数理解析研究所講究録
第 1694 巻 2010 年 131-143

Several complex variable's property of harmonic span for Riemann surface

by

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1 Intoduction.

S. Hamano [5] established the variation formulas of the second order for L_1-principal functions $p(t,z)$ on the moving Riemann surface $R(t)$ with complex parameter t in $B=\{|t|<1\}$. We showed in [9] the corresponding formulas for L_0-principal functions $q(t,z)$. Combining two formulas, we give a several complex variable's property of the harmonic span for Riemann surface introduced in Nakai-Sario [13]. This property implies the following: Let $\pi:\mathcal{R}\to B$ be a two-dimensional holomorphic family over B such that $\mathcal{R}$ is a Stein manifold and each fiber $R(t)=\pi^{-1}(t), t\in B$ is irreducible, non-singular in $\mathcal{R}$ and hyperbolic as Riemann surface. Let $\xi: t\in B\to\xi(t)\in R(t)$ and $\eta: t\in B\to\eta(t)\in R(t)$ be holomorphic sections and let $\Gamma(t)$ be a continuous curve connecting $\xi(t)$ and $\eta(t)$ on $R(t)$ such that $\Gamma:=\cup_{t\in B}(t,\Gamma(t))(\subset\mathcal{R})$ is homeomorphic to the product set $B\times\Gamma(0)$. On each $R(t), t\in B$ we consider the Poincaré metric $ds(t,z)^2$ and draw the geodesic curve $\gamma(t)$ connecting $\xi(t)$ and $\eta(t)$ which is homotopic to $\Gamma(t)$ on $R(t)$. Then $\log\cosh l(t)$, where $l(t)=\int_{\gamma(t)}ds(t,z)$, is subharmonic on B. This note continues on [8] in this volume of Report of RIMS of Kyoto Univ.

2 Variation formulas for L_1- and L_0-constants.

Let R be a bordered Riemann surface with smooth boundary $\partial R=C_1+\cdots+C_\nu$ in a larger Riemann surface $\widetilde{R}$, where $C_j, j=1,\ldots,\nu$ is a C^ω smooth contour in $\widetilde{R}$. Fix two distinct points a, b with local coordinates $|z|<\rho$ and $|z-\xi|<\rho$ where $a\,(b)$ corresponds to $0\,(\xi)$. Among all harmonic functions u on $R\setminus\{0,\xi\}$ with logarithmic singularity $\log\frac{1}{|z|}$ at 0 and $\log|z-\xi|$ at ξ normalized $\lim_{z\to 0}(u(z)-\log\frac{1}{|z|})=0$, we uniquely have two special ones p and q with the following boundary conditions: for each C_j, p satisfies $p(z)=$ const. c_j and $\int_{C_j}*dp(z)=0$, and q does $\frac{\partial q(z)}{\partial n_z}=0$ on C_j. Then p and q are called the L_1- *and* L_0-*principal function* for $(R,0,\xi)$. The constant terms $\alpha:=\lim_{z\to\xi}(p(z)-\log|z-\xi|)$ and $\beta:=\lim_{z\to\xi}(q(z)-\log|z-\xi|)$ are called the L_1- *and* L_0-*constant* for $(R,0,\xi)$ (cf: [1] and [13]).

Let $B = \{|t| < \rho\}$ and let $\widetilde{\mathcal{R}}$ be a two-dimensional unramified domain sheeted over $B \times \mathbb{C}_z$. We write $\widetilde{\mathcal{R}} = \cup_{t\in B}(t, \widetilde{R}(t))$, where $\widetilde{R}(t)$ is a fiber of $\widetilde{\mathcal{R}}$ over $t \in B$, i.e., $\widetilde{R}(t) = \{z : (t,z) \in \widetilde{\mathcal{R}}\}$, so that $\widetilde{R}(t)$ is an unramified Riemann surface sheeted over $\mathbb{C}_z$. Consider a subdomain $\mathcal{R}$ in $\widetilde{\mathcal{R}}$ such that, if we put $\mathcal{R} = \cup_{t\in B}(t, R(t))$, where $R(t)$ is a fiber of $\mathcal{R}$ over $t \in B$, then

(1) $\widetilde{R}(t) \Supset R(t) \neq \emptyset$, $t \in B$ such that $R(t)$ is a connected surface of genus $g \geq 0$ such that $\partial R(t)$ in $\widetilde{R}(t)$ consists of a finite number of C^ω smooth contours $C_1(t), \ldots, C_\nu(t)$ in $\widetilde{R}(t)$.

(2) the boundary $\partial\mathcal{R} = \cup_{t\in B}(t, \partial R(t))$ of $\mathcal{R}$ in $\widetilde{\mathcal{R}}$ is C^ω smooth.

Note that g and ν are independent of $t \in B$. We regard the two-dimensional domain $\mathcal{R}$ over $B \times \mathbb{C}_z$ as a C^ω smooth variation with parameter $t \in B$ of Riemann surfaces $R(t)$ over $\mathbb{C}_z$ with C^ω smooth boundary $\partial R(t)$,

$$\mathcal{R} : t \in B \to R(t).$$

Let $\mathcal{R}$ have two holomorphic sections over B: $\Xi_0 : z = 0$ and $\Xi_\xi : z = \xi(t)$ such that $\Xi_0 \cap \Xi_\xi = \emptyset$. Each $R(t), t \in B$ carries the L_1-(L_0-)principal function $p(t,z)(q(t,z))$ for $(R(t), 0, \xi(t))$. Precisely, both functions are harmonic on $R(t) \setminus \{0, \xi(t)\}$ with poles $\log\frac{1}{|z|}$ at $z = 0$ and $\log|z - \xi(t)|$ at $z = \xi(t)$, and continuous on $\overline{R(t)}$ such that $p(t,z)$ satisfies

(1) $\lim_{z\to 0}(p(t,z) - \log\frac{1}{|z|}) = 0$; (2) $p(t,z) = \text{const.}\, c_j(t)$ on $C_j(t)$ and $\int_{C_j(t)} *dp(t,z) = 0$, $j = 1, \ldots, \nu$,

while $q(t,z)$ satisfies

(1) $\lim_{z\to 0}(\, q(t,z) - \log\frac{1}{|z|}\,) = 0$; (2) $\frac{\partial q(t,z)}{\partial n_z} = 0$ on $\partial R(t)$.

We write $\alpha(t)(\beta(t))$ for the L_1-(L_0-)constant for $(R(t), 0, \xi(t))$:

$$\alpha(t) = \lim_{z\to\xi(t)}(p(t,z) - \log|z - \xi(t)|), \quad \beta(t) = \lim_{z\to\xi(t)}(q(t,z) - \log|z - \xi(t)|).$$

Then we have the following variation formulas of the second order for $\alpha(t)$ and $\beta(t)$:

Lemma 2.1. (see Lemma 3.1 in [5] and Lemma 2.2 in [9]) *It holds for $t \in B$ that*

$$\frac{\partial^2\alpha(t)}{\partial t\partial\bar{t}} = \frac{1}{\pi}\int_{\partial R(t)} k_2(t,z)\left|\frac{\partial p(t,z)}{\partial z}\right|^2 ds_z + \frac{4}{\pi}\iint_{R(t)}\left|\frac{\partial^2 p(t,z)}{\partial\bar{t}\partial z}\right|^2 dxdy,$$

and that, if each $R(t), t \in B$ is a **planar** *Riemann surface, then*

$$\frac{\partial^2 \beta(t)}{\partial t \partial \bar{t}} = -\frac{1}{\pi} \int_{\partial R(t)} k_2(t,z) \left|\frac{\partial q(t,z)}{\partial z}\right|^2 ds_z - \frac{4}{\pi} \iint_{R(t)} \left|\frac{\partial^2 q(t,z)}{\partial \bar{t} \partial z}\right|^2 dxdy$$

Here

$$k_2(t,z) = \left(\frac{\partial^2 \varphi}{\partial t \partial \bar{t}} \left|\frac{\partial \varphi}{\partial z}\right|^2 - 2\mathrm{Re}\left\{\frac{\partial^2 \varphi}{\partial \bar{t} \partial z} \frac{\partial \varphi}{\partial t} \frac{\partial \varphi}{\partial \bar{z}}\right\} + \left|\frac{\partial \varphi}{\partial t}\right|^2 \frac{\partial^2 \varphi}{\partial z \partial \bar{z}}\right) / \left|\frac{\partial \varphi}{\partial z}\right|^3$$

on $\partial \mathcal{R}$, where $\varphi(t,z)$ is a defining function of $\partial \mathcal{R}$ and ds_z is the arc length element on $\partial R(t)$ at z.

Theorem 2.1. *Under the same situation in Lemma 2.1 assume that $\mathcal{R}$ is pseudoconvex in $\widetilde{\mathcal{R}}$. Then the L_1-constant $\alpha(t)$ is a C^ω subharmonic function on B, while the L_0-constant $\beta(t)$ is a C^ω superharmonic function on B.*

The superharmonicity of $\beta(t)$ in the theorem does not hold without the assumption that each $R(t), t \in B$ is planar, in general.

3 Harmonic span for Riemann surface.

In this section R is always a domain in the plane $\mathbb{C}_z$ of one complex variable z bounded by a finite number of smooth contours $C_j, j = 1, \ldots, \nu$ such that $R \ni 0$. Fix a point $\xi \in R, \xi \neq 0$. M. Schiffer [15] introduced, so called, the *analytic span* $a(R) = a(R, 0, \xi)$ for $(R, 0, \xi)$ using the vertical and horizontal mappings $w = f(z)$ with $f(0) = \infty$ and $f(\xi) = 0$ on R. By use of the L_1-(L_0-)principal function $p(z)$ ($q(z)$), and L_1-(L_0-)constant α (β) for $(R, 0, \xi)$, Nakai-Sario [13] introduced the *harmonic span* for $(R, 0, \xi)$:

$$s(R) = s(R, 0, \xi) := \tfrac{1}{2}(\alpha - \beta).$$

Two spans belong to different categories, for example, $a(R)$ is not invariant under the conformal mappings $w = T(z)$ from R onto $T(R)$, while $s(R)$ is invariant, i.e., $s(R, 0, \xi) = s(T(R), T(0), T(\xi))$. Thus, $s(R, a, b)$ defines a real-valued function on $R \times R$ with $s(R, a, a) = 0$.

We consider the set $\mathcal{S}(R)$ of all univalent functions $w = f(z)$ on R such that

$$\begin{aligned} &f(z) - \tfrac{1}{z} \text{ is holomorphic near } z = 0, \\ &f(z) = c_1(z - \xi) + c_2(z - \xi)^2 + \ldots \qquad \text{near } z = \xi, \end{aligned}$$

and we write $c(f) = c_1 (\neq 0)$. We draw a Jordan curve l in R from ξ to 0. Let $f \in \mathcal{S}(R)$. Then the image $f(l)$ in $\mathbb{P}_w$ is a simple curve from 0 to ∞, so that each branch $W = \log f(z)$ on $R \setminus l$ is single-valued. We consider the Euclidean area $E_{\log}(f) \geq 0$ of the complemant of $\log f(R \setminus l)$ in $\mathbb{C}_W$, which is independent of the choice of branches. We put

$$E(R) = \sup\{E_{\log}(f) : f \in \mathcal{S}(R)\}.$$

For the L_1- and L_0-principal function $p(z)$ and $q(z)$ for $(R, 0, \xi)$, we choose their harmonic conjugates $p^*(z)$ and $q^*(z)$ on R such that $P(z) = e^{p(z)+ip^*(z)}$ and $Q(z) = e^{q(z)+iq^*(z)}$ on R are of the form

$$\begin{aligned}
&P(z) - \tfrac{1}{z} \text{ and } Q(z) - \tfrac{1}{z} \text{ are holomorphic near } z = 0\\
&P(z) = e^{\alpha+i\theta_1}\ (z-\xi) + \textstyle\sum_{n=2}^{\infty} a_n (z-\xi)^n \quad \text{near } \ z = \xi,\\
&Q(z) = e^{\beta+i\theta_0}\ (z-\xi) + \textstyle\sum_{n=2}^{\infty} b_n (z-\xi)^n \quad \text{near } \ z = \xi,
\end{aligned}$$

where θ_1, θ_0 are constants. Then $w = P(z)$ and $w = Q(z)$ are a circular slit mapping and a radial slit mapping on R, i.e., their images are

$$\mathfrak{R}_1 := \mathbb{P}_w \setminus \cup_{j=1}^{\nu} P(C_j) = \mathbb{P}_w \setminus \cup_{j=1}^{\nu} \operatorname{arc}\{A_j^{(1)}, A_j^{(2)}\},$$

$$\mathfrak{R}_0 := \mathbb{P}_w \setminus \cup_{j=1} Q(C_j) = \mathbb{P}_w \setminus \cup_{j=1}^{\nu} \operatorname{segment}\{B_j^{(1)}, B_j^{(2)}\}.$$

Here

$$\begin{aligned}
\operatorname{arc}\{A_j^{(1)}, A_j^{(2)}\} &= \{r_j e^{i\theta} : \theta_j^{(1)} \leq \theta \leq \theta_j^{(2)}\},\\
\operatorname{segment}\{B_j^{(1)}, B_j^{(2)}\} &= \{r e^{i\theta_j} : 0 < r_j^{(1)} \leq r \leq r_j^{(2)} < \infty\},
\end{aligned} \tag{3.1}$$

where $0 < \theta_j^{(2)} - \theta_j^{(1)} < 2\pi$ and $r_j, \theta_j^{(k)}, \theta_j, r_j^{(k)}$ $(k = 1, 2)$ are constants. For future use we take points $a_j^{(k)}, b_j^{(k)} \in C_j$ $(k = 1, 2)$ such that

$$P(a_j^{(k)}) = A_j^{(k)} \quad \text{and} \quad Q(b_j^{(k)}) = B_j^{(k)}. \tag{3.2}$$

Then $P(z)$ and $Q(z)$ belong to $\mathcal{S}(R)$ such that $E_{\log}(P) = E_{\log}(Q) = 0$ and $|c(P)| = e^{\alpha}$, $|c(Q)| = e^{\beta}$.

Proposition 3.1. (see 13B, Chap. III in [1]) *The circular slit mapping $P(z)$ maximizes $2\pi \ \log|c(f)| + E_{\log}(f)$ and the radial slit mapping $Q(z)$ minimizes $2\pi \ \log|c(f)| - E_{\log}(f)$ among $\mathcal{S}(R)$.*

We have the following lemma which is necessary for our study of variation of Riemann surfaces (see Theorem 4.1).

Lemma 3.1.

1. *$\sqrt{(PQ)(z)}$ consists of two branches $H(z)$ and $-H(z)$ on R such that $H(z) \in \mathcal{S}(R)$ and each $-(\log H)(C_j), j = 1, \dots, \nu$ is a C^ω convex curve which bounds a bounded domain in $\mathbb{C}_W$.*

2. *The function $H(z)$ maximizes $E_{\log}(f)$ among $\mathcal{S}(R)$ such that*
$$E(R) = E_{\log}(H) = \pi s(R).$$

3. *Let R be a simply connectd domain and let $d(0, \xi)$ denote the geodesic distance between 0 and ξ with respect to the Poincaré metric on R. Then we have*
$$s(R) = 2 \log \cosh d(0, \xi).$$

The corresponding results for the analytic span to 1. and 2. in Lemma 3.1 are well-known (see M. Schiffer [15] and 12A, 12F in Chap. III in [1]). Those proofs have some gaps to prove 1. and 2. for the harmonic span in the lemma. We get over them by using the Schottky double Riemann surface $\widehat{R}$ of the domain R. This idea itself will be needed for the proof of 3. in Lemma 5.2 concerning the variation of Riemann surfaces. So, we here give the sketch of the proofs. Due to H. Grunsky [3] we consider the following function

$$W = F(z) := \frac{d \log Q}{d \log P} \quad \text{for } z \in R \cup \partial R, \tag{3.3}$$

which is a single-valued holomorphic function on R such that $\Re\, F = 0$ on ∂R, since $\log P(C_j)$ is a vertical segment and $\log Q(C_j)$ is a horizontal segment. It follows from Schwarz reflexion principle that F is meromorphically extended to the Schottoky double compact Riemann surface $\widehat{R} = R \cup \partial R \cup R^*$ of R such that $F(z^*) = -\overline{F(z)}$, where $z^* \in R^*$ is the reflexion point of $z \in R$. Fix $C_j, j = 1, \dots, \nu$. Since each branch $\log P(z)$ ($\log Q(z)$) (where $\Re \log P(z) = p(z)$ and $\Re \log Q(z) = q(z)$) is single-valued in a tubular neighborhood V_j of C_j, we fix one of them:

$$\log P(z) = u_1(z) + i v_1(z), \quad \log Q(z) = u_0(z) + i v_0(z), \quad z \in V_j,$$

so that $u_1(z)(v_0(z)) = \text{const.}\, c_1\, (c_0)$ on C_j. We put $\mathfrak{C}_j := \frac{1}{2}(\log P(z) + \log Q(z))\big|_{z \in C_j}$, namely

$$\mathfrak{C}_j : w = \frac{1}{2}\,(\,c_1 + v_0(z)) + \frac{i}{2}(\,c_0 + u_1(z)), \quad z \in C_j.$$

Recalling notation (3.2), we can show:

i) $\{a_j^{(k)}, b_j^{(k)}\}_{k=1,2}$ are 4 distinct points on C_j;

ii) the zeros of F are $\{b_j^{(k)}\}_{j=1,\dots,\nu;\,k=1,2}$ of order one, and the poles are $\{a_j^{(k)}\}_{j=1,\dots,\nu;\,k=1,2}$ of order one;

iii) the closed curve $\mathfrak{C}_j$ is simple and non-singular in $\mathbb{C}_w$;

iv) $\Re F(z) > 0$ on R and $\mathrm{Im}\, F'(z) < 0$ on C_j;

v) at any $w \in \mathfrak{C}_j$, the curvature $\frac{1}{\rho_j(w)}$ of $\mathfrak{C}_j$ is negative, precisely,

$$\frac{1}{\rho_j(x)} = \frac{v_1'(x)^2}{(v_1'(x)^2 + u_0'(x)^2)^{3/2}} \cdot \mathrm{Im}\, F'(x).$$

Then the properties $i) \sim v)$ of $W = F(z)$ implies assertion 1. The proof of 2. is standard under 1. Since the harmonic span is invariant under the conformal mappings, asserton 3. follows *Examples* in section 5 in [8].

4 Variation formulas of harmonic spans for moving Riemann surfaces.

We return to the variation of Riemann surfaces. In this section we let $\mathcal{R} : t \in B \to R(t)$ satisfy the conditions in the beginning of section 2. For a fixed $t \in B$, let $p(t,z)$ $(q(t,z))$; $\alpha(t)$ $(\beta(t))$ and $s(t)$ denote the L_1-$(L_0$-$)$principal function; the L_1-$(L_0$-$)$constant and the harmonic span for $(R(t), 0, \xi(t))$. Then, Lemmas 2.1 and 3.1 implies the following

Lemma 4.1. *Assume that $R(t)$, $t \in B$ is planar. Then it holds that*

$$\begin{aligned}\frac{\partial^2 s(t)}{\partial t \partial \bar{t}} &= \frac{1}{2\pi} \int_{\partial R(t)} k_2(t,z) \left(\left|\frac{\partial p(t,z)}{\partial z}\right|^2 + \left|\frac{\partial q(t,z)}{\partial z}\right|^2 \right) ds_z \\ &\quad + \frac{2}{\pi} \iint_{R(t)} \left(\left|\frac{\partial^2 p(t,z)}{\partial \bar{t} \partial z}\right|^2 + \left|\frac{\partial^2 q(t,z)}{\partial \bar{t} \partial z}\right|^2 \right) dx dy.\end{aligned}$$

Theorem 4.1. *Assume that $\mathcal{R} = \cup_{t \in B}(t, R(t))$ is pseudoconvex over $B \times \mathbb{C}_z$ such that each fiber $R(t)$, $t \in B$ is planar. Then we have*

1. *The harmonic span $s(t)$ for $(R(t), 0, \xi(t))$ is C^ω subharmonic on B.*

2. *If $s(t)$ is harmonic on B, then the variation $\mathcal{R} : t \in B \to R(t)$ is equivalent to the trivial variation : $t \in B \to R(0)$, i.e., the total space $\mathcal{R}$ is biholomorphic to the product $B \times R(0)$ (by a fiber preseving transformation).*

In fact, assertion 1. is clear now. To prove 2. we first consider the cirular slit mapping $w = P(t,z)$ for $(R(t), 0, \xi(t))$. Under the condition of 2., we see from Lemma 4.1 that $P(t,z)$ is holomorphic for (t,z) on $\mathcal{R}$. We put $\mathcal{D} := \cup_{t\in B}(t, D(t))(\subset B \times \mathbb{C}_w)$ where $D(t) = P(t, R(t))(\subset \mathbb{P}_w)$. Then $\partial D(t)$ consists of circular slit arc $\{A_j^{(1)}(t), A_j^{(2)}(t)\}, j = 1, \ldots, \nu$, and $\mathcal{R} \approx \mathcal{D}$. Since $\mathcal{D}$ is pseudoconvex, it follows from Kanten Satz in [2] that each edge point $A_j^{(k)}(t), j = 1, \ldots, \nu; k = 1, 2$ is holomorphic for $t \in B$.

We secondly consider the holomorphic map $(t,w) \in \mathcal{D} \to (t, \widetilde{w}) = (t, L(t,w))$ where $L(t,w) = w/A_1^{(1)}(t)$, and put $\widetilde{D} = \cup_{t\in B}(t, \widetilde{D}(t))$ where $\widetilde{D}(t) = L(t, D(t))$, so that $\mathcal{R} \approx \widetilde{\mathcal{D}}$. Each $\widetilde{D}(t), t \in B$ is circular slit domain $\mathbb{P}_{\widetilde{w}} \setminus \widetilde{C}_j(t)$ such that the first circular slit $\widetilde{C}_1(t) =: \widetilde{C}_1$ is indepedent of $t \in B$.

We thirdly consider the function $W = F(t,z)$ defined in (3.3):

$$F(t,z) = \frac{d_z \log Q(t,z)}{d_z \log P(t,z)} = \frac{\partial q(t,z)}{\partial z} / \frac{\partial p(t,z)}{\partial z} \qquad \text{for } z \in R(t) \cup \partial R(t).$$

Then $F(t,z)$ is holomorphic for (t,z) in $\mathcal{R}$ such that $F(t,0) = 1$ and $\Re F(t,z) = 0$ on each $C_j(t), j = 1, \ldots, \nu$. We put $\mathfrak{C}_j(t) = F(t, C_j(t))$. Then we see from $i) \sim iv)$ that $\mathfrak{C}_j(t)$ rounds just twice on the imaginary axis in $\mathbb{P}_W$, so that $W(t) = F(t, D(t))$ is a ramified Riemann surface over $\Re W > 0$ without relative boundary, and, if we put $\mathcal{W} = \cup_{t\in B}(t, W(t))$, then $\mathcal{R} \approx \mathcal{W}$.

We finally consider the following bi-holomorphic mapping

$$(t, \widetilde{w}) \in \widetilde{\mathcal{D}} \to (t, W) = (t, G(t, \widetilde{w})) \in \mathcal{W},$$

where $\widetilde{G}(t, \widetilde{w}) := F(t, P^{-1}(t, L^{-1}(t, \widetilde{w})))$. Thus, $\widetilde{\mathcal{D}} \approx \mathcal{W}$. Since $\Re G(t, \widetilde{w}) = 0$ on the first circular arc $\widetilde{C}_1$, it follows that $G(t, \widetilde{w})$ does not depend on $t \in B$, so that $\mathcal{W}$ is equal to the product $B \times W(0)$, and hence $\mathcal{R}$ is biholomorphic to the product $B \times R(0)$, which proves assertion 2.

Corollary 4.1. *Under the same conditions as in Theorem 4.1, we denote by $s(t,z,\zeta)$ the harmonic span for $(R(t), z, \zeta)$ for each $t \in B$. Then $s(t,z,\zeta)$ is a C^ω plurisubharmonic function on $\cup_{t\in B}(t, R(t) \times R(t))$ such that $s(t,z,\zeta) > 0 (= 0)$ for $z \neq \zeta$ $(z = \zeta)$ and $s(t,z,\zeta) \to \infty$ as $(t,z,\zeta) \to (t_0, z_0, \zeta_0)$ where $(z_0, \zeta_0) \in \partial(R(t_0) \times R(t_0))$ with $z_0 \neq \zeta_0$.*

Variation formulas for analytic (M. Schiffer's) spans $a(t,z,\zeta)$ for moving Riemann surfaces $\mathcal{R} : t \in B \to R(t)$ is studied in [7].

5 Approximation condition.

For any Riemann surface R we can define the L_1-(L_0-)principal function $p(z)(q(z))$ and the harmonic span s for (R, a, b) by the standard approximation argument (see Chap. III in [1]). Using the idea in the 3rd case in the above proof we generalize 2. in Corollary 4.1 as Lemma 5.2.

Let $\mathcal{R} = \cup_{t\in B}(t, R(t))$ be a two-dimensional **Stein** manifold such that each fiber $R(t)$ is irreducible and non-singular in $\mathcal{R}$. Due to Oka-Grauert, $\mathcal{R}$ admits a C^ω strictly plurisubharmonic exhaustion function $\psi(t,z)$. Then we can find an increasing sequence $a_n, n = 1, 2, \ldots$ which tends to ∞ such that, if we put $\mathcal{R}_n := \{\psi(t,z) < a_n\} = \cup_{t\in B}(t, R_n(t))$ where $R_n(t) = \{z \in R(t) : \psi(t,z) < a_n\}$ which consists of a finite number of connected components $\{R'_n(t), \ldots, R_n^{(q)}(t)\}$ (q may depends on t), then

i) $\partial\mathcal{R}_n$ **is a** C^ω **smooth real 3-dimensional surface in** $\mathcal{R}$ **(which does not always induce** C^ω **smoothness of each** $\partial R_n(t), t \in B$**);**

ii) for an arbitrarily fixed $B_0 \Subset B$ there exists a finite number of C^ω smooth arcs $\ell_k, k = 1, \ldots, \mu$ in B_0 which may have a finite number of intersection points $\{t_1, \ldots, t_\tau\}$ such that

a) for any fixed $t^* \in [\cup_{k=1}^{\mu}\ell_k] \setminus \{t_j\}_{j=}^{\tau}$, we find a small disk $B^* \Subset B_0$ centereed at t^* such that the arc ℓ_k passing through t^* divides B^* into two connected parts B' and B'' such that $\partial R_n(t), t \in B' \cup B''$ consists of a finite number of C^ω smooth closed curves in $R(t)$;

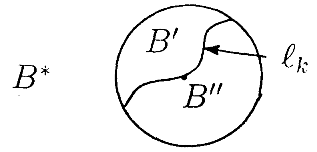
b) any $\partial R_n(t), t \in \ell_k \cap B^*$ is not C^ω smooth in $R(t)$ but it is C^ω smooth except one corner point at which two closed curves transversally itersect (see figures (FI), (FII) below);

c) any $\partial R_n(t_j), j = 1, \ldots, \tau$ is not C^ω smooth in $R(t)$ but it is C^ω smooth except two corner points at which two closed curves transversally itersect.

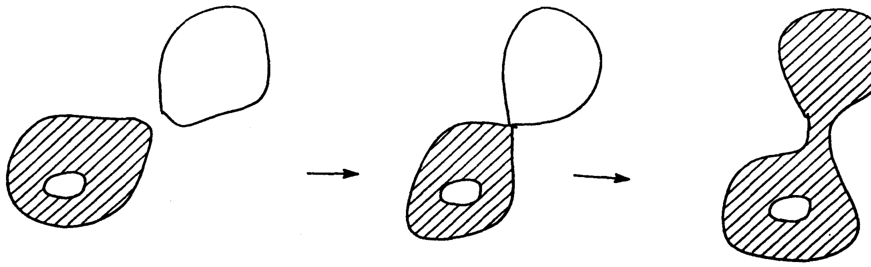
We further assume that each fiber $R(t), t \in B$ is **planar** as Riemann surface, whose connectivity may be ∞. Let ξ, η be two holomorphic sections of $\mathcal{R}$ over B. Fix a disk $B_0 \Subset B$ centered at 0. If we take $N \gg 1$, then each $\xi(t)$, $\eta(t)$, $t \in B_0$ are contained in a connected component of $R(t)$, say, $R'_n(t)$, for $n \geq N$. Thus $R'_n(t) \Subset R'_{n+1}(t)$ and $\lim_{n\to\infty} R'_n(t) = R(t)$. We put $\mathcal{R}'_n = \cup_{t\in B_0}(t, R'_n(t))$. On each $R'_n(t), t \in B_0$ we have the L_1-(L_0-)principal function $p_n(t,z)(q_n(t,z))$; L_1-(L_0-)constant $\alpha_n(t)(\beta_n(t))$ and the harmonic span $s_n(t)$ for $(R'_n(t), \xi(t), \eta(t))$. We use the same notations for $i), ii)$ for $\mathcal{R}_n$ and put

$$B_0^{(1)} = B_0 \setminus [\cup_{k=1}^{\mu}\ell_k] \quad \text{and} \quad B_0^{(2)} = B_0 \setminus \{t_j\}_j^{\tau},$$

where ℓ_k, μ, t_j, τ depend on n. As studied in section 2, $p_n(t,z)$ and $q_n(t,z)$ are of class C^ω for (t,z) in $\mathcal{R}'_n|_{B_0^{(1)}}$, and $s_n(t)$ is C^ω subharmonic on $B_0^{(1)}$.

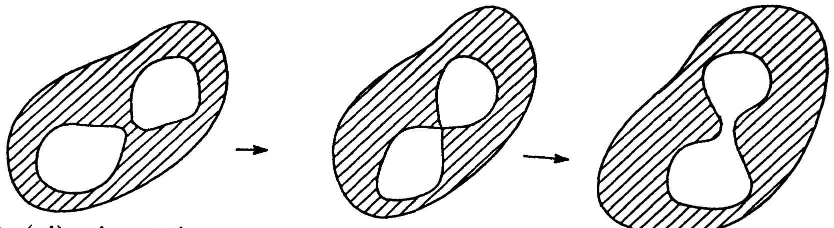


(FI)



$R_n(t'), t' \in B'$ $\qquad$ $R_n(t), t \in \ell_k$ $\qquad$ $R_n(t''), t'' \in B''$

(FII)



$R_n(t'), t' \in B'$ $\qquad$ $R_n(t), t \in \ell_k$ $\qquad$ $R_n(t''), t'' \in B''$

By use of the normal family arguement for the univalent functions we easily see that $p_n(t,z)$ and $q_n(t,z)$ are continuous for (t,z) in $\mathcal{R}'_n$, and hence $s_n(t)$ is continuous on B_0. Improving the proof of Lemma 4.1 in [10] for the variation of the Robin constants, S. Hamano [6] proves the following usuful

Lemma 5.1. *Under the above conditions and notations, if the connectivity of $R'_n(t)$ does not depend on $t \in B_0$ (see the shadowed figures in (FI)), then $p_n(t,z), q_n(t,z)$ are of class C^1 for (t,z) in $\mathcal{R}'|_{B_0^{(2)}}$ and $s_n(t)$ is of class C^1 on $B_0^{(2)}$. Thus, $s_n(t)$ is C^1 subharminic on $B_0^{(2)}$ and is continuous subharmonic on B_0. The converse is also true,i.e., if the connectivity for $R'_n(t)$ does depend on $t \in B_0$ (see the shadowed figures in (FII)), then neither $p_n(t,z)$ nor $q_n(t,z)$ is of class C^1 on B_0, and $s_n(t)$ is not subharmonic on B_0.*

This lemma combined Theorem 4.1 implies the following approximation

Lemma 5.2. *Let $\mathcal{R} = \cup_{t\in B}(t, R(t))$ be a two-dimensional Stein manifold such that each fiber $R(t), t \in B$ is irreducible, non-singular in $\mathcal{R}$ and planar, and let ξ, η be holomorphic sections of $\mathcal{R}$ over B. Assume that there exists a sequence of domains $\mathcal{R}_n = \cup_{t\in B}(t, R_n(t))$ of $\mathcal{R}$ such that*

(i) $\xi, \eta \subset \mathcal{R}_1$; $\mathcal{R}_n \subset \mathcal{R}_{n+1}$ and $\mathcal{R}_n \to \mathcal{R}$ $(n \to \infty)$;

(ii) *each $\mathcal{R}_n$, $n = 1, 2, \ldots$ is pseudoconvex in $\mathcal{R}$;*

(iii) *the connectivity of the connected component $R'_n(t)$ of $R_n(t)$ which contains $\xi(t)$ and $\eta(t)$ is finite and does not depend on $t \in B$ (but may depend on n).*

Then we have

1. *the harmonic span $s(t)$ for $(R(t), \xi(t), \eta(t))$ is subharmonic on B;*

2. *if $s(t)$ is harmonic on B, then $\mathcal{R}$ is simultaneously uniformizable to a univalent domain $\mathcal{D}$ in $B \times \mathbb{P}$ by the circular slit mapping: $(t, z) \in \mathcal{R} \to (t, w) = (t, P(t, z)) \in \mathcal{D}$;*

3. *if $s(t)$ is harmonic on B and if each $R(t), t \in B$ is conformally equivalent to a domain bounded by ν contours, where ν does not depend on $t \in B$, then $\mathcal{R}$ is equivalent to the trivial variation.*

It is known in [13] that a planar Riemann surface R is of class O_{AD}, i.e., there exists no non-constant holomorphic function with finite Dirichlet integral, if and only if the harmonic span $s(R, a, b)$ for (R, a, b) for some $a \neq b$ is equal to zero. The lemma imlpies the following fact: *Under the same conditions as in Lemma 5.2, if the set* $\mathrm{e} = \{t \in B :$ *$R(t)$ is of class $O_{AD}\}$ is of positive logarithmic capacity in $\mathbb{C}_t$, then* $\mathrm{e} = B$ *and $\mathcal{R}$ is uniformaizable to a domain in $B \times \mathbb{P}_w$.* We do not know if this fact is true or not without condition that the connectivity of $R'(t)$ does not depend on $t \in B$ in (iii) in Lemma 5.2.

6 Variations of lengths of Poincaré geodesic curves.

We consider the following variation of general Riemann surfaces: Let $B = \{|t| < \rho\}$ be a disk and $(\mathcal{R}, \pi, B)$ be a holomorphic family such that $\mathcal{R}$ is a two-dimensional manifold; π is a holomorphic projection from $\mathcal{R}$ onto B. The Riemann surface $R(t) = \pi^{-1}(t), t \in B$ may be of genus ∞ and of infinite many ideal boundary components, and the variation $\mathcal{R} : t \in B \to R(t)$ may not be topological trivial. In case $R(t)$ is hyperbolic, $R(t)$ admits the Poincaré metric $ds(t, z)^2$. Given a smooth curve $\gamma(t) \Subset R(t)$ we denote by $l_\gamma(t)$ the Poincaré length of $\gamma(t)$, i.e., $l_\gamma(t) = \int_{\gamma(t)} ds(t, z)$, and define

$$L_\gamma(t) = \log\ \cosh l_\gamma(t).$$

We call $L_\gamma(t)$ the *modified Poincaré length* of $\gamma(t)$ on $R(t)$. In case the universal covering surface of $R(t)$ is conformally equvalent to $\mathbb{C}$, we define that the Poincaré length and hence the modified Poincaré length for any smooth curve $\gamma(t) \Subset R(t)$ is always 0.

Theorem 6.1. *Let $\mathcal{R} = \cup_{t\in B}(t, R(t))$ be a two-dimensional manifold such that each $R(t) = \pi^{-1}(t)$, $t \in B$ is irreducible and non-singular in $\mathcal{R}$. Assume*

(i) *$\mathcal{R}$ is a Stein manifold such that at least one fiber $R(t)$ is hyperbolic; or*

(ii) *each fiber $R(t), t \in B$ is a compact Riemann surface of genus $g \geq 2$.*

Let ξ, η be holomorphic sections of $\mathcal{R}$ over B. For a fixed $t \in B$ let $\Gamma(t)$ be a continuous curve starting at $\xi(t)$ and terminating at $\eta(t)$ in $R(t)$ and put $\Gamma := \cup_{t\in B}(t, \Gamma(t)) \subset \mathcal{R}$. Assume that Γ is homeomorphic to the product set $B \times \Gamma(0)$ by a fiber preseving mapping. For each $t \in B$ we denote by $\gamma(t)$ the Poincaré geodesic curve connectiong $\xi(t)$ and $\eta(t)$ which is homotopic to $\Gamma(t)$ in $R(t)$. Then the modified Poincaré length $L_\gamma(t)$ is subharmonic on B.

Remark 6.1. (1) If $R(t), t \in B$ is simply connected, then the geodesic curve $\gamma(t)$ connecting $\xi(t)$ and $\eta(t)$ on $R(t)$ is unique and $l_\gamma(t)$ is equal to the Poincaré distance $d(t)$ between $\xi(t)$ and $\eta(t)$ on $R(t)$. We call $\delta(t) = \log\cosh d(t)$, the modified Poincaré distance between $\xi(t)$ and $\eta(t)$ on $R(t)$.

(2) Even if $\gamma := \cup_{t\in B}(t, \gamma(t))$ satisfies the above condition in $\mathcal{R}$, the variation $t \in B \to \gamma(t) \subset R(t)$ does not vary continuously in $\mathcal{R}$ with parameter $t \in B$, in general.

The main part of the proof of Theorem 6.1 (togehter with 3. in Lemma 3.1 and Lemma 5.2) is to prove it for the following special $\mathcal{R}$, say $\mathcal{R}_0$: Let D be an unramified domain over $\mathbb{C}_z$ and let ξ, η be holomorphic sections of the product space $B \times \mathbb{C}_z$ over B. Assume that there exists a C^ω smooth strictly plurisubharmonic function $\psi(t, z)$ on $B \times D$ such that $\lim_{(t,z)\to B\times\partial D} \psi(t, z) > \exists m > 0$ and $\widehat{\mathcal{R}}_0 := \{\psi(t, z) < 0\}$ contains ξ, η. We put $\widehat{\mathcal{R}}_0 = \cup_{t,z}(t, \widehat{R}_0(t))$ where $\widehat{R}_0(t) = \{z \in D : \psi(t, z) < 0\}$. For each $t \in B$ consider the connected component $R_0(t)$ of $\widehat{R}_0(t)$ which contains $\xi(t), \eta(t)$. Then the special $\mathcal{R}_0$ is defined by $\mathcal{R}_0 = \cup_{t\in B}(t, R_0(t))$ (see the shadowed figures in (FII)).

This special case is proved by use of the following fact which is based on Theorem III in K. Oka [14] (cf: Lemma 2 in T. Nishino [12]): For each $t \in B$ we construct the universal covering surface $\widetilde{R}_0(t)$ of $R_0(t)$ based on the point $[\xi(t), l]$ where l is a closed curve starting at $\xi(t)$ and returning to $\xi(t)$ on $R_0(t)$ which is homotopic to 0 on $R_0(t)$. If we gather them to obtain $\widetilde{\mathcal{R}}_0 := \cup_{t\in B}(t, \widetilde{R}_0(t))$, then $\widetilde{\mathcal{R}}_0$ becomes a two-dimensional Stein manifold.

In [4] and [11] we find some results related to Theorem 6.1.

Corollary 6.1. *Let $\pi : \mathcal{R} \to S$ be a holomorphic family of compact Riemann surfaces $R(t) = \pi^{-1}(t)$ over a compact Riemann surface S such that each $R(t), t \in S$ is irreducible and of genus ≥ 2. If $\mathcal{R}$ is not equivalent to the trivial variation, then there exist no two holomorphic sections ξ, η of $\mathcal{R}$ over S such that we have a continuous curve $\Gamma(t)$ connecting $\xi(t)$ and $\eta(t)$ on $R(t)$ such that $\Gamma = \cup_{t\in B}(t, \Gamma(t))$ ($\subset \mathcal{R}$) is homeomorphic to $B \times \Gamma(t_0)$, where t_0 is a fixed point in S.*

In fact, assume that there exists two distinct holomorphic sections ξ, η of $\mathcal{R}$ over S satisfying the conditions in the corollary. For $t \in S$ we consider the Poincaré geodesic curve $\gamma(t)$ which is homotopic to $\Gamma(t)$ on $R(t)$. Let e be the finite point set of $t \in S$ such that $R(t)$ is singular in $\mathcal{R}$. We put $S' = S \setminus e$. For $t \in S'$, we denote by $L_\gamma(t)$ the Poincaré modified length of $\gamma(t)$ on $R(t)$. By **(ii)** in Theorem 6.1, $L_\gamma(t)$ is subharmonic on S'. Since S is compact, $L_\gamma(t)$ is extended to be subharmonic on S, so that $L_\gamma(t)$ is constant for $t \in S$, say $L_\gamma(t) \equiv a > 0$ on S. For each $t \in S$ we consider the universal covering surface $\widetilde{R}(t)$ of $R(t)$ based on the point $\mathfrak{o}(t) := [\xi(t), l]$ where l is a closed curve starting at $\xi(t)$ and returning to $\xi(t)$ on $R(t)$ which is homotopic 0, and put $\eta_0(t) = [\eta(t), L(t)] \in \widetilde{R}(t)$, so that $\eta_0 : t \in S \to \eta_0(t) \in \widetilde{R}(t)$ is a holomorphic section of $\widetilde{\mathcal{R}}$ over S. Then the harmonic span $s(t)$ for $(\widetilde{R}(t), \mathfrak{o}(t), \eta_0(t))$ is equal to $L_\gamma(t)$, so that $s(t) \equiv a$ on S. We apply 3. in Lemma 5.2 for the special case $\nu = 1$ and obtain $\widetilde{\mathcal{R}} \approx S \times \Delta$ where $\Delta = \{|W| < 1\}$ in $\mathbb{C}_W$ by a fiber preserving mapping. This implies without difficulty that $\mathcal{R} \approx S \times R(t_0)$, where t_0 is a fixed point in S. This contradicts with the assumption of the corollary.

Remark 6.2. (1) As a particular case of Corollary 6.1 we have the following: *Let $\pi : \mathcal{R} \to S$ be a holomorphic family of compact Riemann surfaces $R(t) = \pi^{-1}(t)$ over a compact Riemann surface S such that each $R(t), t \in S$ is irreducible, non-singular in $\mathcal{R}$ and of genus ≥ 2. Let $\xi : t \in S \to \xi(t) \in R(t)$ be a holomorphic section of $\mathcal{R}$ over S and let $D_\xi(t)$ be the largest Poincaré disk of center $\xi(t)$ on $R(t)$ for each $t \in S$. Then there exists no other holomorphic section $\eta : t \in S \to \eta(t) \in R(t)$ such that $\eta(t) \in D_\xi(t),\ t \in S$.*

(2) Assertion (1) with the elementary normal family argument immediately implies the following famous theorem: *Let $\pi : \mathcal{R} \to S$ be the same as in* (1). *Then there exists no infinite many holomorphic sections of $\mathcal{R}$ over S.*

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