Hamilton-Jacobi equations and Euclidean Sobolev inequality

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1 Introduction

The result of this note is a special case of [3], and the readers should refer to it for more detailed results and their proofs.

Let $\Omega$ be a bounded and Lebesgue measurable set in $\mathbb{R}^n$. Let $0 < \alpha < \beta < \infty$. Then, as is well-known, the following inequality holds:

(1.1) \[ |\Omega|^{-1/\alpha} \| f \|_{\alpha, \Omega} \leq |\Omega|^{-1/\beta} \| f \|_{\beta, \Omega} \leq \| f \|_{\infty, \Omega}, \quad f \in L^\infty(\Omega) \]

where $|\Omega|$ is the Lebesgue measure of $\Omega$ and $\| \cdot \|_{\beta, \Omega}$ is the $L^\beta(\Omega)$-norm ($0 < \beta < \infty$) with respect to the Lebesgue measure in $\mathbb{R}^n$. Furthermore, this inequality is optimal in the sense that all inequalities in (1.1) are reduced to equalities when $f$ is a constant function on $\Omega$. This inequality show a norm-monotone property of $\{ |\Omega|^{-1/\beta} \| f \|_{\beta, \Omega} \}_{0 < \alpha < \infty}$.

However, as far as we know, there is no inequality corresponding to (1.1) when a bounded and Lebesgue measurable set $\Omega$ in $\mathbb{R}^n$ is replaced by the whole domain $\mathbb{R}^n$. A reason for it is that when $\Omega = \mathbb{R}^n$, we have $|\Omega|^{-1/\beta} = 0$ for all $0 < \beta < \infty$.

The goal of this note is to provide an inequality corresponding to (1.1) when a bounded and Lebesgue measurable set $\Omega$ in $\mathbb{R}^n$ is replaced by the whole domain $\mathbb{R}^n$. This inequality is obtained by using the Euclidean logarithmic Sobolev inequality and Hamilton-Jacobi equations. We use the inequalities obtained by [4, 5], and minimize this inequality with respect to some parameter, and finally get the desired inequality by letting another parameter tend to $\infty$.

2 Preliminaries

In this section, we collect some results of [4, 5]. For $p \geq 1$, we denote by $W^{1,p}(\mathbb{R}^n)$ the space of all weakly differentiable functions $f$ on $\mathbb{R}^n$ such that $f$ and $|Df|$ are in $L^p(\mathbb{R}^n)$. Throughout this note, the integral without its domain is understood as the one over $\mathbb{R}^n$.
Lemma 2.1  Let $p \geq 1$. Then, we have the following Euclidean logarithmic Sobolev inequality:

\[(2.1) \int |f|^p \log |f|^p \, dx \leq \frac{n}{p} \log \left( L_p \int |Df|^p \, dx \right) \text{ for } f \in W^{1,p}(\mathbb{R}^n) \text{ with } \int |f|^p \, dx = 1. \]

Here,

\[(2.2) \quad L_p = \frac{p}{n} \left( \frac{p-1}{e} \right)^{p-1} \pi^{-p/2} \left( \frac{\Gamma \left( \frac{n}{2} + 1 \right)}{\Gamma \left( \frac{n}{p} - \frac{1}{2} + 1 \right)} \right)^{p/n}, \]

and this is the best possible constant satisfying (2.1).

We denote by $\| \cdot \|_\alpha$ the $L^\alpha(\mathbb{R}^n)$-norm with respect to the Lebesgue measure in $\mathbb{R}^n$.

Lemma 2.2  Let $p > 1$. For $f \in \text{Lip}(\mathbb{R}^n)$, let $u \in \text{Lip}(\mathbb{R}^n \times [0, \infty))$ be a viscosity subsolution of the Hamilton-Jacobi equation

\[(2.3) \quad u_t(x, t) + \frac{1}{p} |Du(x, t)|^p = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty), \quad u = f \quad \text{on } \mathbb{R}^n \times \{0\}. \]

If there is a constant $\alpha > 0$ such that $e^f \in L^\alpha(\mathbb{R}^n)$, then $e^{u(\cdot, t)} \in L^\beta(\mathbb{R}^n)$ for any $\beta \in (\alpha, \infty)$ and $t \in (0, \infty)$. Furthermore, we have

\[(2.4) \quad \|e^{u(\cdot, t)}\|_\beta \leq \|e^f\|_\alpha \left( \frac{nL_p e^{p-1}(\beta - \alpha)}{p^t} \right)^{\frac{\beta - \alpha}{p}} \frac{\Gamma \left( \frac{n}{p} - \frac{1}{2} + 1 \right)^{p/2}}{\Gamma \left( \frac{n}{p} - \frac{1}{2} + 1 \right)^{p/2}}, \quad t > 0, \]

where $q > 1$ is the exponent conjugate of $p$, i.e., $(1/p) + (1/q) = 1$.

3  A result

Let $\theta > 0$. For $\alpha > 0$, we set

\[(3.1) \quad \mathcal{L}_{\alpha, \theta} = \left\{ f \in \text{Lip}(\mathbb{R}^n) : \text{Lip}(f) \leq \theta, \ e^f \in L^\alpha(\mathbb{R}^n) \right\}, \]

where $\text{Lip}(f)$ is the Lipschitz constant of $f$, i.e., $\text{Lip}(f) = \sup_{x \neq y} |f(x) - f(y)|/|x - y|$. Let us denote by $\omega_{n-1}$ the surface area of the unit ball in $\mathbb{R}^n$. We set

\[(3.2) \quad k_n = \left( \frac{1}{\omega_{n-1}(n-1)!} \right)^{1/n}. \]

Now, we state our result of this note and give a sketch of its proof.

Theorem 3.1  Let $\alpha, \theta > 0$. For $f \in \mathcal{L}_{\alpha, \theta}$, we have the following inequality:

\[(3.3) \quad \|e^f\|_\infty \leq \|e^f\|_\beta \left( k_n \theta \beta \right)^{n/\beta} \leq \|e^f\|_\alpha \left( k_n \theta \alpha \right)^{n/\alpha}, \quad \alpha \leq \beta \leq \infty. \]

Inequality (3.3) is optimal in the sense that equality holds when $f(x) = C - \theta |x|$ for some constant $C \in \mathbb{R}$. 

Remark. Note that \( \lim_{\beta \to \infty} (k_n \theta \beta)^{n/\beta} = 1 \). Hence, the family 
\[ \{ \| e^f \|_\beta \left( k_n \theta \beta \right)^{n/\beta} \}_{\alpha < \beta < \infty} \]
interpolates continuously and monotonically between 
\[ \| e^f \|_\alpha \left( k_n \theta \alpha \right)^{n/\alpha} \quad \text{and} \quad \| e^f \|_\infty. \]

Sketch of Proof. Let \( f \in L_{\alpha, \theta} \). Then, the function \( v(x, t) = f(x) - (\theta^p t / p) \) is a subsolution of (2.3), so that \( v \leq u \) on \( \mathbb{R}^n \times [0, \infty) \) by [7]. By Lemma 2.2, we have, for any \( \beta \in (\alpha, \infty) \) and \( t \in (0, \infty) \),
\[
(3.4) \quad \| e^f \|_\beta \leq \| e^f \|_\alpha \ e^{p t / p} \ t^{- \frac{n \beta - \alpha}{p \alpha}} \times \left( \frac{n L_p e^{p-1}(\beta - \alpha)}{p p} \right)^{\frac{n \beta - \alpha}{p \alpha}} \left( \frac{\alpha^{(\frac{n}{p} + \frac{\beta}{q})}}{\beta^{(\frac{n}{p} + \frac{\alpha}{q})}} \right), \quad t > 0,
\]
where \( q > 1 \) is the exponent conjugate of \( p \), i.e., \( (1/p) + (1/q) = 1 \). By minimizing the right-hand side of (3.4) with respect to the \( t \)-variable, we have
\[
(3.5) \quad \| e^f \|_\beta \leq \| e^f \|_\alpha \ e^{ \left( \frac{\theta e L_{p}^{1/p}}{p} \right)^n \ t^{- \frac{n \beta - \alpha}{p \alpha}} \times \left( \frac{n L_p e^{p-1}(\beta - \alpha)}{p p} \right)^{\frac{n \beta - \alpha}{p \alpha}} \left( \frac{\alpha^{(\frac{n}{p} + \frac{\beta}{q})}}{\beta^{(\frac{n}{p} + \frac{\alpha}{q})}} \right). \quad t > 0,
\]
Hence, we obtain
\[
(3.6) \quad \| e^f \|_\beta \left( k_p^{(n)} \theta \beta \right)^{n/\beta} \leq \| e^f \|_\alpha \left( k_p^{(n)} \theta \alpha \right)^{n/\alpha},
\]
where
\[
(3.7) \quad k_p^{(n)} = \frac{e L_{p}^{1/p}}{p}
\]


