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Quasilinear thermoviscoelastic systems. Large time regular solutions

IRENA PAWŁOW
Systems Research Institute, Polish Academy of Sciences
Newelska 6, 01-447 Warsaw, Poland
and Institute of Mathematics and Cryptology, Cybernetics Faculty,
Military University of Technology,
Kaliskiego 2, 00-908 Warsaw, Poland
e-mail: pawlow@ibspan.waw.pl
WOJCIECH M. ZAJĄCZKOWSKI
Institute of Mathematics, Polish Academy of Sciences
Śniadeckich 8, 00-956 Warsaw, Poland
and Institute of Mathematics and Cryptology, Cybernetics Faculty,
Military University of Technology,
Kaliskiego 2, 00-908 Warsaw, Poland
E-mail:wz@impan.gov.pl

Abstract. The paper reviews recent existence results for two thermoviscoelastic systems arising in materials science: the fourth order system of viscosity-capillarity type describing structural phase transitions in solids and the classical second order system for heat conductive Kelvin-Voigt type materials.

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1 Introduction

In the present paper we review existence results for 3-D thermoviscoelastic systems of the fourth and second order. We point on recent author's results based on the theory of linear IBVP's in Sobolev spaces $W^{2,1}_{p,p_0}(\Omega^T)$, $p, p_0 \in (1,\infty)$ with a mixed space-time norm.

Such theory – referred to as maximal regularity – is the subject of recent theoretical studies; for most significant we refer to Denk-Hieber-Prüss [9], Hieber-Prüss [14], Krylov [15] and the references therein.

As remarked in [9] the development of the maximal regularity theory – apart from the theroretical interest itself – was mainly motivated by the applications to free boundary problems and other nonlinear parabolic problems arising in various physical processes. In fact, the studies of nonlinear problems often ask for optimal space-time estimates for the corresponding linearized equations, see e.g. Escher-Prüss-Simonett [11] for the study of the Stefan problem and Solonnikov [31], Maremonti-Solonnikov [16] for the study of Stokes problems.

In the present paper we discuss the application of the theory of Sobolev spaces with a mixed norm to two systems arising in materials science: the fourth order thermoviscoelastic system of viscosity-capillarity type and the classical second order thermoviscoelastic system. The first one describes structural phase transitions in solids, for example in shape memory materials. The second one describes heat conductive Kelvin-Voigt type materials, for example new generation of viscoelastic materials based on silicon gel.

We review briefly the known results for such systems and present recent author's results on the existence of large time regular solutions in Sobolev spaces with a mixed norm. Our idea behind using the framework of Sobolev spaces with a mixed norm is related to the optimal exploitation of the energy estimates in deriving further regularity estimates. In fact, in deriving a priori estimates for a solution of a system of balance laws it is common to begin with estimates arising from the conservation of a total energy. Such estimates provide L_∞-time regularity for the conserved quantities. Thus, to take advantage of such time regularity in deriving further estimates it is desirable to work in Sobolev spaces with mixed norms, $W^{2,1}_{p,p_0}(\Omega^T)$, where space exponent p is determined by the energy structure and time exponent p_0 may be arbitrarily large.

Thermoviscoelastic system of viscosity-capillarity type

$$
\begin{aligned}
&\boldsymbol{u}_{tt} + \varkappa \boldsymbol{Q}^2\boldsymbol{u} - \nu \boldsymbol{Q}\boldsymbol{u}_t = \nabla \cdot F_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon},\theta) + \boldsymbol{b} && \text{in } \Omega^T = \Omega \times (0,T),\\
&c_v\theta\theta_t - k\Delta\theta = \theta F_{,\theta\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon},\theta)\cdot \boldsymbol{\varepsilon}_t + \nu(\boldsymbol{A}\boldsymbol{\varepsilon}_t)\cdot\boldsymbol{\varepsilon}_t + g && \text{in } \Omega^T,\\
&\boldsymbol{u} = \boldsymbol{Q}\boldsymbol{u} = \boldsymbol{0}, \quad \boldsymbol{n}\cdot\nabla\theta = 0 && \text{on } S^T = S\times(0,T),\\
&\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \quad \boldsymbol{u}_t|_{t=0} = \boldsymbol{u}_1, \quad \theta|_{t=0} = \theta_0 && \text{in } \Omega.
\end{aligned}
\tag{1.1}
$$

Here $\Omega \subset \mathbb{R}^3$ is a bounded domain with a smooth boundary S, $\boldsymbol{u} = (u_i)$ denotes the displacement, $\boldsymbol{u}_t = ((u_t)_i)$ – the velocity, $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T)$ – the linearized strain, $\boldsymbol{\varepsilon}_t = \boldsymbol{\varepsilon}(\boldsymbol{u}_t) = \frac{1}{2}(\nabla\boldsymbol{u}_t + (\nabla\boldsymbol{u}_t)^T)$ – the strain rate, $\theta > 0$ – the absolute temperature, and $F(\boldsymbol{\varepsilon},\theta)$ – the elastic energy.

In accord with the well-known shape memory models due to Falk [12] in one-dimensional case and Falk-Konopka[13] in three-dimensional case, the elastic energy is expressed in the form

$$F(\boldsymbol{\varepsilon}, \theta) = \theta F_1(\boldsymbol{\varepsilon}) + F_2(\boldsymbol{\varepsilon}) \tag{1.2}$$

where the first term is in charge of temperature induced changes in the qualitative behaviour of F from the convex at high temperature into a multiwell below some critical value. For example, in one-dimensional Falk model the elastic energy is given by

$$F(\varepsilon, \theta) = \alpha_1(\theta - \theta_c)\varepsilon^2 - \alpha_2\varepsilon^4 + \alpha_3\varepsilon^6$$

where $\alpha_i > 0$, $i = 1, 2, 3$, are constant material parameters and $\theta_c > 0$ is a critical temperature.
The three-dimensional alk-Konopka model, which we take as a prototype example, has the structure (1.2) with $F_1(\boldsymbol{\varepsilon})$ and $F_2(\boldsymbol{\varepsilon})$ being appropriately constructed polynomials in strain components ε_{ij}. Like in 1-D case, $F_2(\boldsymbol{\varepsilon})$ is there of the sixth order but otherwise the temperature dependent part $F_1(\boldsymbol{\varepsilon})$ is of the fourth order.

The opertor $\boldsymbol{Q}$, with domain $D(\boldsymbol{Q}) = \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}_0^1(\Omega)$, stands for the second order linear elasticity map given by

$$\boldsymbol{Q}\boldsymbol{u} = \nabla \cdot (\boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u})) = \mu\Delta\boldsymbol{u} + (\lambda + \mu)\nabla(\nabla \cdot \boldsymbol{u}) \tag{1.3}$$

where

$$\boldsymbol{A} = (A_{ijkl})_{i,j,k,l=1,2,3} \quad \text{with} \quad A_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

is the standard elasticity tensor with the Lamé constants λ, μ satisfying

$$\mu > 0 \quad \text{and} \quad 3\lambda + 2\mu > 0. \tag{1.4}$$

This ensures that the operator $\boldsymbol{Q}$ is strongly elliptic and satisfies the inequality

$$c\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} \leq \|\boldsymbol{Q}\boldsymbol{u}\|_{\boldsymbol{L}_2(\Omega)} \quad \text{for} \quad \boldsymbol{u} \in D(\boldsymbol{Q}) \tag{1.5}$$

with a positive constant c.
Moreover, conditions (1.4) ensure the coercivity and boundedness of tensor $\boldsymbol{A}$:

$$a_*|\boldsymbol{\varepsilon}|^2 \leq (\boldsymbol{A}\boldsymbol{\varepsilon}) \cdot \boldsymbol{\varepsilon} \leq a^*|\boldsymbol{\varepsilon}|^2 \tag{1.6}$$

with $a_* = \min\{3\lambda + 2\mu, 2\mu\}$, $a^* = \max\{3\lambda + 2\mu, 2\mu\}$.

Above and hereafter the summation convention over the repeated indices is used, vectors and tensors are denoted by bold letters, and the dot denotes the inner product of tensors, e.g.

$$\boldsymbol{\varepsilon} \cdot \tilde{\boldsymbol{\varepsilon}} = \varepsilon_{ij}\tilde{\varepsilon}_{ij}.$$

Moreover, the following notation is used

$$F_{,\boldsymbol{\varepsilon}} = \left(\frac{\partial F}{\partial \varepsilon_{ij}}\right)_{i,j=1,2,3}, \quad \boldsymbol{A}\boldsymbol{\varepsilon} = (A_{ijkl}\varepsilon_{kl})_{i,j=1,2,3},$$

$$\nabla \cdot F_{,\boldsymbol{\varepsilon}} = \left(\frac{\partial}{\partial x_j}\frac{\partial F}{\partial \varepsilon_{ij}}\right)_{i=1,2,3}$$

Equations $(1.1)_1$ and $(1.1)_2$ represent balance laws of linear momentum and energy with mass density assumed to be constant, normalized to unity, $\varrho = 1$:

$$
\begin{aligned}
&\boldsymbol{u}_{tt} - \nabla \cdot \boldsymbol{S} = \boldsymbol{b}, \\
&e_t + \nabla \cdot \boldsymbol{q} - \boldsymbol{S} \cdot \boldsymbol{\varepsilon}_t = g,
\end{aligned} \tag{1.7}
$$

with the following gradient-type constitutive relations for the internal energy e, the stress tensor $\boldsymbol{S}$ and the energy flux $\boldsymbol{q}$ (accounting for the interfacial effects):

$$
\begin{aligned}
&e = \frac{1}{2} c_v \theta^2 + F_2(\boldsymbol{\varepsilon}) + \frac{\varkappa}{2} |\boldsymbol{Q}\boldsymbol{u}|^2, \\
&\boldsymbol{S} = F_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, \theta) - \varkappa \boldsymbol{A} \boldsymbol{\varepsilon}(\boldsymbol{Q}\boldsymbol{u}) + \nu \boldsymbol{A} \boldsymbol{\varepsilon}_t, \\
&\boldsymbol{q} = \boldsymbol{q}_0 - \varkappa \boldsymbol{\varepsilon}_t(\boldsymbol{A}\boldsymbol{Q}\boldsymbol{u}), \quad \boldsymbol{q}_0 = -k \nabla \theta.
\end{aligned} \tag{1.8}
$$

The internal energy includes in addition to the usual volumetric terms the strain gradient term $\frac{\varkappa}{2}|\boldsymbol{Q}\boldsymbol{u}|^2$ with the coefficient $\varkappa > 0$ representing interfacial effects.
The corresponding interfacial terms in the formulas for $\boldsymbol{S}$ and $\boldsymbol{q}$ represent the hyperstress, $-\varkappa \boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{Q}\boldsymbol{u})$, and the nonequilibrium phase interface flux, $-\varkappa \boldsymbol{\varepsilon}_t(\boldsymbol{A}\boldsymbol{Q}\boldsymbol{u})$; $\boldsymbol{q}_0$ is the standard heat flux according to the linear Fourier law.

The coefficients ν and k are positive constants denoting the viscosity and heat conductivity, respectively; $\boldsymbol{b}$ and g stand for external body forces and heat sources.

The function $c_v\theta$ with a positive constant c_v corresponds to a temperature dependent specific heat.

For more details on the model (1.1) we refer to [19], [27], [21], [22], [23], [39] where problem (1.1) with constant specific heat $c_v > 0$ was considered and to [40], [24] where the modified energy equation $(1.1)_2$ was assumed.

We remark that the model (1.1) with temperature dependent specific heat is consistent not only with the first two laws of thermodynamics but also with the third law, namely the Nernst-Planck principle. In view of that the model is particularly justified at the range of very low temperatures. We mention also that the modified energy equation $(1.1)_2$ was introduced with the purpose to avoid mathematical obstacles observed in the case of constant specific heat, for a detailed discussion we refer to [39], [24].

In order to apply the regularity results for second order parabolic systems (see Section 2) we shall assume that coefficients $\varkappa$ and ν satisfy so called viscosity-capillarity relation

$$
0 < 2\sqrt{\varkappa} \le \nu. \tag{1.9}
$$

Let us recall (see [27]) that under such condition elasticity system $(1.1)_1$ with boundary and initial conditions $(1.1)_3$, $(1.1)_4$ admits the decomposition into the following two second order parabolic systems

$$
\begin{aligned}
&\boldsymbol{w}_t - \beta \boldsymbol{Q}\boldsymbol{w} = \nabla \cdot (\theta F_{1,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}) + F_2(\boldsymbol{\varepsilon})) && \text{in } \Omega^T, \\
&\boldsymbol{w} = \boldsymbol{0} && \text{on } S^T, \\
&\boldsymbol{w}|_{t=0} = \boldsymbol{w}_0 \equiv \boldsymbol{u}_1 - \alpha \boldsymbol{Q}\boldsymbol{u}_0 && \text{in } \Omega,
\end{aligned} \tag{1.10}
$$

$$
\begin{aligned}
&\boldsymbol{u}_t - \alpha \boldsymbol{Q}\boldsymbol{u} = \boldsymbol{w} && \text{in } \Omega^T, \\
&\boldsymbol{u} = \boldsymbol{0} && \text{on } S^T, \\
&\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0 && \text{in } \Omega,
\end{aligned} \tag{1.11}
$$

where α, β are numbers satisfying

$$\alpha + \beta = \nu, \quad \alpha\beta = \varkappa.$$

Due to condition (1.9) these numbers are real and positive.

Thermoviscoelastic system of Kelvin-Voigt type

$$(1.12) \qquad \begin{aligned} & \boldsymbol{u}_{tt} - \nu \boldsymbol{Q}\boldsymbol{u}_t = \nabla \cdot \boldsymbol{F}_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, \theta) + \boldsymbol{b} && \text{in } \Omega^T, \\ & c_v \theta\theta_t - k\Delta\theta = \theta F_{,\theta\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, \theta) \cdot \boldsymbol{\varepsilon}_t + \nu(\boldsymbol{A}\boldsymbol{\varepsilon}_t) \cdot \boldsymbol{\varepsilon}_t + g && \text{in } \Omega^T, \\ & \boldsymbol{u} = \boldsymbol{0}, \quad \boldsymbol{n} \cdot \nabla\theta = 0 && \text{on } S^T, \\ & \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \quad \boldsymbol{u}_t|_{t=0} = \boldsymbol{u}_1, \quad \theta|_{t=0} = \theta_0 && \text{in } \Omega \end{aligned}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with a smooth boundary S, and the quantities $\boldsymbol{u}, \boldsymbol{\varepsilon}, \theta, F(\boldsymbol{\varepsilon}, \theta), \boldsymbol{b}, g$ have the some meaning as in system (1.1).

The fourth order tensor

$$(1.13) \qquad \boldsymbol{A} \equiv \boldsymbol{A}_1 = ((A_1)_{ijkl}) \quad \text{with} \quad (A_1)_{ijkl} = \lambda_1 \delta_{ij}\delta_{kl} + \mu_1(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

and constants λ_1, μ_1 satisfying condition (1.4), refers to the viscosity tensor. As in (1.3), the second order opeator $\boldsymbol{Q}$ is defined by

$$\boldsymbol{Q}\boldsymbol{u} = \nabla \cdot (\boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u})) = \mu_1 \Delta \boldsymbol{u} + (\lambda_1 + \mu_1)\nabla(\nabla \cdot \boldsymbol{u}).$$

The elastic energy in system (1.12) has the form

$$(1.14) \qquad F(\boldsymbol{\varepsilon}, \theta) = F_1(\boldsymbol{\varepsilon}) + \theta F_2(\boldsymbol{\varepsilon}) \equiv \frac{1}{2}\boldsymbol{\varepsilon} \cdot (\boldsymbol{A}_2\boldsymbol{\varepsilon}) - \theta\boldsymbol{\varepsilon} \cdot (\boldsymbol{A}_2\boldsymbol{\alpha})$$

where $\boldsymbol{A}_2 = ((A_2)_{ijkl})$ is the fourth order elasticity tensor, defined like $\boldsymbol{A}_1$, with Lamé constants λ_2, μ_2 satisfying (1.4). The second order symmetric tensor $\boldsymbol{\alpha} = (\alpha_{ij})$ with constant α_{ij} represents thermal expansion.

After inserting (1.14) system (1.12) takes on the form

$$(1.15) \qquad \begin{aligned} & \boldsymbol{u}_t - \nu \boldsymbol{Q}\boldsymbol{u}_t = \nabla \cdot (\boldsymbol{A}_2(\boldsymbol{\varepsilon} - \theta\boldsymbol{\alpha})) + \boldsymbol{b} && \text{in } \Omega^T, \\ & c_v \theta\theta_t - k\Delta\theta = -\theta(\boldsymbol{A}_2\boldsymbol{\alpha}) \cdot \boldsymbol{\varepsilon}_t + \nu(\boldsymbol{A}_1\boldsymbol{\varepsilon}_t) \cdot \boldsymbol{\varepsilon}_t + g && \text{in } \Omega^T, \\ & \boldsymbol{u} = \boldsymbol{0}, \quad \boldsymbol{n} \cdot \nabla\theta = 0 && \text{on } S^T, \\ & \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \quad \boldsymbol{u}_t|_{t=0} = \boldsymbol{u}_1, \quad \theta|_{t=0} = \theta_0 && \text{in } \Omega. \end{aligned}$$

Equation $(1.15)_1$ represents linear momentum balance $(1.7)_1$ with the stress tensor given by the linear thermoviscoelastic law of Kelvin-Voigt type (see [10], Chap. 5.4)

$$\boldsymbol{S} = \boldsymbol{F}_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, \theta) + \nu\boldsymbol{A}\boldsymbol{\varepsilon}_t \equiv \boldsymbol{A}_2(\boldsymbol{\varepsilon} - \theta\boldsymbol{\alpha}) + \nu\boldsymbol{A}\boldsymbol{\varepsilon}_t.$$

Equation $(1.15)_2$ represents energy balance $(1.7)_2$ with the internal energy

$$e = \frac{1}{2}c_v\theta^2 + \frac{1}{2}\boldsymbol{\varepsilon} \cdot (\boldsymbol{A}_2\boldsymbol{\varepsilon}),$$

the linear Fourier law for the heat flux

$$q = q_0 = -k\nabla\theta$$

where $k > 0$ is a constant heat conductivity, and with temperature-dependent specific heat $c_v\theta$.

Both thermoviscoelastic systems (1.1) and (1.15) comply with the second law of thermodynamics in the form of the classical Clausius-Duhem inequality

$$\eta_t + \nabla \cdot \frac{q_0}{\theta} \geq 0$$

where η is the entropy given by

$$\begin{aligned} \eta &= c_v\theta - F_1(\varepsilon) && \text{in system (1.1)}, \\ \eta &= c_v\theta + (A_2\alpha)\cdot\varepsilon && \text{in system (1.15)}, \end{aligned}$$

and $q_0 = -k\nabla\theta$ is the Fourier heat flux.

The paper is organized as follows. In Section 2 we present some basic results on Besov spaces and Sobolev spaces with a mixed norm. Besides, we record two lemmas on solvability of linear parabolic IBVP's in Sobolev spaces with a mixed norm which play a key role in the proofs of the presented existence results for systems (1.1) and (1.15). In Section 3 and 4 we recall from [24], [25] recent author's results on the solvability of systems (1.1) and (1.15) in Sobolev spaces with a mixed norm. We review briefly other known results as well.

2 Parabolic problems in Sobolev spaces with a mixed norm

By Sobolev space with a mixed norm $W^{k,k/2}_{p,p_0}(\Omega^T$, $\Omega \subset \mathbb{R}^n$, $k, k/2 \in N \cup \{0\} \equiv N_0$, $p, p_0 \in [1,\infty]$, we denote a completion of $C^\infty(\Omega^T)$-functions under the finite norm

$$\|u\|_{W^{k,k/2}_{p,p_0}(\Omega^T)} = \left(\int_0^T \left(\sum_{|\alpha|+2a\leq k} \int_\Omega |D^\alpha_x \partial^a_t u|^p dx \right)^{p_0/p} dt \right)^{1/p_0}, \tag{2.1}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multiindex, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha_i \geq 0$, $i = 1, \dots, n$, $D^\alpha_x = \partial^{\alpha_1}_{x_1} \dots \partial^{\alpha_n}_{x_n}$.

If $p = p_0$, then the notation

$$W^{k,k/2}_p(\Omega^T) = W^{k,k/2}_{p,p_0}(\Omega^T)$$

is used.

By Besov space $B^\lambda_{p,p_0}(\Omega)$, $\Omega \subset \mathbb{R}^n$, $\lambda \in \mathbb{R}_+$, $p, p_0 \in [1,\infty]$, we denote a set of functions with the finite norm

$$\|u\|_{B^\lambda_{p,p_0}(\Omega)} = \|u\|_{L_p(\Omega)} + \left(\sum_{i=1}^n \int_{\mathbb{R}_+} \frac{\|\Delta^m_i(h,\Omega)\partial^l_{x_i} u\|^{p_0}_{L_p(\Omega)}}{h^{1+(\lambda-l)p_0}} dh \right)^{1/p_0}, \tag{2.2}$$

where $m > \lambda - l > 0$, $m, l \in N_0$ and $\Delta_i^k(h,\Omega)u(x)$ is a finite difference of function $u(x)$ of the order k with respect to x_i:

$$\Delta^1(h,\Omega)u \equiv \Delta_i(h,\Omega)u = u(x_1,\dots,x_{i-1},x_i+h,x_{i+1},\dots,x_n) - u(x_1,\dots,x_n),$$

and $\Delta_i(h,\Omega)u = 0$ for $x+h \notin \Omega$,

$$\Delta_i^k(h,\Omega)u = \Delta_i(h,\Omega)\Delta_i^{k-1}(h,\Omega)u, \quad k \in N.$$

We recall from [6] the trace and the inverse trace theorems for Sobolev spaces with a mixed norm.

Lemma. 1 *Let* $u \in W_{p,p_0}^{k,k/2}(\Omega \times \mathbb{R}_+)$. *Then* $u(x,t_0) = u(x,t)|_{t=0} \in B_{p,p_0}^{k-2/p_0}(\Omega)$ *and*

$$\|u(\cdot,t_0)\|_{B_{p,p_0}^{k-2/p_0}(\Omega)} \le c\|u\|_{W_{p,p_0}^{k,k/2}(\Omega\times\mathbb{R}_+)}. \tag{2.3}$$

Moreover, for a given $v \in B_{p,p_0}^{k-2/p_0}(\Omega)$, *there exists a function* $\tilde{v} \in W_{p,p_0}^{k,k/2}(\Omega \times \mathbb{R}_+)$ *such that* $\tilde{v}(x,t)|_{t=0} = v(x)$ *and*

$$\|\tilde{v}\|_{W_{p,p_0}^{k,k/2}(\Omega\times\mathbb{R}_+)} \le c\|v\|_{B_{p,p_0}^{k-2/p_0}(\Omega)}. \tag{2.4}$$

If $l > 1/p$ then every function from $B_{p,p_0}^l(\Omega)$ has a trace on the boundary S belonging to $B_{p,p_0}^{l-1/p}(S)$ and

$$\|u\|_{B_{p,p_0}^{l-1/p}(S)} \le c\|u\|_{B_{p,p_0}^l(\Omega)}.$$

We use theorems of imbeddings between Besov spaces and Besov and Sobolev spaces from [2], [18].

Let us consider the parabolic non-diagonal problem

$$\begin{aligned} &\boldsymbol{u}_t - \boldsymbol{Q}\boldsymbol{u} = \boldsymbol{f} && \text{in } \Omega^T = \Omega \times (0,T), \\ &\boldsymbol{u} = \boldsymbol{0} && \text{on } S^T = S \times (0,T), \\ &\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0 && \text{in } \Omega, \end{aligned} \tag{2.5}$$

where $\Omega \subset \mathbb{R}^3$, $S = \partial\Omega$ and

$$\boldsymbol{Q}\boldsymbol{u} = \mu\Delta\boldsymbol{u} + \nu\nabla(\nabla\cdot\boldsymbol{u}) \tag{2.6}$$

with positive constants μ, ν.

We recall now a lemma which plays a key role in the existence proofs for systems (1.1) and (1.15). This lemma generalizes the result by Krylov [15] from the single parabolic equation to the parabolic system (2.5).

Lemma. 2 *(see [9], [15], [24])*
(i) Assume that $\boldsymbol{f} \in \boldsymbol{L}_{p,p_0}(\Omega^T)$, $\boldsymbol{u}_0 \in \boldsymbol{B}_{p,p_0}^{2-2/p_0}(\Omega)$, $p, p_0 \in (1,\infty)$, $S \in C^2$. *If* $2-2/p_0-1/p > 0$ *the compatibility condition* $\boldsymbol{u}_0|_S = \boldsymbol{0}$ *is assumed. Then there exists a unique solution to problem (2.5) such that* $\boldsymbol{u} \in \boldsymbol{W}_{p,p_0}^{2,1}(\Omega^T)$ *and*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_{p,p_0}^{2,1}(\Omega^T)} \le c(\|\boldsymbol{f}\|_{\boldsymbol{L}_{p,p_0}(\Omega^T)} + \|\boldsymbol{u}_0\|_{\boldsymbol{B}_{p,p_0}^{2-2/p_0}(\Omega)}) \tag{2.7}$$

with the constant c from the Calderon-Zygmund estimate, dependent on Ω, T, S, p, p_0.
(ii) Assume that $f = \nabla\cdot g + b$, $g, b \in L_{p,p_0}(\Omega^T)$, $u_0 \in B^{2-2/p_0}_{p,p_0}(\Omega)$. *Assume the compatibility condition*

$$u_0|_S = 0 \quad if \quad 1 - 2/p_0 - 1/p > 0.$$

Then there exists a unique solution to (2.5) such that $u \in W^{1,1/2}_{p,p_0}(\Omega^T)$ *and*

$$\|u\|_{W^{1,1/2}_{p,p_0}(\Omega^T)} \le c(\|g\|_{L_{p,p_0}(\Omega^T)} + \|b\|_{L_{p,p_0}(\Omega^T)} + \|u_0\|_{B^{2-2/p_0}_{p,p_0}(\Omega)}) \tag{2.8}$$

with the constant c from the Calderon-Zygmund estimate.

In the existence proofs we need also the following regularity result which is a special case of a more general result in [9], Theorem 2.3.

Lemma. 3 *(see [9]) Let us consider the problem*

$$\begin{aligned} &\theta_t - \varrho\Delta\theta = g && in\ \Omega^T, \\ &n\cdot\nabla\theta = 0 && on\ S^T, \\ &\theta|_{t=0} = \theta_0 && in\ \Omega, \end{aligned} \tag{2.9}$$

where $\varrho(x,t)$ *is a continuous function on* Ω^T *such that* $\inf_\Omega \varrho > 0$. *Assume that* $g \in L_{p,p_0}(\Omega^T)$, $\theta_0 \in B^{2-2/p_0}_{p,p_0}(\Omega)$, $p, p_0 \in (1,\infty)$, $S \in C^2$ *and the corresponding compatibility conditions are satisfied. Then there exists a unique solution to problem (2.9) such that* $\theta \in W^{2,1}_{p,p_0}(\Omega^T)$ *and*

$$\|\theta\|_{W^{2,1}_{p,p_0}(\Omega^T)} \le c(\|g\|_{L_{p,p_0}(\Omega^T)} + \|\theta_0\|_{B^{2-2/p_0}_{p,p_0}(\Omega)}) \tag{2.10}$$

with constant c dependent on Ω, T, S, $\inf_{\Omega^T}\varrho$ *and* $\|\varrho\|_{C(\Omega^T)}$.

3 Existence results for thermoviscoelastic system of viscosity-capillarity type

Thermoviscoelastic system (1.1) represents a 3-D extension of the well-known one-dimensional Falk model [12] for martensitic phase transitions in solids. There exists an extensive literature on mathematical analysis of various variants of 1-D Falk model, see e.g. [5], [1], [35], [32], [33], [34] and the references therein.

Thermoviscoelastic system (1.1) has been firstly introduced and studied in [26], [27]. Its thermodynamic derivation has been given in [19]. In the setting addmitting various forms of the specific heat system $(1.1)_{1,2}$ reads as follows

$$\begin{aligned} &u_{tt} + \varkappa Q^2 u - \nu Q u_t = \nabla\cdot F_{,\varepsilon}(\varepsilon,\theta) + b && \text{in } \Omega^T, \\ &c_0(\varepsilon,\theta)\theta_t - k\Delta\theta = \theta F_{,\theta\varepsilon}(\varepsilon,\theta)\cdot\varepsilon_t + \nu(A\varepsilon_t)\cdot\varepsilon_t + g && \text{in } \Omega^T, \end{aligned} \tag{3.1}$$

where the function

$$c_0(\varepsilon,\theta) = c_*(\theta) - \theta F_{,\theta\theta}(\varepsilon,\theta) \tag{3.2}$$

represents the specific heat. It consists of two parts: the caloric specific heat

$$c_*(\theta) = -\theta F_{*,\theta\theta}(\theta) \tag{3.3}$$

where F_* is the caloric energy, and the specific heat due to elastic effects, $-\theta F_{,\theta\theta}(\varepsilon,\theta)$. Two typical examples of F_* are

$$F_*(\theta) = -c_v\theta\log\frac{\theta}{\theta_1} + c_v\theta + \tilde{c} \tag{3.4}$$

and

$$F_*(\theta) = -\frac{1}{2}c_v\theta^2, \tag{3.5}$$

where $c_v, \theta_1, \tilde{c}$ are positive physical constants.

The form (3.4) is associated with constant caloric heat

$$c_*(\theta) = c_v \tag{3.6}$$

whereas (3.5), with temperature-dependent one

$$c_*(\theta) = c_v\theta. \tag{3.7}$$

In case of constant caloric heat (3.6), system (3.1) with boundary and initial conditions $(1.1)_{3,4}$ has been studied in a line of papers [26], [27], [20]–[23], [36]–[38], [39].

In all above mentioned references the following form of the elastic energy has been postulated

$$F(\varepsilon,\theta) = G(\theta)F_1(\varepsilon) + F_2(\varepsilon) \tag{3.8}$$

with the function $G(\theta)$ assumed to satisfy the growth condition with an exponent strictly less than 1, i.e.,

$$G(\theta) \sim \theta^r \quad \text{with} \quad r < 1 \quad \text{for large } \theta.$$

Moreover, the growth conditions on $F_1(\varepsilon)$, $F_2(\varepsilon)$, like

$$\begin{aligned} &|F_{1,\varepsilon}(\varepsilon)| \le C|\varepsilon|^{K_1-1}, \quad |F_{1,\varepsilon\varepsilon}(\varepsilon)| \le C|\varepsilon|^{K_1-2}, \quad |F_{1,\varepsilon\varepsilon\varepsilon}(\varepsilon)| \le C|\varepsilon|^{K_1-3},\\ &|F_{2,\varepsilon}(\varepsilon) \le C|\varepsilon|^{K_2-1}, \quad |F_{2,\varepsilon\varepsilon}(\varepsilon)| \le C|\varepsilon|^{K_2-2}, \quad |F_{2,\varepsilon\varepsilon\varepsilon}(\varepsilon)| \le C|\varepsilon|^{K_2-3} \end{aligned}$$

for large $|\varepsilon|$ with certain numbers of order K_1, K_2 have been assumed. Under such structure assumptions it has been proved that system (1.1) has a unique large time solution $(u,\theta) \in W_p^{4,2}(\Omega^T) \times W_q^{2,1}(\Omega^T)$ for any $5 < p \le q < \infty$ and $T > 0$.

The best result has been obtained in [39] where the existence of solutions was proved under the following powers of nonlinearity

$$0 \le r < \frac{5}{6}, \quad 0 \le K_1, K_2 < 6, \quad 6r + K_1 < 6$$

in 3-D case, and

$$0 \le r < 1, \quad 0 \le K_1, K_2 < \infty$$

in 2-D case.

The existence proof was based on the Leray-Schauder fixed point theorem and the parabolic theory in the Sobolev spaces $W_p^{4,2}(\Omega^T)$ and $W_q^{2,1}(\Omega^T)$.
The key issue to solve the problem was concerned with deriving $L_\infty(\Omega^T)$-norm and the Hölder-norm estimates for temperature. This was accomplished by applying the parabolic DeGiorgi method.

It has to be noted that the existence of solutions in case $G(\theta) = \theta$, i.e. for the exponent $r = 1$ as in Falk-Konopka model (1.2), still remains an open problem

In order to admit the function $G(\theta)$ linear in θ the thermoviscoelastic system (1.1) with a modified energy equation has been studied in [40] and recently in [24].

In [24] the framework of parabolic theory in Sobolev spaces with a mixed norm has been adopted. Such approach turned out to be advantageous for deriving regularity estimates and allowed to generalize the results in [40] by admitting stronger thermomechanical nonlinearity and a broader class of solution spaces.

We recall here the existence result for system (1.1), proved in [24].
The following structure assumptions are postulated:

(N1) $F(\boldsymbol{\varepsilon},\theta) = \theta F_1(\boldsymbol{\varepsilon}) + F_2(\boldsymbol{\varepsilon})$
with $F_1, F_2 \in C^3$ (Sym $(3,\mathbb{R}),\mathbb{R})$, where Sym $(3,\mathbb{R})$ denotes the set of all symmetric second order tensors in $\mathbb{R}^3$; F_2 satisfies the bounds

$$-c_3 \le F_2(\boldsymbol{\varepsilon}) \le c_2(|\boldsymbol{\varepsilon}|^{K_2} + 1)$$

with a number $0 \le K_2 < \infty$, c_2 is a positive constant and c_3 a real constant;

(N2) $F_1(\boldsymbol{\varepsilon})$ and $F_2(\boldsymbol{\varepsilon})$ satisfy the growth conditions

$$\begin{aligned}
&|F_{1,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon})| \le c|\boldsymbol{\varepsilon}|^{K_1-1}, \quad |F_{1,\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon})| \le c|\boldsymbol{\varepsilon}|^{K_1-2}, \quad |F_{1,\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon})| \le c|\boldsymbol{\varepsilon}|^{K_1-3},\\
&|F_{2,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon})| \le c|\boldsymbol{\varepsilon}|^{K_2-1}, \quad |F_{2,\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon})| \le c|\boldsymbol{\varepsilon}|^{K_2-2}, \quad |F_{2,\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon})| \le c|\boldsymbol{\varepsilon}|^{K_2-3}
\end{aligned}$$

for large $|\boldsymbol{\varepsilon}|$, where $0 < K_1 < \infty$.
Moreover, it is assumed that $\varkappa$ and ν satisfy (1.9).

The existence result has been proved for the following values of exponents K_1 and K_2:

$$0 \le K_1 < 3, \quad 0 \le K_2 < 6.$$

Theorem. 1 *(see [24], Thm 1.1) Assume (N1), (N2) hold with* $0 \le K_1 < 3$, $0 \le K_2 < 6$. *Let* $T > 0$ *and the numbers* $p, p_0, q, q_0 \in (1,\infty)$ *satisfy the conditions*

$$\frac{3}{p} + \frac{2}{p_0} < 1, \quad \frac{3}{q} + \frac{2}{q_0} < 1, \quad p \le q, \quad p_0 \le q_0. \tag{3.9}$$

Then for any

$$(\boldsymbol{u}_0, \boldsymbol{u}_1, \theta_0) \in B^{4-2/p_0}_{p,p_0}(\Omega) \times B^{2-2/p_0}_{p,p_0}(\Omega) \times B^{2-2/q_0}_{q,q_0}(\Omega)$$

with $0 < \underline{\theta} \le \theta_0 \le \bar{\theta}$, *where* $\underline{\theta}, \bar{\theta}$ *are positive constants, there exists a unique solution* $(\boldsymbol{u},\theta)$ *to system (1.1) satisfying*

$$\begin{aligned}
&(\boldsymbol{u},\theta) \in W^{4,2}_{p,p_0}(\Omega^T) \times W^{2,1}_{q,q_0}(\Omega^T),\\
&0 < \theta_* \le \theta \le \theta^* < \infty \quad \text{a.e. in } \Omega^T,
\end{aligned}$$

where θ_*, θ^* depend on $T, \underline{\theta}, \bar{\theta}$ and on

$$\|(u_0, u_1, \theta_0)\|_{B^{4-2/p_0}_{p,p_0}(\Omega)\times B^{2-2/p_0}_{p,p_0}(\Omega)\times B^{2-2/q_0}_{q,q_0}(\Omega)}.$$

For a direct comparison with a corresponding result in Sobolev spaces with a homogeneous norm with respect to space and time variables let us recall theorem proved in [40].

Theorem. 2 *(see [40], Thm 1.1) Assume (N1), (N2) hold with $0 \le K_1 < \frac{12}{7}$, $0 \le K_2 < 6$ and with any positive constants $\varkappa, \nu$. Let $T > 0$ and the numbers p, q satisfy*

$$5 < p \le q < \infty. \tag{3.10}$$

Then for any $(u_0, u_1, \theta_0) \in B^{4-2/p}_{p,p}(\Omega) \times B^{2-2/p}_{p,p}(\Omega) \times B^{2-2/q}_{q,q}(\Omega)$ such that $0 < \underline{\theta} \le \theta_0 \le \bar{\theta}$, where $\underline{\theta}$, $\bar{\theta}$ are positive constants, there exists a unique solution (u, θ) to system (1.1) satisfying

$$\begin{aligned} &(u, \theta) \in W^{4,2}_p(\Omega^T) \times W^{2,1}_q(\Omega^T),\\ &0 < \theta_* \le \theta \le \theta^* < \infty \quad \text{a.e. in } \Omega^T, \end{aligned}$$

where θ_* and θ^* depend on $T, \underline{\theta}, \bar{\theta}$ and on

$$\|(u_0, u_1, \theta_0)\|_{B^{4-2/p}_{p,p}(\Omega)\times B^{2-2/p}_{p,p}(\Omega)\times B^{2-2/q}_{q,q}(\Omega)}.$$

Theorem 1 generalizes Theorem 2 in two aspects. Firstly, it allows for a higher power of nonlinearity $F_1(\varepsilon)$ which is in charge of the thermomechanical coupling, namely $K_1 < 3$ in place of $K_1 < 12/7$ required in Theorem 2. Secondly, it admits a broader class of solution space $W^{4,2}_{p,p_0}(\Omega^T) \times W^{2,1}_{q,q_0}(\Omega^T)$ with p, p_0, q, q_0 satisfying (3.9), in place of $W^{4,2}_{p,p}(\Omega^T) \times W^{2,1}_{q,q}(\Omega^T)$ with p, q satisfying (3.10).

Unfortunately, either Theorem 1 or Theorem 2 do not admit the nonlinearities of the prototype Falk-Konopka model (1.2) in which $K_1 = 4$ and $K_2 = 6$.

It should be underlined that the result in Theorem 2 is more general from the point of view of the assumption on the coefficientls $\varkappa, \nu$ where they are required to be any positive constants, not restricted by the condition (1.9). This is due to the maximal regularity result for the linear fourth order parabolic system

$$\begin{aligned} &u_{tt} + \varkappa Q^2 u - \nu Q u_t = f && \text{in } \Omega^T,\\ &u = Qu = 0 && \text{on } S^T,\\ &u|_{t=0} = u_0, \quad u_t|_{t=0} = u_1 && \text{in } \Omega \end{aligned} \tag{3.11}$$

in Sobolev space $W^{4,2}_p(\Omega^T)$, which has been proved by Yoshikawa [36] without using decomposition (1.10), (1.11).

The proof of Theorem 1 is based on the Leray-Schauder fixed point theorem and the decomposition of the viscoelasticity system $(1.1)_1$ into two second order parabolic systems (1.10) and (1.11). Such decomposition requires assumption (1.9).

The central part of the proof consitute global a priori estimates which are derived by a recursive improvement of energy estimates. The key role in this derivation plays Lemma 2

on the solvability of parabolic non-diagonal system (2.5) in Sobolev space $W^{2,1}_{p,p_0}(\Omega^T)$ with a mixed norm.
This lemma combined with the parabolic DeGiorgi method allows to derive $L_\infty(\Omega^T)$-norm and Hölder – norm estimates for temperature. Then the regularity result stated in Lemma 3 is aplied to get the estimate for θ in $W^{2,1}_{q,q_0}(\Omega^T)$-norm.

4 Existence results for thermoviscoelastic system of Kelvin-Voigt type

As noted in the recent paper by Roubíček [28] – and according to our best knowledge as well – the existence of large-time solutions to a thermoviscoelastic system of Kelvin-Voigt type with constant specific heat and heat conductivity is, in spite of great effort through many decades, still open in dimensions $n \geq 2$. In dimension $n = 1$ it was established in the pioneering papers by Slemrod [30], Dafermos [7] and Dafermos-Hsiao [8].

The local in time existence and globaluniqueness of a weak solution to 3-D thermoviscoelastic system with constant specific heat and heat conductivity has been proved by Bonetti-Bonfanti [4]. Other known results on multidimensional thermoviscoelasticity deal with a modified energy equation. Modifications involve either nonconstant specific heat or nonconstant heat conductivity. Thermoviscoelastic system with temperature-dependent specific heat has been addressed by Blanchard-Guibé [3] where the existence of large-time, weak-renormalized solutions has been proved, and recently in [28] where the existence of a very weak solution has been established.

We mention also that the framework of renormalized solutions has been applied in [41] for 3-D thermoviscoelastic system arising in structural phase transitions.

In a more general setting allowing for large strains 3-D thermoviscoealstic sytem with temperature-dependent specific heat has been studied by Shibata [29] under small data assumption.

For thermoviscoelastic problems with modified heat conductivity we refer to Eck-Jarušek-Krbec [10] and the references therein.

We mention also that energy equation with temperature-dependent specific heat has been considered by Miranville-Schimperna [17] in the context of phase transition model based on a microforce balance.

Motivated by the above mentioned results on problems with a modified energy equation and a recent result on thermoviscoelastic system (1.1) (see [24]) we have considered thermoviscoelastic system of Kelvin-Voigt type (1.12) in the framework of Sobolev spaces with a mixed norm.

The following existence and uniqueness result has been proved.

Theorem. 3 *(see [25]) Let $T > 0$ and the numbers $p, p_0, q, q_0 \in (1, \infty)$ satusfy the conditions*

$$\frac{3}{p} + \frac{2}{p_0} \leq 1, \quad \frac{3}{q} + \frac{2}{q_0} < 1 \quad p \leq q, \quad p_0 \leq q_0. \tag{4.1}$$

Moreover, let

$$
\begin{aligned}
(4.2)\qquad & (\boldsymbol{u}_0, \boldsymbol{u}_1, \theta_0) \in (\boldsymbol{W}_p^2(\Omega) \cap \boldsymbol{H}_0^1(\Omega)) \times (\boldsymbol{B}_{p,p_0}^{2-2/p_0}(\Omega) \cap \boldsymbol{H}_0^1(\Omega)) \\
& \quad \times (B_{q,q_0}^{2-2/q_0}(\Omega) \cap H^1(\Omega)) =: \mathcal{U}, \\
& (\boldsymbol{b}, g) \in \boldsymbol{L}_{p,p_0}(\Omega^T) \times (L_1(0,T; L_\infty(\Omega)) \cap L_{q,q_0}(\Omega^T)) =: \mathcal{V}
\end{aligned}
$$

and

$$
\begin{aligned}
& 0 < \underline{\theta} \le \theta_0 \le \bar{\theta} \quad \text{a.e. in } \Omega, \\
& g \ge 0 \qquad\qquad\; \text{a.e. in } \Omega^T,
\end{aligned}
$$

where $\underline{\theta}$, $\bar{\theta}$ *are positive constants.*
Then there exists a unique solution $(\boldsymbol{u}, \theta)$ *to system (1.15) such that*

$$
(4.3)\qquad (\boldsymbol{u}_t, \theta) \in \boldsymbol{W}_{p,p_0}^{2,1}(\Omega^T) \times W_{q,q_0}^{2,1}(\Omega^T),
$$

satisfying

$$
\begin{aligned}
(4.4)\qquad & \|\boldsymbol{u}_t\|_{\boldsymbol{W}_{p,p_0}^{2,1}(\Omega^T)} + \|\theta\|_{W_{q,q_0}^{2,1}(\Omega^T)} \le c, \\
& 0 < \theta_* \le \theta \le \theta^* \quad \text{a.e. in } \Omega^T,
\end{aligned}
$$

with constants c, θ_*, θ^* *depending on*

$$
\begin{aligned}
& \|(\boldsymbol{u}_0, \boldsymbol{u}_1, \theta_0)\|_{\mathcal{U}} = \|\boldsymbol{u}_0\|_{\boldsymbol{W}_p^2(\Omega)} + \|\boldsymbol{u}_1\|_{\boldsymbol{B}_{p,p_0}^{2-2/p_0}(\Omega)} + \|\theta_0\|_{B_{q,q_0}^{2-2/q_0}(\Omega)}, \\
& \|(\boldsymbol{b}, g)\|_{\mathcal{V}} = \|\boldsymbol{b}\|_{\boldsymbol{L}_{p,p_0}(\Omega)} + \|g\|_{L_{q,q_0}(\Omega)},
\end{aligned}
$$

and exponentially on T.

The novelty of the above result concerns the regularity of 3-D large-time solution corresponding to appropriately smooth but arbitrary in size initial data.

The proof is based on the successive approximation method. More precisely, we construct successive approximations $(\boldsymbol{u}_t^n, \theta^n)$, $n \in \boldsymbol{N} \cup \{0\}$, by solving the following linear thermoviscoelastic system:

$$
\begin{aligned}
& \boldsymbol{u}_{tt}^{n+1} - \nabla \cdot (\boldsymbol{A}_1 \boldsymbol{\varepsilon}(\boldsymbol{u}_t^{n+1})) = \nabla \cdot [\boldsymbol{A}_2 \boldsymbol{\varepsilon}(\boldsymbol{u}^n) - (\boldsymbol{A}_2 \boldsymbol{\alpha})\theta^n] + \boldsymbol{b} && \text{in } \Omega^T, \\
& c_v \theta_0 \theta_t^{n+1} - k \Delta \theta^{n+1} = c_v(\theta_0 - \theta^n)\theta_t^{n+1} \\
& \quad - \theta^n (\boldsymbol{A}_2 \boldsymbol{\alpha}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}_t^n) + \nu(\boldsymbol{A}_1 \boldsymbol{\varepsilon}(\boldsymbol{u}_t^n)) \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}_t^n) + g && \text{in } \Omega^T, \\
& \boldsymbol{u}^{n+1} = \boldsymbol{0}, \quad \boldsymbol{n} \cdot \nabla \theta^{n+1} = 0 && \text{on } S^T, \\
& \boldsymbol{u}^{n+1}|_{t=0} = \boldsymbol{u}_0, \quad \boldsymbol{u}_t^{n+1}|_{t=0} = \boldsymbol{u}_1, \quad \theta^{n+1}|_{t=0} = \theta_0 && \text{in } \Omega
\end{aligned}
$$

where $\boldsymbol{u}^n$, θ^n are treated as given.
Moreover, the approximation $(\boldsymbol{u}^0, \theta^0)$ is constructed by an extension of the initial data in such a way that

$$
\boldsymbol{u}^0|_{t=0} = \boldsymbol{u}_0, \quad \boldsymbol{u}_t^0|_{t=0} = \boldsymbol{u}_1, \quad \theta^0|_{t=0} = \theta_0 \quad \text{in } \Omega
$$

and

$$
\boldsymbol{u}^0 = \boldsymbol{0}, \quad \boldsymbol{n} \cdot \nabla \theta^0 = 0 \quad \text{on } S^T.
$$

We prove that the sequence (u^n, θ^n) is uniformly bounded and convergent on a sufficiently small time interval. This way we prove the existence of a local solution to system (1.15).

To prove the existence of a solution on the interval $[0,T]$ we repeat the procedure step by step in time. To accomplish this we need global a priori estimates. Like in the proof of Theorem 1 (by the Leray-Schauder fixed point theorem) global a priori estimates constitute the central part of the proof. They are derived by a recursive improvement of energy estimates. Again the basic tool for getting such regularity estimates are the results on parabolic problems in Sobolev space with a mixed norm, presented in Section 2.

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