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Singular diffusion equations – fundamentals, applications and open problems

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1 Intorduction

This is an informal note on singular diffusion equations whose diffusion effect is so strong that the speed of evolution becomes a nonlocal quantity. Typical examples include the total variation flow as well as crystalline flow. Our goal in this paper shows current status of fields including open problems. There is already a review article [G1] on a similar topic so we mainly focus development after 2004.

2 Well-posedness

We begin with two examples. The first example is a total variation flow equation for $u=u(t,x)$, $x=(x_{1}, \cdots, x_{n})$ of the form

$$u_{t}= \mathrm{div}\ (\nabla u/|\nabla u|)+\lambda(f-u), \tag{1}$$

where f is a given function and $\lambda\in \mathbf{R}$. The second example is an anisotropic mean curvature flow equation for the graph of a function u of the form

$$u_{t}=\sqrt{1+|\nabla u|^{2}}\ M(\vec{n})(\mathrm{div}\ ((DW)\ (\nabla u))+\sigma(x,u)), \tag{2}$$

where W is a given convex function in $\mathbf{R}^{n}$ which may not be C^{1} and $M(>0)$ is a given positive function; σ is also a given function; $\vec{n}$ denotes the upward normal of $y=u(t,x)$, i.e., $\vec{n}=(-\nabla u/\sqrt{(1+|\nabla u|^{2})},\ 1/\sqrt{1+|\nabla u|^{2}})$. Of course, (1) is a particular example of (2) by taking $M(\vec{n})=1/\sqrt{1+|\nabla u|^{2}},\ W(p)=|p|$ and $\sigma(x,u)=\lambda(f(x)-u)$.

The total variation flow can be formally viewed as an L^{2}-gradient flow of

$$E(u)=\int_{\Omega}(|\nabla u|+\frac{\lambda}{2}(u-f)^{2})dx,\ (\Omega : n-\text{dimensional manifold})$$

i.e., $u_{t}\in-\partial E(n)$, where ∂E denotes the L^{2}-subdifferential of E. There is a nice abstract theory for the existence of solutions for the initial value problem $\partial_t u \in -\partial E(u)$ when E is a lower semicontinuous function defined in a Hilbert space initiated by Y. Komura [Ko] and developed by H. Brezis and others in 1960s; see e.g. [B]. For our problem (1) it guarantees the unique solvability of the initial value problem for (1) with $\lambda \geq 0$ under, for example, periodic boundary condition (by taking $\Omega = \mathbf{T}^n$, a flat torus) with $f \in L^2(\Omega)$. Surprisingly, the speed (the right time derivative) is determined uniqeuly although the evolution law $u_t \in -\partial E(u)$ looks ambiguous.

Let us give a simple example to see that the speed is a nonlocal quantity. We take $\lambda = 0$ in (1) and consider one-dimensional problem to get

$$u_t = (\mathrm{sgn}\ u_x)_x.$$

If one considers $u_0(x) = \cos x$ as an initial data, it is expected that a flat part instanteneously develops from maximum points and minimum points. The other part is not expected to move. Heuristically, the place where slope equals zero feels very strong diffusion while other part stops. In fact, the solution is of the form

$$u(x,t) = \begin{cases} \cos x, & d(t) \leq |x| \leq \pi/2, \\ \cos d(t), & |x| \leq d(t) \end{cases}$$

and $d(t)$ is determined by

$$u_t(x,t) = 2/(\text{length of flat part})\ (= 2/2d(t)), |x| < d(t)$$

with $d(0) = 0$. This fact is observed for example by T. Fukui and Y. Giga [FG] and R. Hardt and X. Zhou [HZ] a long time ago. It is clear that the speed is determined by a nonlocal quantity.

If we consider (2), it cannot be viewed as a gradient flow so general theory for $u_t \in -\partial E(u)$ does not apply. The notion of a solution itself should be studied. A viscosity like theory [CIL] developed by M.-H. Giga and Y. Giga [GG98Ar], [GG99], [GG01Ar] during 1998-2001 covers the case that the space dimension is one and σ equals a constant with some technical assumptions on W (which applies a piecewise linear convex function W). The underlining structure of their theory is that flat position (with slope corresponding to a jump of W) stays as flat so that the speed is constant. This hypotheses (which is verifyed by the subdifferential theory if it applies) is also fundamental to construct admissible crystalline flows by J. Taylor [T1], and independently by S. B. Angenent and M. E. Gurtin [AG]. Actually, the speed is determermied by solving an obstacle problem [KS] which may not be a constant when σ is not spatially constant (as observed in the work of [GG98DS]) or the space dimension $n \geq 3$ (as observed by G. Bellettini M. Novaga and M. Paolini [BNP]).

In fact, G. Bellettini and M. Novaga [BN] consider a surface evolution by singular energy of the form

$$V = \gamma \ \mathrm{div}_\Gamma(\nabla_p \gamma)(\vec{n}),$$

where γ is an interfacial energy and V is the normal velocity. Here γ is extended in $\mathbf{R}^n$ as a positively 1-homogemeous function and $\nabla_p \gamma$ is its gradient. the quantity $\nabla_p \gamma(\vec{n})$ is often called the Cahn-Hoffman vector field. G. Bellettini and M. Novaga introduced a notion of solution and established a uniqueness result [BN]. However, the existence of solution is known only for convex initial data [BCCN]. Flat portion such that the speed is constant is called a calibrable set for E or a Cheeger set studied by people including B. Kawohl [KSch], [KL]. Its relation of the Rayleigh quotient of 1-Laplacian i.e., $\int |\nabla u| dx / \int |u| dx$ is well-studied [KSch], [KL]. It is interesting to charaterize calibrable sets in a general setting [BNP1](cf. [BNP2]).

The case when σ is not constant is handled recently by M.-H. Giga and Y. Giga [GG09], where they study dependence of solutions of the obstacle problems with respect to the domain (flat portion) we consider. However, the theory is still for one-dimensional problem. However, this is a fundamental key to develop a level set theory [CGG], [ES], [G2] for all planer singular curvature flow including crystalline flow with nonuniform driving force term. There is another approach to construct an explicit solution by solving a free boundary problem ([GGR], [GR4], [GR5]). We also note that the speed agrees with the speed proposed by A. R. Roosen [R].

Well-posedness problem is widely open for 1-harmonic map flow:

$$u_t = \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) + |\nabla u| u,$$

where u is a mapping to a unit sphere in $\mathbf{R}^n$. A global existence of weak solution is recently established by J. W. Barrett, X. Feng and A. Prohl [BFP]. However, its uniqueness is not known even locally (except one dimensional case see Y. Giga and R. Kobayashi [GK]). For surface diffusion like type problems unless problem is formulated like $u_t \in -\partial E(u)$ where subdifferential is taken in H^{-1} norm, the well-posedness is widely open (cf. Y. Kashima [Ka]).

3 Applications

Although there are several applications we just give here a rather unusual applications. It is summarized as a application of vertical singular diffusion to have solutions with shocks.

We consider the graph Γ_t of solution u of the Burgers equation

$$u_t + uu_x = 0. \tag{3}$$

If we use upward normal velocity V of Γ_t in $\mathbf{R}^2 = \{(x,y)|x,y \in \mathbf{R}\}$ it is formally of the form.

$$V = y \text{ on } \Gamma_t \tag{4}$$

However, solution of (4) may overturn and may not be the graph of a function. For (3) we often consider an entropy solution with jump discontinuities instead of a multivalued (overturned) solution. What is a good equation instead of (4) to track an entropy solution? Singular diffusion equation gives an answer as rigorously discussed by M.-H. Giga-Y. Giga [GG03]. The main idea is to consider.

$$V = y + A \text{ div } (\nabla\gamma(\vec{n})), \tag{5}$$

where $\gamma(p_1, p_2) = |p_2|$ and $A > 0$ instead of (4). It is useful to calculate numerically (cf. T.-H. R. Tsai, Y. Giga and S. Osher [TGO]). The last term of (5) is called a vertical singular diffusion term. If $A > 0$ is taken sufficiently large with respect to size of jumps (but still finite), then the solution of (5) is the graph of entropy solutions. It is curious whether such singular diffusion can be used for sand pile problems studied by L. C. Evans, M. Feldman and R. F. Gariepy [EFG]. If the slope bound is 1 i.e., $|\nabla u| \leq 1$, a suggestive equation is

$$u_t = f(x) + A \int_{|\omega|=1} (\text{sgn}(1 - \nabla u \cdot \omega))_x \, d\omega, \ A > 0,$$

when the original equation is $u_t = f(x)$.

4 Approximation

It is important to develop several ways of approximation of solutions not only for construction of a solution but also to study properties of a solution. In fact, Bellettini and et al [BCCN] constructed crystalline surface evolution flow by using Almgren-Taylor-Wang [ATW] scheme for convex initial data.

General tendency is that various approximation is uniform with respect to mollification way of singular interfacial energy. This is actually proved for the anistoropic Allen-Cahn equation by Y. Giga, T. Ohtsuka and R. Schätzle [GOS] when the driving force term is spatially homogeneous but for arbitrary dimension. It is interesting to study recent differential game approximation by R. V. Kohn and S. Serfaty [KS1], [KS2] of a solution although approximation scheme for singular interfacial energy is not yet given. For example it is interesting to give a differential game interpretation for crystalline curvature flow.

5 Anisotropic version of constant mena curvature surface

It is well-known that a constant mean curvature embedded hypersurface in $\mathbf{R}^n$ must be a sphere and is known as Alexandrov's theorem [A]. This result is recently extended for anisotropic case for smooth strictly convex interfacial energy by Y. He, H. Li, H. Ma, and J. Ge [HLMG] and by J. Zhai [Z]. The problem is that if the anisotropic curvature $\mathrm{div}_\Gamma \nabla_p \gamma(\vec{n})$ is constant everywhere on an embedded hypersurface, it is the boundary a Wulff shape

$$W_\gamma = \{x \in \mathbf{R}^n \mid x \cdot m \leq \gamma(m) \text{ for all } m \text{ such that } |m| = 1\}$$

up to dilation and translation. This problem (which is solved by [HLMG] for smooth strictly convex energy) was posed in the preprint version of [G2] and later by R. Morgan [M]. For singular interfacial energy it is known only for $n = 2$, the case of curves by P. Mucha and P. Rybka [MR]. For $n \geq 2$ it is widely open. For development of this subject for strictly convex smooth energy the reader is referred to [HLMG], [KP1], [KP2] and references cited there.

Even if the surface in $\mathbf{R}^3$ is immersed, if it is defeomorphic to the sphere S^2, the constant anisotropic curvature surface is the boundary of the Wulff shape up to dilation and translation. This is first proved by Y. Giga and J. Zhai [GZ] where interfacial energy in close to isotropic. A general result is proved by Y. He and H. Li [HL] recently for strictly convex smooth interfacial energy. However, it is an open problem for singular energy.

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