(u)$ in $\mathcal{S}_T^s$. It is easy to verify that such a solution is the unique limit in $\mathcal{S}_T^s$ of smooth solutions starting from initial data smoothly approximating the original datum u_0 in H^s.

The above two estimates are actually enough to show that Φ_{u_0} is contractive on $\{\, u \in \mathcal{S}_T^s \mid \|u\|_{\mathcal{S}_T^s} \le 2C\|u_0\|_{H^s} \,\}$ if u_0 is sufficiently small. We thus obtain a solution as the unique fixed point of Φ_{u_0}, and the Lipschitz continuous dependence of solutions on data also follows naturally. Note that the KdV equation has the scaling invariance, that is, the scaling transform

$$u(t,x) \quad \mapsto \quad u^\lambda(t,x) := \lambda^{-2}u(\lambda^{-3}t, \lambda^{-1}x), \qquad \lambda > 0$$

maps a solution of (1) to the solution with initial datum $u_0^\lambda(x) := \lambda^{-2}u_0(\lambda^{-1}x)$. Since we have

$$\|u_0^\lambda\|_{H^s(\mathbb{R})} = O(\lambda^{-3/2-\min\{\,0,s\,\}})$$

as $\lambda \to \infty$, the problem for general initial data is reduced to solving the equation on the time interval $[-1,1]$ for any sufficiently small data as long as we treat $s > -3/2$. From now on, we consider the case $T=1$.

We have seen that the linear solution $e^{-t\partial_x^3}u_0$ is defined clearly through the spatial Fourier transform; however, it is instructive to compute the *space-time* Fourier transform of the linear solution. The result for a smooth u_0 is $c\delta(\tau - \xi^3)\widehat{u_0}(\xi)$, where δ denotes the Dirac delta function. We find a remarkable property of the linear solution that it is supported in the space-time frequency space on the cubic curve $\{\, \tau = \xi^3 \,\}$.

In order to take advantage of the space-time Fourier transform in the context of nonlinear equations, we need to deal with a solution as a function on the entire space $\mathbb{R}^2$ rather than on the space-time slab $[-1,1] \times \mathbb{R}$. Therefore, we introduce a C^∞

bump function ψ on $\mathbb{R}$ satisfying $\psi \equiv 1$ on $[-1,1]$ and supp $\psi \subset [-2,2]$, and then seek a solution to the global-in-time integral equation

$$u(t) = \psi(t)e^{-t\partial_x^3}u_0 + \psi(t)\int_0^t e^{-(t-t')\partial_x^3}\partial_x(u(t')^2)\,dt', \quad t \in \mathbb{R},$$

instead of the previous local-in-time equation. A simple computation implies that the space-time Fourier transform of the truncated linear solution $\psi(t)e^{-t\partial_x^3}u_0$ with a smooth u_0 is equal to $\widehat{\psi}(\tau - \xi^3)\widehat{u_0}(\xi)$, which is now a smooth function mainly supported *near* the cubic curve (in fact it is rapidly decreasing in the variable $\tau - \xi^3$). As long as the nonlinear equation can be thought of as a perturbation of the linear equation, it is expected that the nonlinear solution also concentrates near the characteristic hypersurface.

From this point of view, it is quite natural to introduce the Bourgain spaces $X^{s,b}$, or the *Fourier restriction spaces*, defined as the completion of space-time Schwartz functions with respect to the norm

$$\|u\|_{X^{s,b}} := \|\langle\xi\rangle^s\langle\tau - \xi^3\rangle^b\widetilde{u}(\tau,\xi)\|_{L^2_{\tau,\xi}},$$

where $\widetilde{u}$ denotes the space-time Fourier transform of u. If the real parameter b is greater than 1/2, then the continuous embedding $X^{s,b} \hookrightarrow C(\mathbb{R}; H^s)$ holds. In this case, $X^{s,b}$ effectively captures functions supported in frequency near the cubic curve from the space $C(\mathbb{R}; H^s)$. Note also that $X^{s,b}$ can be regarded as the product Sobolev spaces twisted by the linear evolution; in fact, we have

$$\|u\|_{X^{s,b}} = \|e^{t\partial_x^3}u(t)\|_{H^b_t(H^s_x)}.$$

The Bourgain spaces $X^{s,b}$, named after J. Bourgain who introduced it to study the nonlinear Schrödinger and KdV equations [2, 3], provided substantial progress in the well-posedness theory for a wide variety of nonlinear dispersive equations. Especially, it is a quite powerful tool to establish the local well-posedness in Sobolev spaces with very low (perhaps *negative*) regularities.

If the space $X^{s,b}$ is used for the resolution space $\mathcal{S}^s$, the estimates required to make Φ_{u_0} contractive will be described as

$$\|\psi(t)e^{-t\partial_x^3}u_0\|_{X^{s,b}} \le C\|u_0\|_{H^s}, \tag{3}$$

$$\|\psi(t)\int_0^t e^{-(t-t')\partial_x^3}\partial_x(u(t')v(t'))\,dt'\|_{X^{s,b}} \le C\|u\|_{X^{s,b}}\|v\|_{X^{s,b}}. \tag{4}$$

We usually divide the second estimate (4) into the linear Duhamel estimate

$$\|\psi(t)\int_0^t e^{-(t-t')\partial_x^3}F(t')\,dt'\|_{X^{s,b}} \le C\|F\|_{X^{s,b-1}} \tag{5}$$

and the *bilinear estimate*

$$\|\partial_x(uv)\|_{X^{s,b-1}} \leq C\|u\|_{X^{s,b}}\|v\|_{X^{s,b}}. \tag{6}$$

The choice of auxiliary space $X^{s,b-1}$ for nonlinearity seems natural if we recall that the parameter b denotes the regularity with respect to the differential operator $\partial_t+\partial_x^3$, and that the solutions $u = (\partial_t + \partial_x^3)^{-1}\partial_x(u^2)$ should be in $X^{s,b}$.

Then, it is enough for the local well-posedness in H^s to establish the estimates (3), (5), and (6). It turns out that two linear estimates (3) and (5) hold for any $s \in \mathbb{R}$ with appropriate values of b; see [8] for instance. In contrast, the bilinear estimate (6) is known to fail for any b if we consider regularities below a certain threshold. This fact suggests that if the data become rougher, the nonlinear effect will get stronger and the nonlinear equation will behave less as a perturbation of the linear equation. Therefore, the bilinear estimate (6) controlling nonlinearity is directly connected to the well-posedness and plays a crucial role in the iteration argument.

2. Previous results and the main theorem

The Cauchy problem (1) has been extensively studied. We first recall that Kenig, Ponce, and Vega [11] established the bilinear estimate (6) for $s > -3/4$ with some $b > 1/2$, which implies the local well-posedness of (1) in the corresponding regularity. Their local result was improved to the global well-posedness in $H^s(\mathbb{R})$ with $s > -3/4$ in the real-valued case by Colliander, Keel, Staffilani, Takaoka, and Tao [6]. The proof was based on the *I-method*, which we shall review in Section 5.

It is natural to try to verify the bilinear estimate for $s \leq -3/4$ if one wishes to obtain the well-posedness for that regularity. However, it is known that (6) fails for any $b \in \mathbb{R}$ if $s \leq -3/4$ ([11, 17]). Moreover, when $s < -3/4$ the data-to-solution map for (1) fails to be uniformly continuous as a map from H^s to $C_t(H^s_x)$ ([12, 4]). This result does not necessarily imply the *ill-posedness* of the Cauchy problem, but the iteration method would naturally provide the Lipschitz continuity, so it will not work for regularities $s < -3/4$. It is an important open problem whether the local well-posedness with a merely continuous data-to-solution map holds in these regularities.

We now focus on the case $s = -3/4$. As seen above, this is the critical regularity for the iteration method (but far above the scaling critical regularity $s = -3/2$). Since we do not have the bilinear estimate in $X^{-3/4,b}$, we have to iterate in a different space, or abandon the direct iteration method, to obtain well-posedness in $H^{-3/4}$.

The latter approach was taken in [4]. They obtained the local-in-time result for (1) in $H^{-3/4}(\mathbb{R})$ by combining (slightly modified) Miura transform with the corresponding local well-posedness for the modified KdV equation in $H^{1/4}(\mathbb{R})$ obtained in [10]. The Cauchy problem of the *modified KdV equation*,

$$\begin{cases} \partial_t v + \partial_x^3 v = \pm\partial_x(v^3), & v : [-T,T]\times\mathbb{R}\to\mathbb{R}\ , \\ v(0,x) = v_0(x), \end{cases} \tag{7}$$

is also well-studied and linked with the KdV equation through the *Miura transform*; if v is a smooth solution to (7), then $u := c_1\partial_x v + c_2 v^2$ with suitable constants c_1, c_2 solves the KdV equation. Since the Miura transform acts roughly as a derivative, many results for KdV have counterparts for modified KdV at one higher regularity; for instance, the regularity threshold for validity of the iteration method is $s = 1/4$, exactly one higher than $s = -3/4$ for KdV.

We point out that the above result for KdV in $H^{-3/4}$ is relatively weak, compared with that for $s > -3/4$. Firstly, the uniqueness of solutions was obtained only in the image of the Miura transform. In fact, for the case $s > -3/4$ it was shown that solutions are unique in $X^{s,b}$. Since the Miura transform is a nonlinear mapping, we find it not so easy to analyze the structure of its image, or verify whether a given function is in its image or not. Secondly, we do not have the control of their local solutions in a function space well adapted to the I-method, such as $X^{s,b}$. This is why the global well-posedness for real-valued (1) in $H^{-3/4}(\mathbb{R})$ was left open.

From this point of view, it is quite interesting to investigate the strong local well-posedness for (1) in $H^{-3/4}(\mathbb{R})$ by the iteration method. Our main result precisely deals with this issue. Of course, we have to change the working space from $X^{-3/4,b}$. We shall construct a Banach space X as the working space $\mathcal{S}^{-3/4}$, which is some Besov-like generalization of the Bourgain space $X^{-3/4,1/2}$ with slight modification in low frequency. See the definition in the next section. The space X possesses the bilinear estimate similar to (6), but fails to be embedded into $C(\mathbb{R}; H^{-3/4})$, which forces us to introduce an auxiliary space Y defined by the norm

$$\|u\|_Y := \|\langle\xi\rangle^{-3/4}\widetilde{u}\|_{L^2_\xi(L^1_\tau)}.$$

This space Y has also appeared in a number of previous works (originally in [8]). For these spaces we have the following bilinear estimate:

Proposition 1. *We have*

$$\left\|\langle\partial_t + \partial_x^3\rangle^{-1}\partial_x(uv)\right\|_X + \left\|\langle\partial_t + \partial_x^3\rangle^{-1}\partial_x(uv)\right\|_Y \le C\|u\|_X\|v\|_X,$$

where $\langle\partial_t + \partial_x^3\rangle^{-1}$ is the space-time Fourier multiplier corresponding to $\langle\tau - \xi^3\rangle^{-1}$.

A standard iteration argument then implies our main theorem.

Theorem 1. *The Cauchy problem* (1) *(real-valued or complex-valued) is locally well-posed in $H^{-3/4}(\mathbb{R})$. In particular, solutions are unique in X to be defined in Section 3.*

We remark that the uniqueness in the above theorem is precisely as follows: the solutions of (1) on the time interval $[-T, T]$ are unique in the class $X_{[-T,T]}$, where for an interval I we define the space X_I as the restriction to the time interval I of functions in X, which is equipped with the restricted norm

$$\|u\|_{X_I} := \inf\{\, \|U\|_X \,|\, U \in X,\, U(t) = u(t) \text{ for } t \in I \,\}.$$

We also use this restricted norm for a global-in-time function u under the convention of $\|u\|_{X_I} := \|u|_{t\in I}\|_{X_I}$.

This theorem combined with the I-method yields the global results. Since our function space X is very close to the usual Bourgain space $X^{s,b}$ (in fact satisfies the embedding $X^{-3/4,b} \hookrightarrow X \hookrightarrow X^{-3/4,1/2}$ for any $b > 1/2$), proof is almost identical with the case of $X^{s,b}$ for $s > -3/4$.

Theorem 2. *The real-valued Cauchy problem* (1) *is globally well-posed in* $H^{-3/4}(\mathbb{R})$.

Note that these global results do not hold for the complex-valued case. In fact, several finite-time blow-up solutions have been discovered. For instance, see [1] and references therein.

In the next section, we will discuss how to construct the space X which yields the crucial bilinear estimate. The proof of Proposition 1 is quite complicated, so we refer to [13] for it. In Section 4, we will show outline of the proof for Theorem 1, especially for the uniqueness of solutions. Section 5 will be devoted to a review of the I-method. We will omit the details for the proof of Theorem 2 and refer to [6]. In the last section, we will recall a recent result by Guo [9] and compare it with ours.

3. Construction of the working space

Let us recall some counterexamples to the bilinear estimate in $X^{-3/4,b}$,

$$\|\partial_x(uv)\|_{X^{-3/4,b-1}} \le C\|u\|_{X^{-3/4,b}}\|v\|_{X^{-3/4,b}}, \tag{8}$$

and then see how to modify the Bourgain spaces so that these examples may be suitably controlled.

We first prepare some notations for convenience. Let us fix a smooth function $q_0 : \mathbb{R} \to [0,1]$ which is equal to 1 on $[-5/4, 5/4]$ and supported in $[-8/5, 8/5]$. For $N > 0$ and $j = 1, 2, \dots$, define

$$q_N(\xi) := q_0(\tfrac{\xi}{N}) - q_0(\tfrac{2\xi}{N}), \quad p_0 := q_0, \quad p_j := q_{2^j},$$

and then denote the Fourier multipliers with respect to x corresponding to q_0, q_N, p_0, and p_j by Q_0, Q_N, P_0, and P_j, respectively. Note that $\{\, P_j \,\}_{j=0}^{\infty}$ is an inhomogeneous Littlewood-Paley decomposition, and that Q_N with $N > 0$ is the frequency-localizing operator satisfying $\operatorname{supp} q_N \subset \{\, \frac{5N}{8} \le |\xi| \le \frac{8N}{5} \,\}$, $q_N \equiv 1$ on $\{\, \frac{4N}{5} \le |\xi| \le \frac{5N}{4} \,\}$.

We begin with the following examples.

Proposition 2 ([11]). *Let $b \in \mathbb{R}$, then there exists $c>0$ such that the following holds.*
(i) For any $N \gg 1$, there exist u_N, v_N satisfying $Q_N u_N = u_N$, $Q_N v_N = v_N$, and

$$\left\|Q_0 \partial_x (u_N v_N)\right\|_{X^{-3/4,b-1}} \geq c N^{\frac{3}{2}b-\frac{3}{4}} \left\|u_N\right\|_{X^{-3/4,b}} \left\|v_N\right\|_{X^{-3/4,b}}.$$

(ii) For any $N \gg 1$, there exist u_N, v_N satisfying $Q_N u_N = u_N$, $Q_0 v_N = v_N$, and

$$\left\|\partial_x (u_N v_N)\right\|_{X^{-3/4,b-1}} \geq c N^{\frac{3}{4}} \langle N^{\frac{3}{2}b}\rangle^{-1} \left\|u_N\right\|_{X^{-3/4,b}} \left\|v_N\right\|_{X^{-3/4,b}}.$$

This proposition says that (8) fails to hold for $b > 1/2$ (from (i)), and for $b < 1/2$ (from (ii)). These examples are sketched in Figure 1. We observe that the example in (i) consists of high-frequency functions supported in the frequency space along the curve $\tau = \xi^3$, and their product (or, in frequency, their convolution) is concentrated near the frequency origin (thus in the low-frequency region). We call such interactions *high-high-low*. On the other hand, the example in (ii) is the interaction between functions of high frequency and low frequency, which produces a high-frequency component near the curve, so we call it *high-low-high* interaction.

The bilinear estimate (8) also fails in the case $b = 1/2$, but the divergence order is *logarithmic* in N rather than *power* in N as Proposition 2.

Proposition 3 ([17]). *Let $0<\alpha<1/2$, then there exists $c > 0$ such that the following holds: For any $N \gg 1$, there exist u_N, v_N satisfying $Q_N u_N = u_N$, $Q_N v_N = v_N$, and*

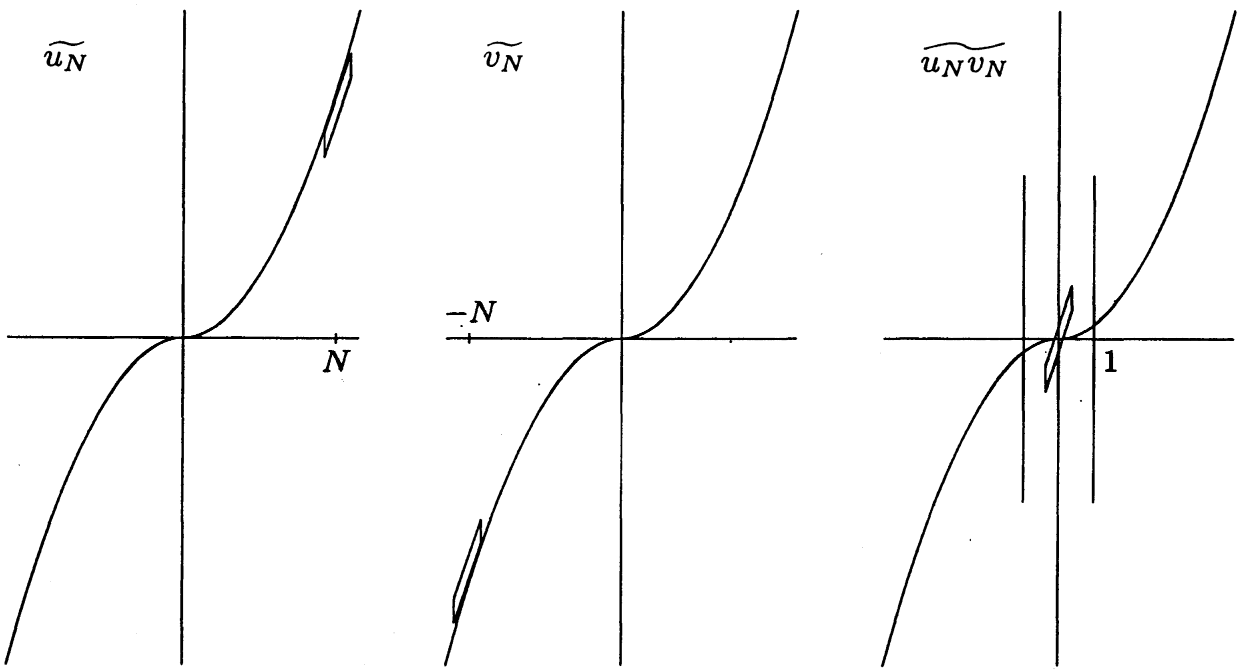
$$\left\|Q_0 \partial_x (u_N v_N)\right\|_{X^{-3/4,-1/2}} \geq c(\log N)^{\alpha} \left\|u_N\right\|_{X^{-3/4,1/2}} \left\|v_N\right\|_{X^{-3/4,1/2}}.$$

As sketched in Figure 2, this example of high-high-low type is much more complicated than the previous one. We point out that the high-frequency function is supported also in the region distant from the curve $\tau = \xi^3$ in contrast to the counterexamples in Proposition 2. In fact, u_N consists of $\varepsilon \log N$ components dyadically supported away from the curve ($0 < \varepsilon \ll 1$). Each of these components $u_{N,j}$ ($1 \leq j \leq \varepsilon \log N$), which has some positive $X^{-3/4,1/2}$ norm a_j, interacts with v_N and outputs the component, whose norm is $\|\partial_x(u_{N,j} v_N)\|_{X^{-3/4,-1/2}} \gtrsim a_j \|v_N\|_{X^{-3/4,1/2}}$, at almost the *same* part of the low-frequency region $\{\,|\xi| \leq 1\,\}$. The norm of the total output is then at least $\|v_N\|_{X^{-3/4,1/2}} \sum a_j$, while the norm of u_N is equal to the ℓ^2 sum of those of $u_{N,j}$'s; $\|u_N\|_{X^{-3/4,1/2}} \sim (\sum a_j^2)^{1/2}$. Putting $a_j = j^{\alpha-1}$ ($0 < \alpha < 1/2$) for instance, we have the divergence of $O((\log N)^{\alpha})$.

We have seen that the bilinear estimate in $X^{-3/4,b}$, (8), fails for any $b \in \mathbb{R}$, and that the divergence in the case $b = 1/2$ is logarithmic, better than the other cases. Therefore, we shall start from $X^{-3/4,1/2}$ and modify it to endure the nonlinear interaction described in Proposition 3.

In the analysis with the Bourgain spaces, in fact, logarithmic divergences of nonlinear estimates often occur in such a limiting regularity. One standard way to avoid this is the ℓ^1-*Besov* modification (in the temporal direction) of the Bourgain spaces.

(i) *High-high-low* interaction



(ii) *High-low-high* interaction

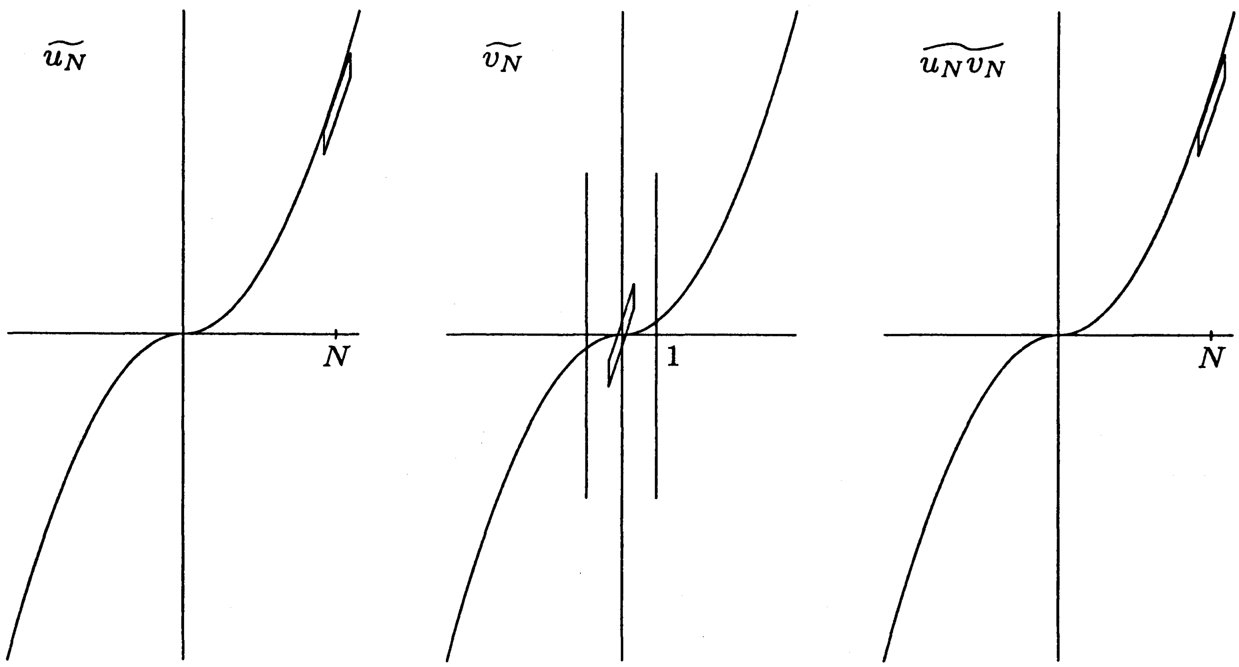


Figure 1. Two typical nonlinear interactions described in Proposition 2. In the context of the bilinear estimate (8) for $b \neq 1/2$, they produce some *power* divergences in N.

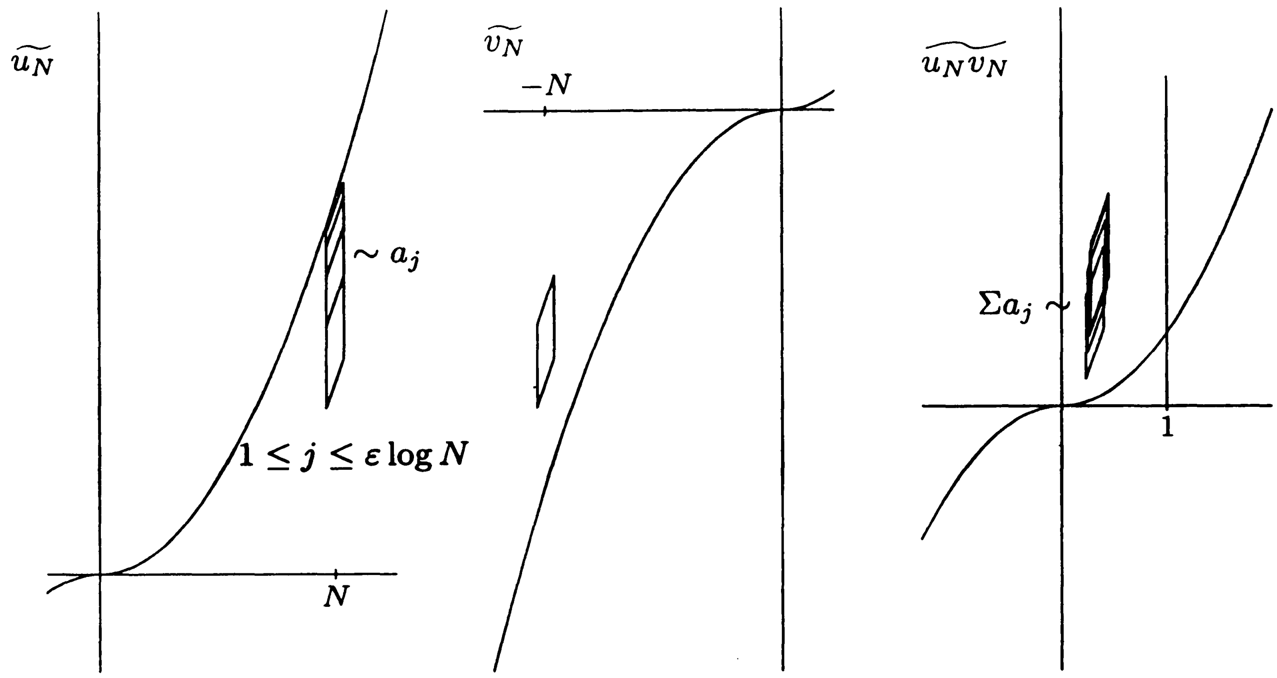


Figure 2. The example of high-high-low interaction described in Proposition 3, which breaks the bilinear estimate in $X^{-3/4,1/2}$ with *logarithmic* divergence.

This is similar to the space $B^{1/2}_{2,1}(\mathbb{R})$ as a modification of $H^{1/2}(\mathbb{R})$, which has many good properties such as the embedding into the space of bounded continuous functions. ℓ^1-Besov structure is also convenient for the summation of dyadic frequency pieces: For example, if we have a frequency-localized bilinear estimate

$$\|B(P_j u, P_k v)\| \le C\|P_j u\|\,\|P_k v\|$$

for some bilinear operator B, then the bilinear estimate

$$\|B(u,v)\| \le C\|u\|\,\|v\|$$

immediately follows from the triangle inequality and the ℓ^1 nature of the norm.

Such Besov-type Bourgain spaces were used first in the context of the wave map equations ([18]), and have appeared in a number of literature.

In our context, the ℓ^1-Besov Bourgain spaces $X^{s,b,1}$ is defined by the norm

$$\|u\|_{X^{s,b,1}} := \Big(\sum_{j=0}^{\infty} 2^{2sj}\Big(\sum_{k=0}^{\infty} 2^{bk}\|p_j(\xi)p_k(\tau-\xi^3)\widehat{u}\|_{L^2_{\tau,\xi}}\Big)^2\Big)^{1/2}.$$

The usual $X^{s,b}$ norm is equivalent to the above norm with the ℓ^1_k sum replaced by ℓ^2_k. We see that $X^{s,b,1}$ is slightly stronger than $X^{s,b}$.

Note that the counterexample in Proposition 3 can be well controlled if we measure the high-frequency function in $X^{-3/4,1/2,1}$ instead of $X^{-3/4,1/2}$. When we work at

	$X^{-3/4,1/2+\varepsilon}$	$X^{-3/4,1/2-\varepsilon}$	$X^{-3/4,1/2}$	$X^{-3/4,1/2,1}$	X_*
high-high-low	N^α Prop 2 (i)		$(\log N)^\alpha$ Prop 3	$(\log N)^\alpha$ Prop 4 (i)	
high-low-high		N^α Prop 2 (ii)	$(\log N)^\alpha$ Prop 4 (ii)		$(\log N)^\alpha$ Prop 4 (ii)

Table 1. Various divergences in the bilinear estimates for $s = -3/4$.

the bottom regularity, similar issues may arise, and the space $X^{s,1/2,1}$ is considered generally as a good substitute for $X^{s,1/2}$. However, for the KdV case, we can not restore the bilinear estimate in $X^{-3/4,b}$ by just making the ℓ^1-Besov modification; counterexamples are given in the following Proposition 4, which is our second main result. That seems to be the main reason why this problem of much interest had been left open since the bilinear estimate for $s > -3/4$ was established in [11].

Proposition 4. *Let $0<\alpha<1/2$, then there exists $c > 0$ such that the following holds:*
(i) For any $N \gg 1$, there exist u_N, v_N satisfying $Q_N u_N = u_N$, $Q_N v_N = v_N$, and

$$\left\|Q_0\langle\partial_t + \partial_x^3\rangle^{-1}\partial_x(u_N v_N)\right\|_{X^{-3/4,1/2,1}} \ge c(\log N)^\alpha \left\|u_N\right\|_{X^{-3/4,1/2,1}} \left\|v_N\right\|_{X^{-3/4,1/2,1}}.$$

(ii) For any $N \gg 1$, there exist u_N, v_N satisfying $Q_N u_N = u_N$, $Q_0 v_N = v_N$, and

$$\left\|\langle\partial_t + \partial_x^3\rangle^{-1}\partial_x(u_N v_N)\right\|_{X^{-3/4,1/2,1}} \ge c(\log N)^\alpha \left\|u_N\right\|_{X^{-3/4,1/2,1}} \left\|v_N\right\|_{X^{-3/4,1/2}},$$

$$\left\|\langle\partial_t + \partial_x^3\rangle^{-1}\partial_x(u_N v_N)\right\|_{X^{-3/4,1/2}} \ge c(\log N)^\alpha \left\|u_N\right\|_{X^{-3/4,1/2}} \left\|v_N\right\|_{X^{-3/4,1/2}}.$$

(i) shows that the $X^{-3/4,1/2,1}$ norm is too strong in low frequency to control the *high-high-low* interaction. Then, it seems natural to consider the space X_* defined via the norm

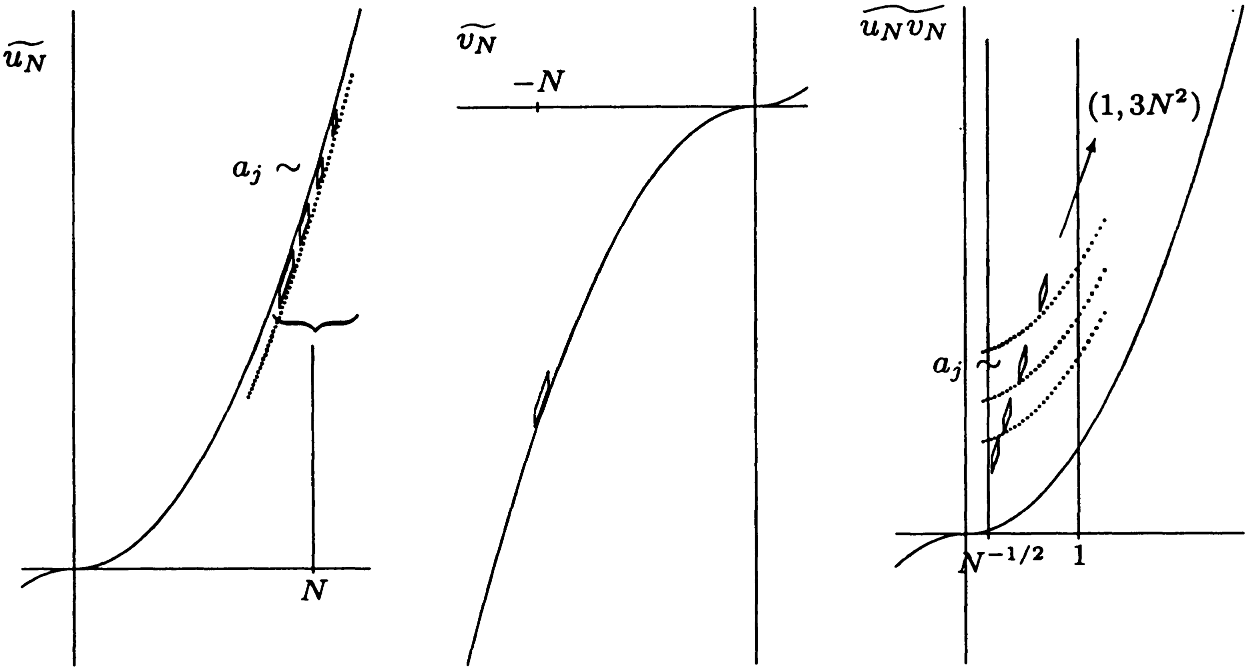
$$\|u\|_{X_*} := \|P_0 u\|_{X^{-3/4,1/2}} + \|(1-P_0)u\|_{X^{-3/4,1/2,1}},$$

which has the stronger structure $X^{-3/4,1/2,1}$ in high frequency and the weaker structure $X^{-3/4,1/2}$ in low frequency. However, the first estimate in (ii) says that this space is too weak in low frequency to control the *high-low-high* interaction. The same example also re-proves that (8) with $b = 1/2$ does not hold. Thus, we can sum up the divergences in bilinear estimates for the regularity $s = -3/4$ as Table 1.

To overcome this difficulty, we have to take a real look at these counterexamples, which are described in Figure 3.

For (i), $\varepsilon \log N$ components of the function u_N are all supported *near* the curve $\tau = \xi^3$, unlike the example given in Proposition 3. Thus, the stronger $X^{-3/4,1/2,1}$ norm of u_N is still given by the ℓ^2 sum. On the other hand, we see that the output

(i) Logarithmic divergence of the bilinear estimate in $X^{-3/4,1/2,1}$ (*high-high-low*)



(ii) Logarithmic divergence of the bilinear estimate in X_* or $X^{-3/4,1/2}$ (*high-low-high*)

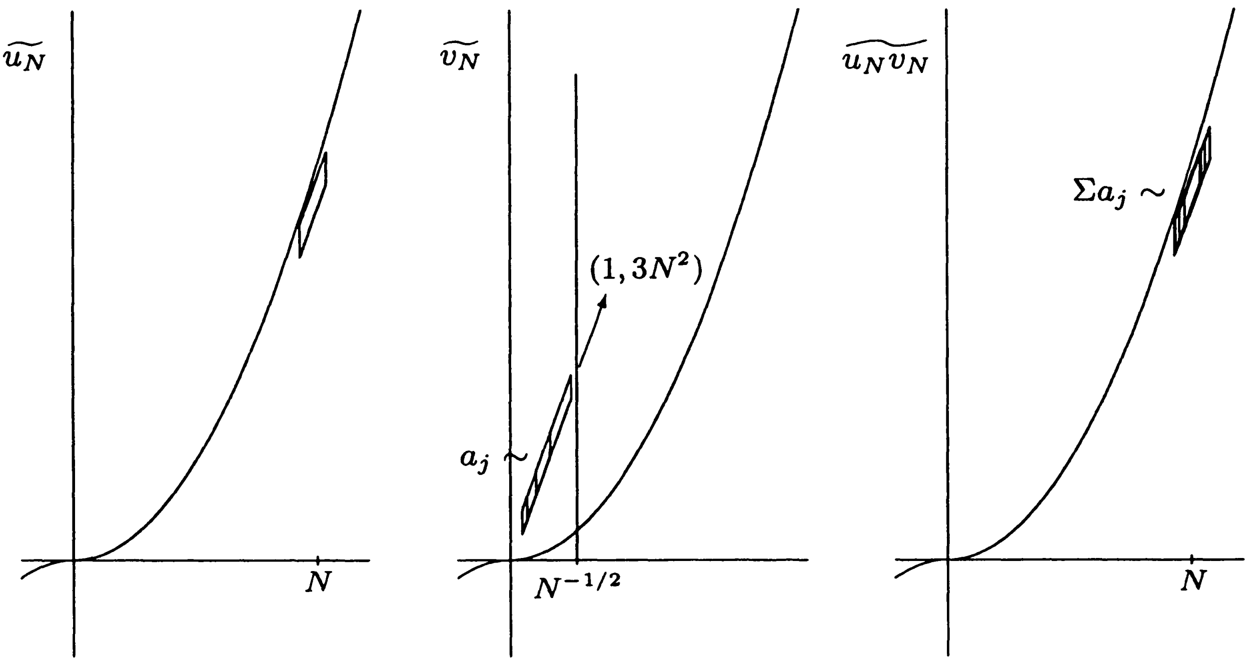


Figure 3. Counterexamples to the bilinear estimates given in Proposition 4.

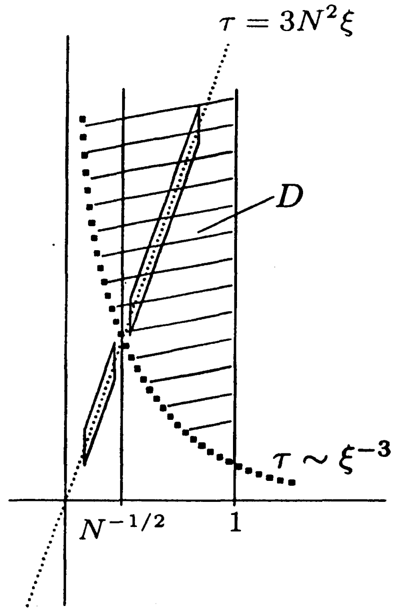


Figure 4. The 'middle-frequency' region D.

components are supported in the low-frequency region between $N^{-1/2}$ and 1, and also dyadically separated with respect to $\tau - \xi^3$. Thus, the norm of the output amounts to the ℓ^1 sum if we employ the ℓ^1-Besov structure in low frequency. We remark that all the ℓ^p_k norms are equivalent if a function is restricted near the curve $\tau = \xi^3$ since there is no summation over k in such a case, and that the modification from ℓ^2_k to ℓ^1_k has an effect only when a function is supported away from the curve.

For (ii), the low-frequency function v_N has $\varepsilon \log N$ components between 0 and $N^{-1/2}$, and its $X^{-3/4,1/2}$ norm is given by the ℓ^2 sum. We see that outputs of the interaction of these components with u_N fall onto almost the same frequency position near the curve. Therefore, the norm of the output is the ℓ^1 sum, no matter which structure we use in high frequency.

It is worth noting that these serious interactions come from different parts of the low-frequency region separated by the fuzzy boundary $|\xi| \sim N^{-1/2}$. Note also that both of them are supported along the line $\tau = 3N^2\xi$. This suggests that we may use $X^{-3/4,1/2}$ in the middle frequency region

$$D := \{\, (\tau,\xi) \in \mathbb{R}^2 \mid |\xi| < 1,\ |\tau| > |\xi|^{-3} \,\},$$

and use $X^{-3/4,1/2,1}$ in very low frequency $\{\, |\xi| < 1 \,\} \setminus D$; see Figure 4. In fact, it turns out that the high-high-low interaction can be controlled in very low frequency even if we assume the stronger structure $X^{-3/4,1/2,1}$ there, and that the high-low-high can be still controlled under the weaker structure $X^{-3/4,1/2}$ in middle frequency.

Our working space X is defined by

$$\|u\|_X := \|P_D u\|_{X^{-3/4,1/2}} + \|(1 - P_D)u\|_{X^{-3/4,1/2,1}},$$

where P_D is the frequency-localizing operator to the set D. For this X we can establish the desired bilinear estimate, Proposition 1.

4. Local well-posedness

In addition to the bilinear estimate, the following linear estimates are verified.

Lemma 1. *We have the following estimates*

$$\|e^{-t\partial_x^3}u_0\|_{X_{[-1,1]}} + \sup_{-1\le t\le 1}\|e^{-t\partial_x^3}u_0\|_{H^{-3/4}(\mathbb{R})} \le C\|u_0\|_{H^{-3/4}(\mathbb{R})},$$

$$\Big\|\int_0^t e^{-(t-t')\partial_x^3}F(t')\,dt'\Big\|_{X_{[-1,1]}} + \sup_{-1\le t\le 1}\Big\|\int_0^t e^{-(t-t')\partial_x^3}F(t')\,dt'\Big\|_{H^{-3/4}(\mathbb{R})}$$
$$\le C\|\langle\partial_t+\partial_x^3\rangle^{-1}F\|_X + \|\langle\partial_t+\partial_x^3\rangle^{-1}F\|_Y.$$

Proof is much easier than the bilinear estimate (see [13]). Remark that the time-restricted norm plays the same role as the cutoff function in the estimates (3), (5).

From Proposition 1 and Lemma 1, we can iterate the equation in the space $X_{[-1,1]}\cap C([-1,1];H^{-3/4}(\mathbb{R}))$ and obtain Theorem 1 except for the uniqueness; note that these estimates only imply the uniqueness in a small ball of the working space.

It remains to extend the uniqueness to the whole working space. Recall that in the case $s>-3/4$, Kenig, Ponce, and Vega [11] showed the uniqueness of solutions in $X^{s,b}_{[-T,T]}$, essentially using the following stronger bilinear estimate: There exists $\alpha=\alpha(s)>0$ such that

$$\|\partial_x(\psi(\tfrac{t}{\delta})u\cdot\psi(\tfrac{t}{\delta})v)\|_{X^{s,b}} \le C\delta^\alpha\|u\|_{X^{s,b}}\|v\|_{X^{s,b}} \tag{9}$$

for any $\delta\in(0,1]$.

If (9) is valid, then we can derive the uniqueness in $X^{s,b}_{[-T,T]}$ as follows. Let $u,v\in X^{s,b}_{[-T,T]}$ be two solutions of the integral equation on a time interval $[-T,T]$ with the same initial datum u_0. Then, $\psi(t/\delta)u$ and $\psi(t/\delta)v$ solve the equation on $[-\delta',\delta']$, where $\delta'=\min\{\delta,T\}$. Therefore, we have

$$u(t)-v(t)=\int_0^t e^{-(t-t')\partial_x^3}\partial_x\big[(\psi(\tfrac{t'}{\delta})u(t'))^2-(\psi(\tfrac{t'}{\delta})v(t'))^2\big]\,dt'$$

on $[-\delta',\delta']$. We see from (5) and (9) that

$$\|u-v\|_{X^{s,b}_{[-\delta',\delta']}} \le C\delta'^\alpha\|u+v\|_{X^{s,b}_{[-\delta',\delta']}}\|u-v\|_{X^{s,b}_{[-\delta',\delta']}}$$

for $\delta'\in(0,1]$. Since $\|u+v\|_{X^{s,b}_{[-\delta',\delta']}}\le\|u\|_{X^{s,b}_{[-T,T]}}+\|v\|_{X^{s,b}_{[-T,T]}}$, we can choose δ so small that $C\delta'^\alpha\|u+v\|_{X^{s,b}_{[-\delta',\delta']}}\le 1/2$. If $T=\delta'$, we have the desired uniqueness. In the case $T > \delta' = \delta$, the uniqueness in $X^{s,b}_{[-T,T]}$ will be obtained by repeating this procedure.

In our context, however, estimate like (9) is not available. One of the reasons for this is the criticality of the problem; in fact, the exponent $\alpha(s)$ given in (9) tends to 0 as $s \to -3/4$. We employ the argument of Muramatu and Taoka [16], who considered the local well-posedness for nonlinear Schrödinger equations with quadratic nonlinearities. In this argument, the following fact is essential:

$$\lim_{\delta\to 0} \|u\|_{Z_{[-\delta,\delta]}} = 0 \tag{10}$$

for $u \in Z_{[-T,T]}$ with some $T > 0$ satisfying $u\big|_{t=0} = 0$, where $Z := X \cap C(\mathbb{R}; H^{-3/4}(\mathbb{R}))$. For the proof of (10), we refer to [16, 13].

Let $u, v \in Z_{[-T,T]}$ be as above. Using Lemma 1 and Proposition 1, we see that

$$\|u - v\|_{Z_{[-\delta',\delta']}} \le C\|u + v\|_{Z_{[-\delta',\delta']}} \|u - v\|_{Z_{[-\delta',\delta']}},$$

so it suffices to make $\|u + v\|_{Z_{[-\delta',\delta']}}$ small. We split it between $\|u + v - 2e^{-t\partial_x^3}u_0\|_{Z_{[-\delta',\delta']}}$ and $2\|e^{-t\partial_x^3}u_0\|_{Z_{[-\delta',\delta']}}$. Then, the first one can be arbitrarily small with the aid of (10), since $u + v - 2e^{-t\partial_x^3}u_0\big|_{t=0} = 0$. On the other hand, Lemma 1 bounds the second term by $C\|u_0\|_{H^{-3/4}}$, so we can make it small by the scaling argument, and then obtain the uniqueness in $Z_{[-\delta',\delta']}$ for sufficiently small δ. The desired uniqueness follows after repeating it.

5. Global well-posedness and the I-method

Here, we briefly review the argument in [5], which established the global well-posedness in $H^s(\mathbb{R})$ for $s > -3/10$, to see the essence of the I-method.

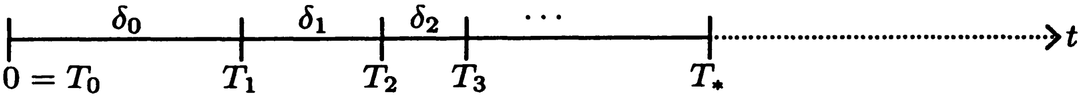
In general, global well-posedness is obtained by pasting the local results. However, the basic local result, which gives the existence time $\delta \sim \|u_0\|^{-\alpha}$ with some $\alpha > 0$ and the estimate $\sup_{-\delta\le t\le\delta} \|u(t)\| \le C\|u_0\|$, is not sufficient by itself, because in each step, the initial datum may grow exponentially and provide the exponentially-decaying existence time. Therefore, we need some *a priori* estimate on the growth of the solution which bounds the data uniformly in each step; see Figure 5. For instance, the L^2 conservation of the real-valued KdV solution together with LWP in L^2 immediately yields GWP of (1) in L^2 in the real-valued setting.

However, when we consider negative Sobolev regularities, there is no conservation law or a priori estimate on the H^s norm of solutions. We now introduce an *almost conserved quantity* which controls the time of local existence in place of the H^s norm.

Let $N \gg 1$ and $s < 0$. We define $I = I_{s,N}$ as the spatial Fourier multiplier with the symbol $m_{s,N}(\xi) = m_s(|\xi|/N)$, where $m_s(r) : \mathbb{R}_+ \to \mathbb{R}_+$ is a smooth monotone function which equals 1 for $r \le 1$ and r^s for $r \ge 2$. We have $C^{-1}\|\phi\|_{H^s} \le \|I\phi\|_{L^2} \le CN^{-s}\|\phi\|_{H^s}$. Furthermore, the following variant of local well-posedness holds.

- Local results $\not\Rightarrow$ global solutions in general.

$$\|u(T_{n+1})\|_{H^s} \le C\|u(T_n)\|_{H^s}, \quad \delta_n \sim \|u(T_n)\|_{H^s}^{-\alpha}$$



- Local results + *a priori estimate* $\Rightarrow$ global solutions.

$$\|u(T_n)\|_{H^s} \le C(u_0), \quad \delta_n \sim \delta = \delta(u_0) > 0$$

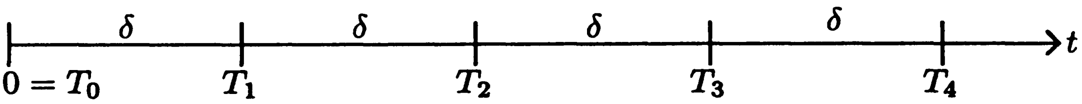


Figure 5. A priori estimate and global solutions.

Lemma 2 ([5]). *Let $s > -3/4$. Then, there exists $b > 1/2$ such that for any $u_0 \in H^s$, a solution $u(t) \in C([-\delta,\delta]; H^s)$ to (1) exists on $[-\delta,\delta]$ with $\delta \ge c\|Iu_0\|_{L^2}^{-\alpha}$ and satisfies $\|Iu\|_{X^{0,b}_{[-\delta,\delta]}} \le C\|Iu_0\|_{L^2}$. Here c, C, and α are some positive constants independent of N.*

Another important feature of the operator I is *almost conservation* of $\|Iu(t)\|_{L^2}$.

Lemma 3 ([5]). *Let $u(t)$ be a real-valued solution to the KdV equation on the time interval $[-\delta,\delta]$. Then, for any $\varepsilon > 0$ and $b > 1/2$ there exists $C > 0$ independent of N such that*

$$\|Iu(t)\|_{L^2}^2 \le \|Iu(0)\|_{L^2}^2 + CN^{-3/4+\varepsilon}\|Iu\|_{X^{0,b}_{[-\delta,\delta]}}^3$$

for $-\delta \le t \le \delta$.

It follows from Lemmas 2 and 3 that if $s > -3/4$ and the real-valued initial datum u_0 satisfies $\|Iu_0\|_{L^2} \le 1$, then we can iterate the local theory $O(N^{3/4-\varepsilon})$ times until the norm $\|Iu(t)\|_{L^2}$ becomes greater than 2. We thus obtain solutions at least up to $t = O(N^{3/4-\varepsilon})$ from such initial data.

For general data, we utilize the scaling argument. If the datum satisfies $\|Iu(0)\|_{L^2} \le M$, then we first rescale it so that

$$\|Iu^\lambda(0)\|_{L^2} \le CM\lambda^{-3/2-s}N^{-s} = 1 \quad \Leftrightarrow \quad \lambda \sim (MN^{-s})^{2/(3+2s)},$$

and solve the equation from the rescaled datum. Rescaling back to the original one, we obtain a solution up to the time $t = O(\lambda^{-3}N^{3/4-\varepsilon})$. Therefore, we can solve the equation on an arbitrarily large time interval, by taking N sufficiently large, if $\lim_{N\to\infty} \lambda^{-3}N^{3/4-\varepsilon} = \infty$. This condition is equivalent to $-6s/(3+2s) < 3/4$, or $s > -3/10$, and the global well-posedness for these s follows.

To show the global results in $H^{-3/4}$, we have to add some correction terms to the almost conserved quantity $\|Iu(t)\|_{L^2}$ and improve the decay with respect to N in Lemma 3. See [6] for details.

6. Remark

Recently, Guo [9] obtained the same well-posedness results independently. The function space in the work of Guo [9] is identical with our space in high frequency. The only difference is in low frequency $\{\,|\xi|\leq 1\,\}$; the space in [9] has the maximal function norm $\|P_0u\|_{L^2_x(L^\infty_t)}$, while our space is defined by

$$\|P_Du\|_{X^{0,1/2}}+\|P_0(1-P_D)u\|_{X^{0,1/2,1}}\quad(\,+\,\|P_0u\|_{L^\infty_t(L^2_x)}\,).$$

These structures share some common properties; for instance, both are weaker than $X^{-3/4,1/2,1}$ for the high frequency part and stronger than $C(H^{-3/4})$. However, there is no inclusion relation between two spaces.

On the other hand, in contrast to the space in [9] defined on the physical space $\mathbb{R}^2_{t,x}$ in low frequency, we define our space X totally on the Fourier space $\mathbb{R}^2_{\tau,\xi}$ similarly to the standard $X^{s,b}$.

This feature of our space allows us to define an auxiliary space for the estimate of nonlinearity simply as $\langle\partial_t+\partial_x^3\rangle X$, and completely separate the estimate for the Duhamel term of the integral equation, like (4), into the linear Duhamel estimate, like (5), and the bilinear estimate, like (6). The same reduction would be nontrivial for function spaces including the norm on the physical space. Moreover, the space in [9] should be considered in the time-restricted form, i.e. with a temporal bump function, because the $L^2_t(L^\infty_x)$ maximal function estimate does not hold globally in time. Such restriction in time is not needed for our space in proving the bilinear estimate.

We should also make a crucial remark that our space X has the monotonicity in frequency, namely, $|\widetilde{u}|\leq|\widetilde{v}|$ implies $\|u\|_X\leq\|v\|_X$, which does not hold in the space defined on the physical space $\mathbb{R}^2_{t,x}$. We actually use this property in the proof of required linear estimates Lemma 1. Such structure is also compatible with the I-method and admits the identical proof for the global well-posedness as the previous one ([6]) working on the standard $X^{s,b}$.

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