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VARIATION FORMULAS FOR PRINCIPAL FUNCTIONS AND HARMONIC SPANS

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ABSTRACT. The purpose of this article is to give a summary of the seminar lecture with title *Variation formulas for principal functions and harmonic spans* by the author in the conference held at RIMS, Kyoto, Japan, December 2009. The former half of this article is in Hamano [4] and the latter half is in the manuscript [6] which is a joint work with Maitani and Yamaguchi.

1. INTRODUCTION

Let $B=\{|t|<\rho\}$ and let $\widetilde{\mathcal{R}}$ be an unramified (two-dimensional) Riemann domain sheeted over $B\times\mathbb{C}_z$. We write $\widetilde{\mathcal{R}}=\bigcup_{t\in B}(t,\widetilde{R}(t))$, where $\widetilde{R}(t)$ is a fiber over each $t\in B$, i.e., $\widetilde{R}(t)=\{z:(t,z)\in\widetilde{\mathcal{R}}\}$, so that each $\widetilde{R}(t)$ consists of unramified Riemann surfaces sheeted over $\mathbb{C}_z$. Consider a subdomain $\mathcal{R}$ in $\widetilde{\mathcal{R}}$ such that, if we put $\mathcal{R}=\bigcup_{t\in B}(t,R(t)))$, where $R(t)$ is a fiber of $\mathcal{R}$ over $t\in B$, then

(1) $\widetilde{R}(t)\Supset R(t)\neq\emptyset$, $t\in B$;
(2) the boundary $\partial\mathcal{R}=\cup_{t\in B}(t,\partial R(t))$ of $\mathcal{R}$ in $\widetilde{\mathcal{R}}$ is C^ω smooth in $\widetilde{\mathcal{R}}$;
(3) each $R(t), t\in B$ is a connected Riemann surface of genus $g\geq 0$, where g is independent of $t\in B$;
(4) each $\partial R(t)$, $t\in B$ in $\widetilde{R}(t)$ consists of a finite number of C^ω smooth contours $C_j(t), j=0,1,\dots,\nu$, where ν is independent of $t\in B$. We give the orientation of $C_j(t)$ such that $\partial R(t)=C_0(t)+C_1(t)+\cdots+C_\nu(t)$.

We usually regard two-dimensional Riemann domain $\mathcal{R}$ over $B\times\mathbb{C}_z$ as a C^ω smooth variation of Riemann surface $R(t)$ over $\mathbb{C}_z$ with C^ω smooth boundary $\partial R(t)$ with complex parameter $t\in B$:

$$\mathcal{R}:t\in B\rightarrow R(t).$$

Variation formula of the Green function for $(R(t),0)$ We assume that $\mathcal{R}$ contains $B\times\{0\}$, precisely, there exists at least one constant section $\mathbf{O}$ of $\mathcal{R}$ over $B\times\{0\}$. We consider the Green function

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$g(t,z)$ with pole at $z=0$ and the Robin constant $\lambda(t)$ for $(R(t),0)$, so that

$$g(t,z)=\log\frac{1}{|z|}+\lambda(t)+h(t,z).$$

Here $h(t,z)$ is harmonic for z in a neighborhood of $z=0$ in $R(t)$ such that

$$h(t,0)=0 \qquad \text{for } t\in B.$$

Let $\varphi(t,z)$ be a defining function of $\partial\mathcal{R}$ in $B\times\mathbb{C}_z$. For $(t,z)\in\partial\mathcal{R}$, we consider the following quantities:

$$k_1(t,z)=\frac{\partial\varphi}{\partial t}\Big/\left|\frac{\partial\varphi}{\partial z}\right|,$$

$$k_2(t,z)=\left(\frac{\partial^2\varphi}{\partial t\partial\bar{t}}\left|\frac{\partial\varphi}{\partial z}\right|^2-2\mathrm{Re}\left\{\frac{\partial^2\varphi}{\partial\bar{t}\partial z}\frac{\partial\varphi}{\partial t}\frac{\partial\varphi}{\partial\bar{z}}\right\}+\left|\frac{\partial\varphi}{\partial t}\right|^2\frac{\partial^2\varphi}{\partial z\partial\bar{z}}\right)\Big/\left|\frac{\partial\varphi}{\partial z}\right|^3.$$

We note that they do not depend on the choice of defining functions $\varphi(t,z)$ of $\partial\mathcal{R}$. We denote by ds_z the arc length element of $\partial R(t)$ at z. The function $k_2(t,z)$ on $\partial\mathcal{R}$ is due to Maitani-Yamaguchi in [8] which is based on [7], which is called the *Levi curvature* for $\partial\mathcal{R}$. Then the following variation formulas for the Robin constans are shown in Hadamard [3], Maitani-Yamaguchi [8].

Fact. It holds for $t\in B$ that

$$\frac{\partial\lambda(t)}{\partial t}=-\frac{1}{\pi}\int_{\partial R(t)}k_1(t,z)\left|\frac{\partial g(t,z)}{\partial z}\right|^2 ds_z,$$

$$\frac{\partial^2\lambda(t)}{\partial t\partial\bar{t}}=-\frac{1}{\pi}\int_{\partial R(t)}k_2(t,z)\left|\frac{\partial g(t,z)}{\partial z}\right|^2 ds_z-\frac{4}{\pi}\iint_{R(t)}\left|\frac{\partial^2 g(t,z)}{\partial\bar{t}\partial z}\right|^2 dxdy.$$

2. Variation formula of L_1-principal function for $(R(t),0,C_0(t))$

This and the next sections are quoted from S. Hamano [4]. Under the same conditions for the unramified domain $\mathcal{R}=\bigcup_{t\in B}(t,R(t))$ in $\widetilde{\mathcal{R}}$ over $B\times\mathbb{C}_z$ and $\partial R(t)=\sum_{j=0}^{\nu}C_j(t)$, we assume that the total space $\mathcal{R}$ contains $B\times\{0\}$. For each $t\in B$, we conventionally write 0 for the point $\mathbf{O}\cap R(t)$.

Let $t\in B$ be fixed. In the theory of one complex variable, it is known that there exists a unique real-valued function $u(t,z)$ on $R(t)\setminus\{0\}$ satisfying the following four conditions:

(1) $u(t,z)$ is harmonic on $R(t)\setminus\{0\}$ and is continuous on $\overline{R(t)}$;
(2) $u(t,z)-\log\frac{1}{|z|}$ is harmonic at $z=0$;
(3) $u(t,z)=0$ on $C_0(t)$;

(4) for each $i = 1, \dots, \nu$, we have

(i) $u(t,z) = a_i(t)$: constant on $C_i(t)$;

(ii) $\displaystyle\int_{C_i(t)} *du(t,z) = 0.$

We note that $u(t,z)$ extends harmonically across $\partial R(t)$ as a harmonic function on $V(t)$ such that $\partial R(t) \Subset V(t) \Subset \widetilde{R}(t)$. By (2), we find a neighborhood $U_0(t)$ of $z = 0$ such that

$$u(t,z) = \log \frac{1}{|z|} + \gamma(t) + h(t,z) \qquad \text{on } U_0(t),$$

where $\gamma(t)$ is the constant term and $h(t,z)$ is harmonic for z on $U_0(t)$ such that

$$h(t,0) = 0, \qquad t \in B.$$

The function $u(t,z)$ is called the *L_1-principal function* on $R(t)$ with logarithmic pole at 0 with respect to $C_0(t)$, and $\gamma(t)$ is called the *L_1-constant* on $R(t)$ with logarithmic pole at 0 with respect to $C_0(t)$ (cf: [1]). In this article, we simply call $u(t,z)$ the L_1-principal function for $(R(t), 0, C_0(t))$, and $\gamma(t)$ the L_1-constant for $(R(t), 0, C_0(t))$. We note that $u(t,z) > 0$ in $R(t) \setminus \{0\}$ and $a_i(t) > 0$ $(i = 1, \dots, \nu)$.

Then we have the following variation formula for the L_1-constant $\gamma(t)$ for $(R(t), 0, C_0(t))$.

Lemma 1. *It holds for $t \in B$ that*

$$\frac{\partial \gamma(t)}{\partial t} = -\frac{1}{\pi} \int_{\partial R(t)} k_1(t,z) \left| \frac{\partial u(t,z)}{\partial z} \right|^2 ds_z,$$

$$\frac{\partial^2 \gamma(t)}{\partial t \partial \bar{t}} = -\frac{1}{\pi} \int_{\partial R(t)} k_2(t,z) \left| \frac{\partial u(t,z)}{\partial z} \right|^2 ds_z - \frac{4}{\pi} \iint_{R(t)} \left| \frac{\partial^2 u(t,z)}{\partial \bar{t} \partial z} \right|^2 dxdy.$$

This variation formula is formally the same as that for the Robin constant $\lambda(t)$ in section 1. The essential difference of the proofs for $\gamma(t)$ and $\lambda(t)$ comes from the fact that $u(t,z)$ is not a defining function of $\partial \mathcal{R}$ contrary to the case of the Green function $g(t,z)$.

Theorem 2. *Under the same conditions in Lemma 1, if $\mathcal{R}$ is pseudoconvex over $B \times \mathbb{C}_z$, then $\gamma(t)$ is a C^ω superharmonic function on B.*

Remark 1. For Lemma 1, we assumed that $\mathcal{R}$ is unramified over $B \times \mathbb{C}_z$. However, even if each $R(t), t \in B$ has a finite number of branch points $\zeta_k(t)$ $(k = 1, \dots, m)$ for $t \in B$ such that $\zeta_k(t)$ is a holomorphic function on B with $\zeta_k(t) \neq \zeta_l(t)$ $(k \neq l)$, $t \in B$, then Lemma 1 and hence Theorem 2 hold. For, this case can be reduced to Lemma 1 by the standard method by use of Y. Nishimura's theorem [10].

In the special case when $R(t)$ is a planar Riemann surface, the L_1-principal function $u(t,z)$ induces a circular slit mapping $f(t,z)$. That is, if we choose a branch $u^*(t,z)$ of harmonic conjugate function of $u(t,z)$ on $R(t), t \in B$ such that

$$f(t,z) = e^{\gamma(t)-(u(t,z)+iu^*(t,z))}$$

is of the form

$$w = f(t,z) = z + \sum_{j=2}^{\infty} b_j(t) z^j \quad \text{on } U_0(t),$$

then $f(t,z)$ conformally maps $R(t)$ onto a circular slit domain $\{|w| < e^{\gamma(t)}\} \setminus (\cup_{i=1}^{\nu} \ell_i)$, where $\ell_i(t) = f(t, C_i(t))$ (an arc of the circle $\{|w| = e^{\gamma(t)-a_i(t)}\}$). If $\mathcal{R}$ is pseudoconvex over $B \times \mathbb{C}_z$, then $e^{\gamma(t)}$ is logarithmic superharmonic on B, so that the total space $\bigcup_{t\in B}\{|w| < e^{\gamma(t)}\}$ is a Hartogs pseudoconvex domain in $B \times \mathbb{C}_w$.

Remark 2. In the theory of one complex variable, the circular slit mapping and the radial slit mapping have good correspondence. But *the same result for the corresponding radius of the radial slit mapping does not hold.* In fact, we have the following counterexamples (i) and (ii) of pseudoconvex domains $\mathcal{R}$ in $B \times \mathbb{C}_z$ such that the radii of radial slit mappings are not logarithmic superharmonic or not logarithmic subharmonic on B:

(i) The radius of radial slit mapping is not logarithmic superharmonic on B: Let

$$\mathcal{R} = \{|t| < \frac{1}{2}\} \times \{|z| < 1\} \setminus \{(t,z) : |z - \frac{1}{2}| \le |t| < \frac{1}{2}\},$$
$$B = \{|t| < \frac{1}{2}\}, \quad R(t) = \{|z| < 1\} \setminus \{|z - \frac{1}{2}| \le |t|\},$$

so that $\partial R(t) = C_0(t) + C_1(t)$ where $C_0(t) = \{|z| = 1\}$ and $C_1(t) = \{|z - 1/2| = |t|\}$.

(ii) The radius of radial slit mapping is not logarithmic subharmonic on B: Let

$$\mathcal{R} = \bigcup_{t\in B}\{|z| < r(t)\} \setminus B \times (C_1 \cup C_2),$$

where $C_1 = [\frac{1}{2}, \frac{2}{3}]$, $C_2 = [\frac{i}{2}, \frac{2i}{3}]$ in $\mathbb{C}_z$, $r(t) > 1$ and $\log r(t)$ is superharmonic on B. Thus $\partial R(t) = C_0(t) + C_1(t) + C_2(t)$ where $C_0(t) = \{|z| = r(t)\}$, $C_i(t) = C_i, i = 1, 2$.

3. Variation formula of L_1-principal function for $(R(t), 0, \xi(t))$

Under the same conditions for the unramified domain $\mathcal{R} = \bigcup_{t\in B}(t, R(t))$ in $\widetilde{\mathcal{R}}$ over $B \times \mathbb{C}_z$ and $\partial R(t) = \sum_{j=0}^{\nu} C_j(t)$, we assume that there exist two holomorphic sections:

$$\Xi_0 : z = 0 \quad \text{and} \quad \Xi_1 : z = \xi(t)$$

of $\mathcal{R}$ over B such that $\Xi_0 \cap \Xi_1 = \emptyset$. Let $t \in B$ be fixed. In the theory of one complex variable, there exists a unique real-valued function $p(t,z)$ on $R(t) \setminus \{0, \xi(t)\}$ satisfying the following four conditions:

(1) $p(t,z)$ is harmonic on $R(t) \setminus \{0, \xi(t)\}$ and continuous on $\overline{R(t)}$;
(2) $p(t,z) - \log 1/|z|$ is harmonic at $z = 0$ and

$$\lim_{z \to 0} \left(p(t,z) - \log \frac{1}{|z|} \right) = 0;$$

(3) $p(t,z) - \log|z - \xi(t)|$ is harmonic at $z = \xi(t)$;
(4) for each $j = 0, 1, \ldots, \nu$, we have

(i) $p(t,z) = a_j(t)$: constant on $C_j(t)$;

(ii) $\displaystyle\int_{C_j(t)} *dp(t,z) = 0.$

We note that $p(t,z)$ extends harmonically across $\partial R(t)$ as a harmonic function on $V(t)$ such that $\partial R(t) \Subset V(t) \Subset \widetilde{R}(t)$, $-\infty < p(t,z) < +\infty$, and $-\infty < a_j(t) < +\infty$.

By (2), we find a neighborhood $U_0(t)$ of $z = 0$ such that

$$p(t,z) = \log \frac{1}{|z|} + h_0(t,z) \qquad \text{on } U_0(t),$$

where $h_0(t,z)$ is harmonic for z on $U_0(t)$ and

$$h_0(t,0) = 0, \qquad t \in B.$$

By (3), we find a neighborhood $U_\xi(t)$ of $z = \xi(t)$ such that

$$p(t,z) = \log|z - \xi(t)| + \alpha(t) + h_\xi(t,z) \qquad \text{on } U_\xi(t),$$

where $\alpha(t)$ is a real constant and $h_\xi(t,z)$ is harmonic for z on $U_\xi(t)$ and

$$h_\xi(t, \xi(t)) = 0, \qquad t \in B.$$

In this article, we simply call $p(t,z)$ the L_1-*principal function* for $(R(t), 0, \xi(t))$, and $\alpha(t)$ the L_1-*constant* for $(R(t), 0, \xi(t))$.

Under these situations, we have

Lemma 3. *It holds for $t \in B$ that*

$$\frac{\partial \alpha(t)}{\partial t} = \frac{1}{\pi} \int_{\partial R(t)} k_1(t,z) \left| \frac{\partial p(t,z)}{\partial z} \right|^2 ds_z + 2 \left. \frac{\partial h_\xi}{\partial z} \right|_{(t,\xi(t))} \cdot \xi'(t),$$

$$\frac{\partial^2 \alpha(t)}{\partial t \partial \bar{t}} = \frac{1}{\pi} \int_{\partial R(t)} k_2(t,z) \left| \frac{\partial p(t,z)}{\partial z} \right|^2 ds_z + \frac{4}{\pi} \iint_{R(t)} \left| \frac{\partial^2 p(t,z)}{\partial \bar{t} \partial z} \right|^2 dxdy$$

If $\mathcal{R}$ is pseudoconvex in $\widetilde{\mathcal{R}}$, then $k_2(t,z) \geq 0$ on $\partial\mathcal{R}$ (Levi condition), and the converse is also true. We then apply Lemma 1 to obtain the following

Theorem 4. *Under the same conditions in Lemma 3, if $\mathcal{R}$ is pseudoconvex over $B \times \mathbb{C}_z$, then $\alpha(t)$ is a C^ω subharmonic function on B. This is also true under the same condition for $\mathcal{R}$ as in Remark 1.*

Application of Theorem 4 As an application of Theorem 4, we show that the following fact. Let B be a simply connected domain in $\mathbb{C}_t$. Let $\pi : \mathcal{S} \to B$ be a holomorphic family of compact Riemann surfaces $S(t) = \pi^{-1}(t)$ over B such that each fiber $S(t)$ is of genus ≥ 2 and non-singular in $\mathcal{S}$. For a fixed $t \in B$, we consider the Schottky covering $\widetilde{S}(t)$ of each $S(t)$ (cf: Sec. 101 in p.266 in Ford [2], and 19F in p.241 in Ahlfors-Sario [1]). We denote by $\widetilde{\mathcal{S}}$ the total space of the variation: $t \in B \to \widetilde{S}(t)$, namely, $\widetilde{\mathcal{S}} = \bigcup_{t\in B}(t, \widetilde{S}(t))$. Then we have:

Theorem 5. *The total space $\widetilde{\mathcal{S}}$ consisting of the Schottky covering $\widetilde{S}(t)$ of compact Riemann surfaces $S(t)$ with complex parameter $t \in B$ is holomorphically uniformized to a univalent domain on $B \times \mathbb{P}^1$.*

In [8], Maitani and Yamaguchi proved that, if $\mathcal{R} = \bigcup_{t\in B}(t, R(t))$ is an unramified pseudoconvex domain over $B \times \mathbb{C}_z$ such that each $R(t)$, $t \in B$ is planar and parabolic, then $\mathcal{R}$ is holomorphically uniformizable to a domain in $B \times \mathbb{P}^1$. Since the Schottky covering $\widetilde{S}(t)$ of a compact Riemann surface $S(t)$ of genus $g \geq 2$ is planar but not parabolic, their theorem and method cannot be applicable to our case. In [12], Yamaguchi wrote a resumé about Theorem 5 with a rough sketch of the proof. However his sketch had a "gap". Then I bridge the gap by establishing the variation formula for L_1-principal function (Lemma 3), and obtain Theorem 5.

4. Variation formula of L_0-principal function for $(R(t), 0, \xi(t))$

This section is quoted from S. Hamano, F. Maitani and H. Yamguchi [6]. Under the same conditions for the unramified domain $\mathcal{R} = \bigcup_{t\in B}(t, R(t))$ in $\widetilde{\mathcal{R}}$ over $B \times \mathbb{C}_z$ and $\partial R(t) = \sum_{j=0}^{\nu} C_j(t)$, we assume that $\mathcal{R}$ has two holomorphic section over B:

$$\Xi_0 : z = 0 \quad \text{and} \quad \Xi_1 : z = \xi(t)$$

such that $\Xi_0 \cap \Xi_1 = \emptyset$. Let $t \in B$ be fixed. Then it is known (cf: Sario-Nakai [9]) that $R(t)$ carries the following real-valued function $q(t,z)$.

(1) $q(t,z)$ is harmonic on $R(t) \setminus \{0, \xi(t)\}$ and is continuous on $\overline{R(t)}$;
(2) $q(t,z) - \log 1/|z|$ is harmonic at $z = 0$ and

$$\lim_{z\to 0}\left(q(t,z) - \log 1/|z| \right) = 0;$$

(3) $q(t,z) - \log|z-\xi(t)|$ is harmonic at $z=\xi(t)$;
(4) $\frac{\partial q(t,z)}{\partial n_z} = 0$ on $\partial R(t)$.

We call the function $q(t,z)$ the *L_0-principal function* for $(R(t),0,\xi(t))$.

Note that $q(t,z)$ extends harmonically across $\partial R(t)$ as a harmonic function on $V(t)$ such that $\partial R(t) \Subset V(t) \Subset \widetilde{R}(t)$.

By (2) for $q(t,z)$, we find a neighborhood $U_0(t)$ of $z=0$ such that

$$q(t,z) = \log\frac{1}{|z|} + \mathfrak{h}_0(t,z) \qquad \text{on } U_0(t),$$

where $\mathfrak{h}_0(t,z)$ is harmonic for z on $U_0(t)$ and

$$\mathfrak{h}_0(t,0) = 0, \quad t \in B.$$

By (3) for $q(t,z)$, we find a neighborhood $U_\xi(t)$ of $z=\xi(t)$ such that

$$q(t,z) = \log|z-\xi(t)| + \beta(t) + \mathfrak{h}_\xi(t,z) \qquad \text{on } U_\xi(t),$$

where $\beta(t)$ is a constant and $\mathfrak{h}_\xi(t,z)$ is harmonic for z on $U_\xi(t)$ and

$$\mathfrak{h}_\xi(t,\xi(t)) = 0, \quad t \in B.$$

We call $\beta(t)$ the *L_0-constant* for $(R(t),0,\xi(t))$.

We shall give the variational formulas for L_0-constant $\beta(t)$. In order to proves the formula for $\beta(t)$, we have to add a new idea to the proof for $\alpha(t)$. In fact, the formulas for $\alpha(t)$ do not concern to the genus of $R(t)$ but the variation formula of the second order for $\beta(t)$ does concern to the genus of $R(t)$. It seems to be curious that of the first order does not concern to the genus as below. In case when $R(t)$ is of positive genus $g \geq 1$, we take $\{A_l(t), B_l(t)\}_{1\leq l\leq g}$ be usual A, B cycles on $R(t)$ with intersection number condition: for $k,l = 1,\dots,g$,

$$A_k(t)\times B_l(t) = \delta_{k,l}, \quad A_k(t)\times A_l(t) = 0, \quad B_k(t)\times B_l(t) = 0.$$

Here $\delta_{k,l}$ is Kronecker's delta; $A_k(t)\times B_l(t)$ means that $A_k(t)$ crosses $B_l(t)$ from the left-side to the right-side of the direction $B_l(t)$; and each $A_k(t)(B_k(t))$, $k=1,\dots,g$ varies continuously with parameter $t\in B$ such that $A_k(t), B_k(t)$ do not pass through $\{0,\xi(t)\}$. On each $R(t), t\in B$ we denote by $*dq(t,z)$ the conjugate differential of $dq(t,z)$.

Then we have

Lemma 6. *It holds for $t \in B$ that*

$$\frac{\partial\beta(t)}{\partial t} = -\frac{1}{\pi}\int_{\partial R(t)} k_1(t,z)\left|\frac{\partial q(t,z)}{\partial z}\right|^2 ds_z + 2\left.\frac{\partial \mathfrak{h}_\xi}{\partial z}\right|_{(t,\xi(t))}\cdot\xi'(t),$$

$$\frac{\partial^2\beta(t)}{\partial t\partial\bar t} = -\frac{1}{\pi}\int_{\partial R(t)} k_2(t,z)\left|\frac{\partial q(t,z)}{\partial z}\right|^2 ds_z - \frac{4}{\pi}\iint_{R(t)}\left|\frac{\partial^2 q(t,z)}{\partial\bar t\partial z}\right|^2 dxdy$$
$$-\frac{2}{\pi}\Im\sum_{k=1}^{g}\left(\frac{\partial}{\partial t}\int_{A_k(t)} *dq(t,z)\right)\cdot\left(\frac{\partial}{\partial \bar t}\int_{B_k(t)} *dq(t,z)\right).$$

Lemma 3 implied that if $\mathcal{R}$ is pseudoconvex in $\widetilde{\mathcal{R}}$, then L_1-constant $\alpha(t)$ for $(R(t), 0, \xi(t))$ is a C^ω subharmonic function on B. On the other hand, Lemma 6 implied the following:

Theorem 7. *Under the same conditions in Lemma 6, if $\mathcal{R}$ is pseudoconvex in $\widetilde{\mathcal{R}}$ and each $R(t), t \in B$ is planar, then the L_0-constant $\beta(t)$ for $(R(t), 0, \xi(t))$ is a C^ω superharmonic function on B.*

The contrast between the subaharmincity of $\alpha(t)$ and the superharmonicity of $\beta(t)$ are unified with the notion of the harmonic span $s(t)$ for $(R(t), 0, \xi(t))$ due to M. Schiffer [11].

Variations formula for the harmonic spans. We recall some notions studied in the theory of one complex variable. Let R be a domain in $\mathbb{C}_z$ bounded by a finite number of closed curves $C_j, j = 0, 1, \ldots, \nu$. For simplicity we assume $0 \in R$. For a point $\xi \neq 0$, we consider the L_1- and L_0-function $p(z)$ and $q(z)$ for $(R, 0, \xi)$ and the L_1- and L_0-constant α and β for $(R, 0, \xi)$. In the function theory of one complex variable, it is known (cf: [1] and [9])

$$s(R) = \frac{\pi}{2}(\alpha - \beta)$$

as the *harmonic span* $s(R)$ for $(R, 0, \xi)$.

We return to the variation of Riemann surfaces. Let $\mathcal{R} : t \in B \to R(t)$ satisfy the conditions in the beginning of Sections 3 and 4. For a fixed $t \in B$, we denote by $p(t, z)$ $(q(t, z))$ the L_1-$(L_0$-$)$ principal function, by $\alpha(t)$ $(\beta(t))$ the L_1-$(L_0$-$)$ constant and by $s(t)$ the harmonic span for $(R(t), 0, \xi(t))$. Then combining Lemmas 3 and 6, we immediately have the following:

Lemma 8. *Assume that $R(t)$, $t \in B$ is planar. Then it holds that*

$$\frac{\partial^2 s(t)}{\partial t \partial \bar{t}} = \frac{1}{2} \int_{\partial R(t)} k_2(t, z) \left(\left| \frac{\partial p(t, z)}{\partial z} \right|^2 + \left| \frac{\partial q(t, z)}{\partial z} \right|^2 \right) ds_z$$
$$+ 2 \iint_{R(t)} \left(\left| \frac{\partial^2 p(t, z)}{\partial \bar{t} \partial z} \right|^2 + \left| \frac{\partial^2 q(t, z)}{\partial \bar{t} \partial z} \right|^2 \right) dx dy$$

Lemma 8 implied the following:

Theorem 9. *Under the same conditions in Lemma 6, if $\mathcal{R}$ is pseudoconvex in $\widetilde{\mathcal{R}}$ and each $R(t), t \in B$ is planar, then the harmonic span $s(t)$ for $(R(t), 0, \xi(t))$ is a C^ω subharmonic function on B.*

5. Examples

We begin with a simple example of our general result shown in this article. Let $B = \{|t| < \rho\}$ be a disk in $\mathbb{C}_t$. For each $t \in B$, let $R(t)$ be a disk $\{|z| < r(t)\}$ in $\mathbb{C}_z$, where $\log r(t)$ is a superharmonic function on B. If we set the Hartogs domain of disks $\mathcal{R} = \bigcup_{t\in B}(t, R(t))$, then $\mathcal{R}$ is a pseudoconvex domain in $B \times \mathbb{C}_z$. Assume that there exists a holomorphic section $\xi : t \in B \to \xi(t)(\neq 0) \in R(t)$.

[Example of Theorem 4.] We consider the following function:

$$f(t,z) = -\frac{1}{\xi(t)} \cdot \frac{r(t)^2(z-\xi(t))}{z\,(r(t)^2 - \bar{\xi}(t)z)} \qquad \text{on } R(t).$$

Then f is a circular slit mapping on $R(t)$ with zero at $z = \xi(t)$ and pole at $z = 0$. The L_1-constant $\alpha(t)$ on B is written into

$$\alpha(t) = \log\left|\frac{\partial f}{\partial z}(t,\xi(t))\right| = \log\left|-\frac{1}{\xi(t)^2} \cdot \frac{r(t)^2}{r(t)^2 - |\xi(t)|^2}\right|.$$

Since $\xi(t)$ is holomorphic on B and since $\log r(t)$ is superharmonic on B, $\log\frac{|\xi(t)|}{r(t)}$ is subharmonic on B, so is the second term in the right-hand side. Hence, $\alpha(t)$ is a subharmonic function on B.

[Example of Theorem 7.] We put $\theta(t) = \arg \xi(t)$. Then

$$F(t,z) = \frac{1}{2}\left(\frac{z}{r(t)e^{i\theta(t)}} + \frac{r(t)e^{i\theta(t)}}{z}\right) - \frac{1}{2}\left(\frac{|\xi(t)|}{r(t)} + \frac{r(t)}{|\xi(t)|}\right)$$

is the radial slit mapping on $R(t)$ with zero at $z = \xi(t)$ and pole at $z = 0$. The L_0-constant $\beta(t)$ is written into

$$\beta(t) = \log\left|\frac{\partial Q}{\partial z}(t,\xi(t))\right| = -2\log|\xi(t)| + \log\left[1 - \left(\frac{|\xi(t)|}{r(t)}\right)^2\right],$$

which certainly is superharmonic on B.

[Example of Theorem 9.] We also see that the harmonic span $s(t) = \frac{\pi}{2}(\alpha(t) - \beta(t))$ for $(R(t), 0, \xi(t))$ is

$$s(t) = \log\frac{1}{1 - \left(\frac{|\xi(t)|}{r(t)}\right)^2},$$

which is subharmonic on B.

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