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On an Elementary Approach to Optimal Control with Neumann Boundary Condition

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1 Introduction

This paper is a further study of our previous work [10], which provides a discrete game interpretation for the Neumann boundary problem of motion by curvature. The deterministic optimal control approach is proposed by Kohn and Serfaty ([16, 17]) to connect two-person games and second order PDEs. It turns out that by the convergence argument, a time-optimal problem is related to the Dirichlet problem of an elliptic equation and a time-dependent game corresponds to the Cauchy problem of a parabolic equation; see [9, 15] for generalizations in distinct directions. Our goal rests on the Neumann boundary problems of evolutionary equations. In [10], discrete deterministic games are constructed so that their value functions converge to the unique solution of the Neumann boundary problem of curve shortening flow equation. The introduction of a billiard semiflow is of most importance for the game setting. We here aim to present more applications of billiards and the optimal control or game representations.

The billiard semiflow stems from a very simple law: the angle of incidence equals to the angle of reflection. All of our optimal control or games are then established on this dynamics. It is worth mentioning that there is another more classical way of generating Neumann boundary condition called the Skorokhod problem, which is first used by Lions [18] to pioneer the study of the boundary conditions in the viscosity sense; consult [5, 6, 19, 23] for details about the Skorokhod problem. Although the definition of the billiard semiflow is different from that of the Skorokhod type reflection, they are essentially not that different in the sense that billiard and Skorokhod reflections are still analogous in form (Lemma 2.2 or [10, Lemma 2.3]), which meanwhile makes our arguments more reasonable and easier to understand.

There is a problem left in [10] about the continuity of the billiard semiflow. The stable nature of a state equation is usually indispensable to show that the value function is continuous [1]. The comparison principle certainly guarantees the continuity, but its derivation from games was missing there. We will complement it in this paper by putting the billiards into a simpler background. More precisely, our concentration is focused on the following first order linear equation

$$(1.1)\qquad \begin{cases} u_t(x,t) - v\cdot\nabla u = 0 & \text{in } \Omega\times[0,\infty),\\ \nabla u(x,t)\cdot\nu(x) = 0 & \text{on } \partial\Omega\times[0,\infty),\\ u(x,0) = u_0(x) & \text{in } \overline{\Omega}, \end{cases}$$

where Ω is a smooth domain in $\mathbb{R}^2$ with the unit outward ν, and $v\in \mathbf{S}^1$ and $u_0\in C(\overline{\Omega})$ are given. Then the solution could be expressed as a composite of the billiard dynamics and u_0,

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and thus the sufficient information concerning (1.1) allows us to get the regularity of billiards desired.

Another application of billiards is to explain the fattening of level sets for the mean curvature flow equation with boundary condition. The method is as follows: We compare the game and its inverse, which are both provided in [10]; namely, we study not only the "inf sup" games but also replace all "inf sup" by "sup inf." These changes formally make little difference to the original arguments and thus still yield the same equation. However, we here must emphasize that the optimal trajectories in the original and inverse games can be entirely different. An example in the present work will be given to show such features clearly. It is worth remarking that we employ only the game interpretation without using any parabolic PDE theory, which is usually resorted to when one tries to prove the existence of fattening rigorously. Our computation for the example is quite geometric and elementary. All of the PDE theory we are supposed to use is actually implied by the convergence argument in [10]. Such kind of explanation is not new if we recall the optimal control interpretation of first order fattening due to Barles, Soner and Souganidis [3]. Refer to the forthcoming work [20] for a more general idea and more applications.

We will not discuss much about the existence, uniqueness and stability for solutions of the equations appearing in the following, especially those with Neuamnn type boundary. Their well-posedness in the viscosity sense is investigated thoroughly in [18] for first order cases and [8, 11, 21, 22] for second order ones. Also, we hereafter consider everything only in two dimensions. The generalization to higher dimensions is possible.

2 Planar Billiard Dynamics

We begin with a review of the results about the billiard semiflow. All of the proofs, omitted in this paper, are given in [10]. The domain $\Omega \subset \mathbb{R}^2$, said to be a *billiard table* in the new context, satisfies the following assumption:

(A1) Ω is a bounded and convex domain in $\mathbb{R}^2$ with C^2 boundary.

The billiard flow in Ω, denoted by $T^t : \overline{\Omega} \times \mathbf{S}^1 \to \overline{\Omega}$ $(t \in \mathbb{R})$, describes the billiard motion in the table. By billiard motion, we mean that a mass point is moving along straight-lines in the interior of the domain and following the optic law on the boundary, namely, the angle of incidence equals the angle of reflection. For a fixed pair (x, v), $T^t(x, v)$ represents the ball's position at time t. The set $\{T^t(x, v) \in \overline{\Omega} : t \geq 0\}$ is called a *billiard trajectory* starting from (x, v) and the hitting points on the boundary are called *vertices* of the trajectory. It is obvious that T^t satisfies the group property restricted in $\Omega \times \mathbf{S}^1$ with the identity T^0 and $T^{-t}(x, v) = T^t(x, -v)$ for any $x \in \Omega$ and $v \in \mathbf{S}^1$.

We stress here that such a billiard motion is not always proper. Indeed, a so-called *terminating* phenomenon may occur even in this C^2 domain, or in other words, the sequence of vertices $\{p_n\}_{n \geq 1}$ may converge to a point on $\partial\Omega$. An important property is drawn from [13] to be stated in Lemma 2.1 below.

We hereafter utilize the arc-length parametrization $\Gamma(\cdot) : \mathbb{R} \to \mathbb{R}^2$, a function of class C^2, to represent $\partial\Omega$. Its derivative with respect to s is denoted by Γ_s.

Lemma 2.1. *Suppose that Ω satisfies* (A1). *If a trajectory terminates at a point $\Gamma(s_\infty) \in \partial\Omega$, with a sequence of vertices $\{\Gamma(s_n)\}_{n \geq 1}$ arranged in order, then there exists $N > 0$ such that for $n \geq N$, s_n monotonically converges to s_∞ and $(\Gamma(s_\infty) - \Gamma(s_n))/|s_\infty - s_n|$ converges to a unit tangent, denoted by v_∞, to the boundary at $\Gamma(s_\infty)$.*

We refer readers to [12] for a similar result in higher dimensions and to [4], where the termination is rephrased as *accumulation of collision times*, for general discussions.

A modified billiard dynamics is therefore necessary and can be presented as follows.

Definition 2.1. Let Ω satisfy (A1).

(i) If $x \in \partial\Omega$, and v equals to the tangent of $\partial\Omega$, then

$$S^t(x,v) := \Gamma(t), \quad \text{for any } t \geq 0,$$

where $\Gamma(\cdot)$ is the arc-length parametrization of $\partial\Omega$ such that $\Gamma(0) = x$ and $\Gamma_s(0) = v$;

(ii) If $x \in \Omega$ and v is such that $T^t(x,v)$ terminates on $\partial\Omega$ at time t_0, then

$$S^t(x,v) := \begin{cases} T^t(x,v) & \text{if } 0 \leq t < t_0, \\ S^{t-t_0}(T^{t_0}(x,v), v_\infty) & \text{if } t \geq t_0, \end{cases}$$

where v_∞ is obtained from Lemma 2.1;

(iii) If $x \in \partial\Omega$ and v points inside Ω, then

$$S^t(x,v) := \begin{cases} x & \text{if } t = 0, \\ S^{t-\varepsilon}(x + \varepsilon v, v) & \text{if } t > 0, \end{cases}$$

where $\varepsilon > 0$ is such that $x + \delta v \in \Omega$ for all $\delta \in (0, \varepsilon)$.

It is easily seen that S^t is a semiflow. For $t \geq 0$, $x \in \overline{\Omega}$ and $v \in \mathbf{S}^1$, we set

$$\alpha^t(x,v) = x + tv - S^t(x,v) \tag{2.1}$$

and call it the *boundary adjustor*. A property of our semiflow is given in the next lemma.

Lemma 2.2 ([10, Lemma 2.3]). *Assume that Ω satisfies* (A1). *For any fixed $t \geq 0$, $x \in \overline{\Omega}$ and $v \in \mathbf{S}^1$, let $\alpha^t(x,v)$ be the boundary adjustor of $S^t(x,v)$. Then there exist $d_l \geq 0$ and $y_l \in \partial\Omega \cap B_t(x)$, $l = 1, 2, \ldots$ such that*

$$\alpha^t(x,v) = \sum_{l=0}^{\infty} d_l \nu(y_l), \tag{2.2}$$

where the convergence on the right hand side is in $\mathbb{R}^2$. In addition, the following estimates hold:

$$\left|\alpha^t(x,v)\right| \leq 2t. \tag{2.3}$$

$$\left|\sum_{l=k}^{\infty} d_l \nu(y_l)\right| \leq 4t, \ \textit{for all } k = 1, 2, \ldots \tag{2.4}$$

$$\sum_{l=1}^{\infty} |y_{l+1} - y_l| \leq 2t. \tag{2.5}$$

This lemma tells us that the effect of billiard reflection is nothing but a series of inward normal impacts. Such an observation, resembling the Skorokhod problem, turns out to play a significant role in our game setting.

We conclude this section with another property, which is a direct consequence of the separation theorem for convex sets in $\mathbb{R}^2$.

Lemma 2.3 ([10, Lemma 2.4]). *Assume that* Ω *satisfies* (A1). *Then*

$$|x_0 - S^t(x,v)| \leq |x_0 - (x+tv)| \text{ for any } x, x_0 \in \overline{\Omega},\ v \in \mathbf{S}^1 \text{ and } t \geq 0. \tag{2.6}$$

This lemma is used in second order games to construct barriers for initial data. We discuss in this paper only a convex domain. For more general domains, we need a few additional techniques since the above lemma no longer holds. See [10] for further study.

Another question is about the continuity of this semiflow. It turns out that the the function $\overline{\Omega} \times \mathbf{S}^1 \ni (x,v) \to S^(x,v)$ is not continuous in general, especially when termination occurs in the trajectory. However, in the next section, we can get separately the continuity of $S^(x,v)$ in $(x,t) \in \overline{\Omega} \times [0,\infty)$ and in $v \in \mathbf{S}^1$ by characterizing the billiard dynamics in the setting of a linear partial differential equation with the homogeneous Neumann boundary condition.

3 Linear Neumann Problem

We start with a most simple case to see how billiards can help us realize a Neumann type boundary. Our equation is simply (1.1), whose well-posedness is of no problem and covered in [18]. Letting $u(x,t) = u_0(S^t(x,v))$, we intend to show

Theorem 3.1. *u is the unique viscosity solution of* (1.1).

The dynamic programming principle in this case becomes degenerate and obvious.

Proposition 3.2 (Dynamic programming). *For all* $(x,t) \in \overline{\Omega} \times [0,\infty)$, $u(x,t) = u(S^\tau(x,v), t-\tau)$ *whenever* $0 \leq \tau \leq t$.

We use this proposition to prove Theorem 3.1.

Proof of Theorem 3.1. It is clear by definition that u is bounded and $u(x,0) = u_0(x)$. We next prove u is a subsolution of the equation with Neumann boundary condition. Assume that there exist $(x_0,t_0) \in \overline{\Omega} \times (0,\infty)$ and $\phi \in C^1(\overline{\Omega})$ such that

$$(u-\phi)(x_0,t_0) = \max_{\overline{\Omega} \times (0,\infty)} (u-\phi),$$

which means

$$u(x_0,t_0) - \phi(x_0,t_0) \geq u(x,t) - \phi(x,t) \quad \text{for all } x,t \in \overline{\Omega} \times (0,\infty).$$

For each $0 < \tau \leq t_0$, take $x = S^\tau(x_0,v)$ and $t = t_0 - \tau$. We then use the dynamic programming to obtain

$$\phi(x_0,t_0) - \phi(S^\tau(x_0,v), t_0-\tau) \leq 0.$$

It follows from Taylor expansion and the representation of adjustor (2.2) that

$$\tau\phi_t(x_0,t_0) - \nabla\phi(x_0,t_0) \cdot (v\tau - \alpha^\tau(x_0,v)) \leq o(\tau). \tag{3.1}$$

We discuss all subsequences of α^τ. Due to (2.3), we only need to discuss two cases. If $\alpha^\tau/\tau \to 0$ as $\tau \to 0$, then dividing by τ in (3.1) and sending $\tau \to 0$, we get

$$\phi_t(x_0, t_0) - \nabla\phi(x_0, t_0) \cdot v \le 0.$$

The other case is for the existence of a constant $C > 0$ such that $|\alpha^\tau(x_0, v)|/\tau \to C$ as $\tau \to 0$. Of course x_0 must appear on $\partial\Omega$. We claim in this case that either

$$\phi(x_0, t_0) - \nabla\phi(x_0, t_0) \cdot v \le 0$$

or

$$\nabla\phi(x_0, t_0) \cdot n(x_0) \le 0.$$

Indeed, suppose neither of the above holds, and then by using the regularity of ϕ we can take $\gamma_0 > 0$ satisfying $\nabla\phi(x_0, t_0) \cdot n(y) > \gamma_0$ for all y on the piece of boundary adjacent to x_0, which, together with (2.2), yields

$$\nabla\phi(x_0, t_0) \cdot \frac{\alpha^\tau(x_0, v)}{\tau} = \frac{\sum_{l=0}^{\infty} d_l \gamma_0}{\tau} \ge C\gamma_0,$$

as $\tau \to 0$. This will lead us, if we look back at (3.1), to

$$\phi(x_0, t_0) - \nabla\phi(x_0, t_0) \cdot v \ge C\gamma_0 > 0,$$

which is a contradiction.

Both cases combine to say that u is a subsolution of (1.1) in the viscosity sense. We do not present the verification of supersolution since it is similar. □

The proof of Theorem 3.1 is simple and we next show some of its immediate consequences. Let us first see the regularity of billiard semiflow as we mentioned.

Corollary 3.3. *Assume* (A1). *Fix* $v \in \mathbf{S}^1$. *Then the mapping* $(x, t) \to S^t(x, v)$ *is continuous.*

Proof. We recall the classical result that the solution u is continuous as long as u_0 is continuous. Then $S^t(x, v)$ must be continuous otherwise we can construct u_0 to make u discontinuous. □

Corollary 3.4. *Assume* (A1). *Fix* $(x, t) \in \overline{\Omega} \times [0, \infty)$. *Then the mapping* $v \to S^t(x, v)$ *is continuous.*

Proof. Owing to Theorem 3.1, our straightforward proof is based on the stability theory. □

We remark that the difficulty of the original question on continuity of $S^t(x, v)$ in (x, v) is natural because from a PDE viewpoint we are actually seeking for the continuity of solution with the equation perturbed at the same time. It requires more regularity on the domain than (A1), which is reflected by the counterexample below.

Suppose there is a terminating billiard trajectory in $\overline{\Omega}$ with a sequence of vertices $\{y_n\}_{n=1}^{\infty}$ and a terminating point y_∞ on $\partial\Omega$. From the results we have got before, we know as $k \to \infty$, $a_k = |y_{k+1} - y_k| \to 0$ and $v_k = (y_{k+1} - y_k)/a_k \to v_\infty$, where v_∞ is a unit tangent of $\partial\Omega$ at y_∞. Take a point x on the straight line segment $\overline{y_1 y_2}$ such that the distance $\operatorname{dist}(x, \partial\Omega) > 0$. Moreover, it is possible to pick an open interval (x_1, x_2) on the segment $\overline{y_1 y_2}$, satisfying $x \in (x_1, x_2)$ and

$$\inf_{z \in (x_1, x_2)} \operatorname{dist}(z, \partial\Omega) > 0. \tag{3.2}$$

We assume the total length of the billiard trajectory from x to y_∞ is τ_0. Then it is not hard to see that $S^{\tau_0}(\cdot,\cdot)$ is not continuous at $(y_\infty, -v_\infty)$. Indeed, since $y_\infty \in \partial\Omega$ and $-v_\infty$ is a tangent, we have by our definition $S^{\tau_0}(y_\infty, -v_\infty) \in \partial\Omega$. On the other hand, $S^{\tau_0}(y_k, -v_k) \in (x_1, x_2)$ when k is large. So discontinuity is obtained immediately due to (3.2).

Not only can we obtain the regularity of billiards through the above connection, but we may also look to the reverse, making use of billiards' regularity to study the local regularity of PDE solutions.

Theorem 3.5 ([4, Lemma 2.24]). *Assume Ω is of class C^k and $u_0 \in C^{k-1}$ $(k \geq 3)$. Then the flow T^t is C^{k-1} smooth at points that experience only regular collisions.*

An immediate consequence follows.

Corollary 3.6. *Assume Ω is of class C^k and $u_0 \in C^{k-1}$ $(k \geq 3)$. Let u be the solution of (1.1). Let $x_0 \in \overline{\Omega}$ and $t_0 \in (0,\infty)$. Then the following statements hold:*

(a) *If $x_0 \in \Omega$, then there exists $0 < \delta < t_0$ such that $u \in C^{k-1}(B_\delta(x_0) \times [t_0 - \delta, t_0 + \delta])$.*

(b) *If $x_0 \in \partial\Omega$ and $v \cdot n(x_0) \neq 0$, then there exists $0 < \delta < t_0$ such that*

$$u \in C([t_0 - \delta, t_0 + \delta], C^{k-1}(B_\delta(x_0))).$$

4 Fattening of Neumann boundary problem

A mere representation theorem is far from satisfactory. We attempt to employ billiards to explain the fattening behavior of level set equations with Neumann type boundary. To this end, we turn to second order equations and see that our billiards can also be applied to Neumann boundary problem of curve shortening equation, whose level-set equation is written as follows.

$$(4.1)\qquad \begin{cases} \partial_t u - |\nabla u| \mathrm{div}(\frac{\nabla u}{|\nabla u|}) = 0 & \text{in } \Omega \times (0,T), \\ \nabla u(x,t) \cdot \nu(x) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,T) = u_0(x) & \text{in } \overline{\Omega}. \end{cases}$$

Here Ω denotes a smooth domain in $\mathbb{R}^2$, $\nu(x)$ is the unit outward normal to $\partial\Omega$ at x. For simplicity, we also require Ω to be convex.

The well-posedness of the problem (4.1) is first studied by Giga and Sato [11]; refer also to [14] for its recent development. But the level set of u is known to be fat, or more precisely, develop interior sometimes even where u_0 is correctly chosen; see more analysis in [2, 8].

We reveal that fattening can be studied through a game-theoretic method, with billiards described previously. Reviewing the associated game interpretation proposed in [10], we observe and compare, for a simple example, the distinction between the optimal decisions of players for "min max" and "max min" games, which clarify the formation of fat level sets. Our idea here is extended from a recent work [20] on more detailed game explanation for the fattening of mean curvature flow equation.

Our game starts from an initial position $x \in \overline{\Omega}$ and the time 0. The maturity time given is denoted by t. Let the step size be $\varepsilon > 0$. At each step, a movement of length $\sqrt{2}\varepsilon$ along a billiard trajectory is conducted but time ε^2 is consumed. Then the total number of game steps N can be regarded as $[\frac{t}{\varepsilon^2}]$, where $[a]$ stands for the largest integer less or equal to each real number a. Two players, Paul and Carol participate the game. Paul intends to minimize the value $u_0(y(N))$ while the other, Carol, is to maximize it. To be more precise, in the k-th round,

(1) Paul chooses a direction v_k, i.e., $|v_k| = 1$;
(2) Carol has the right to reverse Paul's choice, which determines $b_k = \pm 1$;
(3) The marker is moved from $y(k-1)$ to $y(k) = S^{\sqrt{2}\varepsilon}(y(k-1), b_k v_k)$.

Then the value functions of the game and its inverse are respectively

$$u_1^\varepsilon(x,t) = \min_{|v_1|=1} \max_{b_1=\pm 1} \dots \min_{|v_N|=1} \max_{b_N=\pm 1} u_0(y(N)) \tag{4.2}$$

and

$$u_2^\varepsilon(x,t) = \max_{|v_1|=1} \min_{b_1=\pm 1} \dots \max_{|v_N|=1} \min_{b_N=\pm 1} u_0(y(N)). \tag{4.3}$$

With the associated dynamic programming equations,

$$u_1^\varepsilon(x,t) = \min_{|v|=1} \max_{b=\pm 1} u_1^\varepsilon(S^{\sqrt{2}\varepsilon}(x, bv), t - \varepsilon^2)$$

and

$$u_2^\varepsilon(x,t) = \max_{|v|=1} \min_{b=\pm 1} u_2^\varepsilon(S^{\sqrt{2}\varepsilon}(x, bv), t - \varepsilon^2),$$

we show that both u_1^ε and u_2^ε converge to the unique solution of (4.1).

Theorem 4.1 ([10]). *Assume that Ω satisfies* (A1). *Let u_0 be a continuous function in $\overline{\Omega}$. Let u_1^ε and u_2^ε be the value functions defined as* (4.2) *and* (4.3) *respectively. Then both u_1^ε and u_2^ε converge, as $\varepsilon \to 0$, to the unique viscosity solution of* (4.1) *uniformly on compact subsets of $\Omega \times (0, \infty)$.*

In [10], only the statement about u_1^ε is proved, but one can actually show the other part by following a symmetric argument. Theorem 4.1 therefore enables us to study the level-set fattening of the Neumman problem. Let us see one example, which is a variant of the one in [2].

Let $\Omega = \{x \in \mathbb{R}^2 : |x| < 2\}$ and then take u_0 as follows.

$$u_0(x) = \min\{\mathrm{dist}(x, \overline{\Omega}) - \mathrm{dist}(x, \Omega^c), 3\}, \quad \text{for all } x \in \mathbb{R}^2. \tag{4.4}$$

It is clear that in this situation $u_1^\varepsilon(x,t) \le 0$ for all $x \in \Omega$ and $t > 0$. On the other hand, the estimate for u_2^ε can be given as follows.

Lemma 4.2. $u_2^\varepsilon(x,t) \ge -\sqrt{2}\varepsilon$, *for all $(x,t) \in \Omega \times (0,\infty)$ such that $|x|^2 + 2t \ge 4$.*

Proof. Paul's only job in this inverse game is to choose his direction each time on the tangent of the concentric circle around $(0,0)$. Then a direct calculation yields that the marker can reach the boundary $\partial\Omega$ by the time $\frac{1}{2}(4 - |x|^2)$. Notice that after the reaching moment, it will keep colliding the boundary $\partial\Omega$ and hence its distance to the boundary will not exceed $\sqrt{2}\varepsilon$, which deduces our conclusion. □

Applying Theorem 4.1, we thus can understand the development of fattening with ease.

Theorem 4.3. *Let $\Omega = \{x \in \mathbb{R}^2 : |x| < 2\}$ and u_0 be as in* (4.4). *Then the zero level set of the solution u of* (4.1), *has interior for every $t > 0$.*

In fact, the fat zero level-set contains $A_t = \{x \in \Omega : |x|^2 + 2t \geq 4\}$ (Figure 4).

The original example in [2] is for a nonconvex domain $\Omega = \{x \in \mathbb{R}^2 : 1 < |x| < 2\}$, to which Theorem 4.1 cannot be applied directly. However, since the concave piece of boundary does not result in any additional terminating of billiard trajectories, or in the terminology of [10], there are no exits on the boundary, we still can get an approximate theorem like Theorem 4.1 and consequently our interpretation of fattening follows as above. There is another example in [7] for Neumann problem, whose game interpretation will not involve anything other than our preceding study.

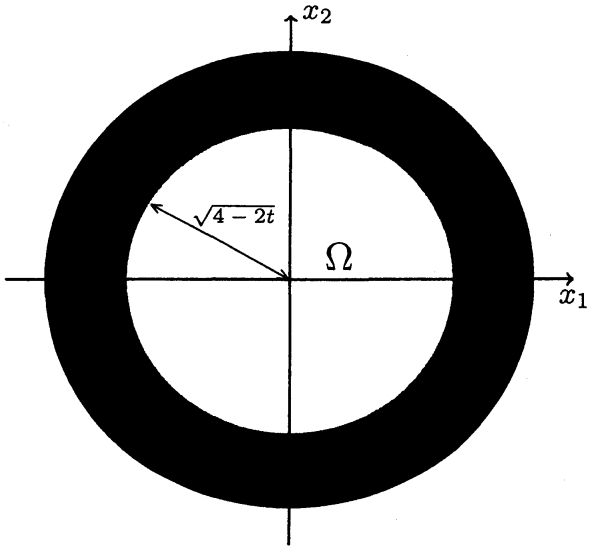

Figure 1: Gray region A_t

References

[1] M. Bardi and I. Capuzzo-Dolcetta, *Optimal Control and viscosity solutions of Hamilton-Jacobi-Bellman equations*, Systems and Control: Foundations and Applications, Birkhäuser Boston, Boston, 1997.

[2] G. Barles, *Nonlinear Neumann boundary conditons for quasilinear degenerate elliptic equations and applications*, J. Differential Equations, 154 (1999), 191-224.

[3] G. Barles, H. M. Soner, P. E. Souganidis, *Front propagation and phase field theory*, SIAM J. Control Optim., 31, (1993), 439-469.

[4] N. Chernov and R. Markarian, *Chaotic Billiards*, Mathematical Surveys and Monographs 127, American Mathematical Society, 2006.

[5] P. Dupuis and H. Ishii, *On Lipschitz continuity of the solution mapping to the Skorokhod problem, with applications*, Stochastics Stochastics Rep., 35 (1991), 31-62.

[6] P. Dupuis and H. Ishii, *SDEs with oblique reflection on nonsmooth domains*, Ann. Prob., 21 (1993), 554-580.

[7] Y. Giga, *Evolving curves with boundary conditions*, Curvature Flows and Related Topics, Mathematical Sciences and Applications, 5 (1995), Gakuto, Tokyo, 99-109.

[8] Y. Giga, *Surface evolution equations, a level set approach*, Monographs in Mathematics 99, Birkhäuser Verlag, Basel, 2006.

[9] Y. Giga and Q. Liu, *A remark on the discrete deterministic game approach for curvature flow equations*, preprint, Hokkaido University Preprint Series in Mathematics #901. http://eprints.math.sci.hokudai.ac.jp/archive/00001842/01/pre901.pdf

[10] Y. Giga and Q. Liu, *A billiard-based game interpretation of the Neumann problem for the curve shortening equation*, Adv. Differential equations, 14 (2009), 201-240.

[11] Y. Giga and M.-H. Sato, *Neumann problem for singular degenerate parabolic equations*, Differential Integral Equations, 6 (1993), 1217-1230.

[12] P. M. Gruber, *Convex Billiards*, Geometriae Dedicata, 33 (1990), 205-226.

[13] B. Halpern, *Strange billiard tables*, Tran. Amer. Math. Soc., 232 (1977), 297-305.

[14] H. Ishii and M.-H. Sato, *Nonlinear oblique derivative problems for a singular degenreate parabolic equations on a general domain*, Nonlinear Analysis, 57 (2004), 1077-1098.

[15] K. Kasai, *Representation of solutions for nonlinear parabolic equations via two-person game with interest rate*, preprint.

[16] R. V. Kohn and S. Serfaty, *A deterministic-control-based approach to motion by curvature*, Comm. Pure Appl. Math., 59 (2006), 344-407.

[17] R. V. Kohn and S. Serfaty, *Second-order PDE's and deterministic games*, preprint.

[18] P. L. Lions, *Neumann type boundary conditions for Hamilton-Jacobi equations*, Duke Math. J., 52 (1985), 793-820.

[19] P. L. Lions and A. S. Sznitman, *Stochastic differential equations with reflecting boundary conditions*, Comm. Pure Appl. Math., 37 (1984), 511-537.

[20] Q. Liu, in preparation.

[21] M.-H. Sato, *Interface evolution with Neumann boundary condition*, Adv. Math. Sci. Appl., 4 (1994), 249-264.

[22] M.-H. Sato, *Capillary problem for singular degenerate parabolic equations on a half space*, Differential Integral Equations, 9 (1996), 1213-1224.

[23] H. Tanaka, *Stochastic differential equations with reflecting boundary conditions in convex regions*, Hiroshima Math. J., 9 (1979), 163-177.