Variational Inequalities with Gradient Constraint and Applications to Optimal Dividend Payments

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1 Variational inequalities arisen from dividend payments

We consider the variational inequality of the form:

(a) $w'(x) \geq 1, \ x > 0, \ w'(0+) > 1,$

(b) $-\alpha w + \frac{1}{2}\sigma^2 w'' + \mu w' \leq 0, \ x > 0,$

(c) $(-\alpha w + \frac{1}{2}\sigma^2 w'' + \mu w')(w' - 1)^+ = 0, \ x > 0,$

(d) $w(0) = 0, \ \mu, \sigma > 0 : \text{constants}.$

Define

$$w(x) = \begin{cases} w_0(x), & x \leq m, \\ x - m + w_0(m), & x > m, \end{cases}$$

where $w_0$ is the solution of

$$\mathcal{A}w_0 := -\alpha w_0 + \frac{1}{2}\sigma w_0'' + \mu w_0 = 0, \ x \leq m,$$

and $m > 0$ is chosen as $w'_0(m) = 1$.

**Theorem 1.1** $w \in C^2(0, \infty) \cap C[0, \infty)$ is a concave solution of the variational inequality (a)-(d).

The variational inequality (a)-(d) is closely related to optimal dividend payments. The reserve $R_t$ of an insurance company at time $t \geq 0$ is assumed to be governed by

$$R_t = \mu t + \sigma B_t - L_t, \ R_0 = x - L_0 \geq 0,$$
where $B_t$ is a standard Brownian motion, $\mu, \sigma > 0$ constants, $x \geq 0$ the initial position of reserve and $L_t$ the rate of dividend payment at time $t$ (0 acts absorbing barrier for $R_t$). Note that $R_0 = x - L_0$ means that if there is a pay-out of dividends at time 0, then $R_t$ instantaneously decreases from $x$ to $x - L_0$. The dividend process $\{L_t\}$ is called admissible if

$$L_t : \mathcal{F}_t := \sigma(B_s, s \leq t)\text{-measurable, } x - L_0 \geq 0,$$

$L_t$ is nonnegative, nondecreasing, continuous,

and we denote by $\mathcal{L}$ the class of all admissible dividend processes $\{L_t\}$.

The objective is to find an optimal dividend payment $\{L_t^*\} \in \mathcal{L}$ so as to maximize the expected total pay-out of dividend

$$J_x(L) = E[\int_0^\tau e^{-\alpha t} dL_t], \ L \in \mathcal{L},$$

where $\alpha > 0$ is the discount rate and $\tau$ the absorption time, $\tau = \inf\{t \geq 0 : R_t = 0\}$.

**Theorem 1.2** We have

$$J_x(L) \leq w(x).$$

Define

$$R_t^* = x + \mu t + \sigma B_t - L^*_t, \ R_0^* = x - L_0^* \geq 0,$$

$$L_t^* = \max_{s \leq t}(x + \mu s + \sigma B_s - m)^+.$$

**Theorem 1.3** We assume that the initial position $x \leq m$. Then $\{L_t^*\}$ is optimal.

**Remark 1.4** Instead of the variational inequality, we consider the Black-Scholes Model:

(a) $w'(x) \geq 1, \ x > 0, \ w'(0+) > 1,$

(b) $-\alpha w + \frac{1}{2} \sigma^2 x^2 w'' + \mu x w' \leq 0, \ x > 0,$

(c) $(-\alpha w + \frac{1}{2} \sigma^2 x^2 w'' + \mu x w')(w' - 1)^+ = 0, \ x > 0,$

(d) $w(0) = 0,$
where $\mu, \sigma > 0$ constants. Then $w(x) = x$ and (a) fails if $\alpha > \mu$.

**Remark 1.5** Consider the following variational inequality:

(a) $w'(x) \geq 1, \ x > 0, \ w'(0+) > 1,$

(b) $-\alpha w + \frac{1}{2} \sigma^2 x^2 w'' + \mu w' \leq 0, \ x > 0,$

(c) $(-\alpha w + \frac{1}{2} \sigma^2 x^2 w'' + \mu w')(w' - 1)^+ = 0, \ x > 0,$

(d) $w(0) = 0.$

Then this variational inequality seems to have no solution.

## 2 Variational inequalities in the Stochastic Ramsey problem

From now on, we consider the variational inequality associated with optimal dividends for the stochastic Ramsey model. We define the following quantities:

$K_t = \text{capital stock of a firm at time } t,$

$K^\gamma = \text{the Cobb-Douglas function for the amount of capital stock } K, \quad 0 < \gamma < 1,$

$B_t = 1$-dim. Brownain motion,

$\mathcal{F}_t = \sigma(B_s, s \leq t),$

$\sigma = \text{diffusion constant, } \sigma > 0$

$x = \text{initial position, } x > 0.$

Dividends are paid from the profit of the firm for shareholders and the remainder accumulates in capital stock. We assume that the flow of dividend payments at time $t$ can be written as $K_t dD_t,$ where $dD_t$ denotes the per capital stock dividend payments. Let $\mathcal{A}$ be the class of all nonnegative, nondecreasing, continuous, $\{\mathcal{F}_t\}$-adapted stochastic processes $D = \{D_t\}$ such that $x_D := x - D_0 > 0.$ Given a policy $D \in \mathcal{A},$ the capital stock process $\{K_t\}$ evolves according to

$$dK_t = K_t^\gamma dt + \sigma K_t dB_t - K_t dD_t, \quad K_0 = x - D_0 > 0.$$
Our objective is to find an optimal policy \( D^* = \{D_t^*\} \) so as to maximize the expected total pay-out functional with discount factor \( \alpha > 0 \):

\[
J(D) = E\left[ \int_0^\infty e^{-\alpha t} K_t dD_t \right], \quad \forall D \in A.
\]

The associated variational inequality is given by

\[
\begin{align*}
&v'(x) \geq 1, \quad x > 0, \quad v'(0+) > 1, \\
&(VI) \quad -\alpha v + \frac{1}{2}\sigma^2 x^2 v'' + x^\gamma v' \leq 0, \quad x > 0, \\
&\quad (-\alpha v + \frac{1}{2}\sigma^2 x^2 v'' + x^\gamma v')(v' - 1)^+ = 0, \quad x > 0.
\end{align*}
\]

For the existence of \( K_t \), we have the following.

**Proposition 2.1** For each \( D \in A \), there exists uniquely a positive solution \( \{K_t\} \) of

\[
dK_t = K_t^{\gamma} dt + \sigma K_t dB_t - K_t dD_t, \quad K_0 = x_D = x - D_0 > 0.
\]

such that

\[
E[K_t] \leq 2^\beta(x_D + t^\beta),
\]

\[
E[K_t^2] \leq 2^{2\beta} e^{\sigma^2 t}(x_D^2 + t^{2\gamma\beta}/\sigma^2),
\]

where \( \beta = 1/(1 - \gamma) \).

**Outline of the proof.** We set \( k_t = K_t^{1-\gamma} \). Then, by Ito's formula

\[
\begin{align*}
dk_t &= (1 - \gamma) K_t^{-\gamma} dK_t + \frac{\sigma^2}{2} K_t^{2(1 - \gamma)}(-\gamma) K_t^{-\gamma - 1} dt \\
&= (1 - \gamma) dt + \sigma K_t^{1-\gamma} dB_t - K_t^{1-\gamma} dD_t \\
&\quad + \frac{\sigma^2}{2} (1 - \gamma) (-\gamma) K_t^{1-\gamma} dt \\
&= (1 - \gamma) \{ (1 - \frac{\sigma^2}{2} \gamma k_t) dt + \sigma k_t dB_t - k_t dD_t \},
\end{align*}
\]

\[
k_0 = x_D^{1-\gamma}.
\]

By linearity, there exists a unique positive solution \( \{k_t\} \).
Proposition 2.2 Assume $\sigma = 0$. Then there exists a concave solution $v_0 \in C^2(0, \infty)$ of (VI).

Outline of the proof. We solve the equation $-\alpha h + x^\gamma h' = 0$ to have
\[ h(x) = Q \exp\{\alpha x^{1-\gamma}/(1-\gamma)\}. \]

Define
\[ v_0(x) = \begin{cases} h(x) & \text{if } x \leq x_*, \\ x - x_* + h(x_*) & \text{if } x_* < x, \end{cases} \]

Choose $x_* = (\gamma/\alpha)^{1/(1-\gamma)}, Q > 0$ such that $h'(x_*) = 1$. Then we have
\[ h''(x_*) = 0, \]

and
\[ -\alpha v_0 + x^\gamma v_0' = -\alpha\{x - x_* + h(x_*)\} + x^\gamma \leq 0 \text{ for } x > x_* . \]

3 Probabilistic solution of the penalty equation

We consider the penalty equation
\[ (p) \quad -\alpha u + \frac{1}{2}\sigma^2 x^2 u'' + x^\gamma u' + \frac{x}{\epsilon}(u' - 1)^- = 0, \quad x > 0, \]

which can be rewritten as
\[ -\alpha u + \frac{1}{2}\sigma^2 x^2 u'' + x^\gamma u' + \frac{x}{\epsilon} \max_{0 \leq c \leq 1} (1 - u')c = 0, \quad x > 0. \]

Let $C$ be the class of all $\{F_t\}$-progressively measurable processes $c = \{c_t\}$ such that $0 \leq c_t \leq 1$, a.s.

for all $t \geq 0$. For any $c \in C$, let $\{X_t\}$ be the solution of
\[ dX_t = X_t^\gamma dt + \sigma X_t dB_t - \frac{1}{\epsilon} c_t X_t dt, \quad X_0 = x > 0. \]

Define
\[ u(x) = \sup_{c \in C} E[\int_0^\infty e^{-\alpha t} \frac{1}{\epsilon} c_t X_t dt], \]

where the supremum is taken over all systems $(\Omega, F, P, \{c_t\}, \{B_t\})$. Then we observe that the penalty equation $(p)$ is a Hamilton-Jacobi-Bellman equation.
Theorem 3.1 We have

\[ 0 \leq u(x) \leq v_0(x) \leq C(1 + x), \quad x > 0, \]

for some constant \( C > 0 \).

Theorem 3.2 For any \( \rho > 0 \), there exists \( C_{\rho, \epsilon} > 0 \) such that

\[ |u(x) - u(y)| \leq C_{\rho, \epsilon}|x - y| + \rho(1 + x + y), \quad x, y > 0. \]

Theorem 3.3 \( u \) is concave on \((0, \infty)\).

4 Solution of the penalty equation

In this section, we show that the probabilistic solution \( u \) is a classical solution of the penalty equation \((p)\).

Definition 4.1 Let \( w \in C(0, \infty) \). Then \( w \) is called a viscosity solution of \((p)\) if

(a) \( w \) is a viscosity subsolution of \((p)\), that is, for any \( \phi \in C^2(0, \infty) \) and any

local maximum point \( z > 0 \) of \( w - \phi \),

\[ -\alpha w + \frac{1}{2} \sigma^2 x^2 \phi'' + x^\gamma \phi' + \frac{x}{\epsilon} (\phi' - 1)^-|_{x=z} \geq 0, \]

and (b) \( w \) is a viscosity supersolution of \((p)\), that is, for any \( \phi \in C^2(0, \infty) \) and any

local minimum point \( \overline{z} > 0 \) of \( w - \phi \),

\[ -\alpha w + \frac{1}{2} \sigma^2 x^2 \phi'' + x^\gamma \phi' + \frac{x}{\epsilon} (\phi' - 1)^-|_{x=\overline{z}} \leq 0. \]

By Theorems 3.1 and 3.2, we can show that the dynamic programming principle holds for \( u \), i.e.,

\[ u(x) = \sup_{c \in {C}} E[\int_0^s e^{-\alpha t} \frac{1}{\epsilon} c_t X_t dt + e^{-\alpha s} u(X_s)] \]

for any \( s \geq 0 \). By the theory of viscosity solutions, taking into account Proposition 2.1, we have the viscosity property of \( u \). For details, we refer to [9].
Theorem 4.2 \( u \) is a viscosity solution of \((p)\).

Theorem 4.3 We have

\[
 u \in C^2(0, \infty).
\]

5 Solution of the variational inequality

In this section, we study the convergence of \( u = u_\epsilon \) to a viscosity solution \( v \) of the variational inequality \((VI)\) as \( \epsilon \to 0 \).

5.1 Limit of the penalized problem

Definition 5.1 Let \( w \in C(0, \infty) \). Then \( w \) is called a viscosity solution of \((VI)\), if the following assertions are satisfied:

(a) For any \( \phi \in C^2 \) and any local minimum point \( \overline{z} > 0 \) of \( w - \phi \),

\[
 \phi' (\overline{z}) \geq 1, \quad -\alpha w + \frac{1}{2} \sigma^2 x^2 \phi'' + x^\gamma \phi' \bigg|_{x=\overline{z}} \leq 0,
\]

(b) For any \( \phi \in C^2 \) and any local maximum point \( z > 0 \) of \( w - \phi \),

\[
 (-\alpha w + \frac{1}{2} \sigma^2 x^2 \phi'' + x^\gamma \phi') (\phi' - 1)^+ \bigg|_{x=z} \geq 0.
\]

By concavity and Theorem 3.1, we get

\[
 0 \leq u_\epsilon'(x)x \leq u_\epsilon(x) - u_\epsilon(0) \leq v_0(x), \quad x > 0.
\]

Hence, for any \( 0 < a < b \),

\[
 \sup_{\epsilon} \| u_\epsilon' \|_{C[a,b]} < \infty.
\]

By the Ascoli-Arzelà theorem and Theorem 4.2, we have the following.

Theorem 5.2 There exists a subsequence \(\{ u_{\epsilon_n} \} \) such that

\[
 u_{\epsilon_n} \to v \in C(0, \infty) \text{ locally uniformly in } (0, \infty) \text{ as } \epsilon_n \to 0.
\]

Furthermore, \( v \) is a viscosity solution of \((VI)\).
5.2 Regularity

In this subsection, we study the regularity of the viscosity solution $v$ of (VI). By concavity, we can show that

$$u_{\epsilon_n}' \geq 1 \text{ on } [a, b].$$

We rewrite the penalty equation as

$$-u'' = \frac{2}{\sigma^2 x^2} \{-\alpha u + x^\gamma u' + \frac{x}{\epsilon} (u' - 1)^-\}.$$

Thus we have:

**Theorem 5.3** For any $0 < a < b$, we have

$$\sup_{n \geq 1} \|u''_{\epsilon_n}\|_{C[a,b]} < \infty.$$

By Theorem 5.3, extracting a subsequence, we have

$$u_{\epsilon_n}' \rightarrow v' \text{ locally uniformly in } (0, \infty) \text{ as } n \rightarrow \infty,$$

and $v'$ is locally Lipschitz on $(0, \infty)$.

**Theorem 5.4** We have

$$v \in C_{loc}^{1,1}(0, \infty), \text{ piecewise } C^2, \quad v' \geq 1 \text{ on } (0, \infty).$$

Furthermore, by using Proposition 2.2, we can state the following.

**Theorem 5.5** We have

$$v'(0+) > 1,$$

and there exists $x^* > 0$ such that

$$x^* = \inf\{x > 0 : v'(x) = 1\}.$$
6 Optimal dividend payments

In this section, we give a synthesis of the optimal policy $D^* \in A$ of the maximization problem.

Consider the SDE with reflecting barrier conditions:

\[(a) \quad dK_t^* = (K_t^*)^\gamma dt + \sigma K_t^* dB_t - K_t^* dD_t^*, \quad K_0^* = x - D_0^* > 0,\]
\[(b) \quad D_t^* = (x - x^*)^+ + \int_0^t 1_{\{K_s^* = x^r\}} dD_s^*,\]
\[(c) \quad D_t^* \text{ is continuous a.s.,}\]
\[(d) \quad K_t^* \in \mathcal{R}, \quad \forall t \geq 0, \quad \text{a.s.,}\]
\[(e) \quad \int_0^t 1_{\{K_s^* = x^r\}} ds = 0, \quad \forall t \geq 0, \quad \text{a.s.,}\]

where $\mathcal{R} := (0, x^*)$ for $x^* = \inf \{x > 0 : v'(x) = 1\}$.

**Theorem 6.1** We assume that the initial position $x \leq x^*$, (by making $D_0 = x - x^*$ if $x > x^*$).

Then the optimal policy $D^* = \{D_t^*\}$ is given by (a) - (e).

**Lemma 6.2** There exists a unique solution $(\{K_t^*\}, \{D_t^*\})$ of (a) - (e).

Proof. There exists a unique solution $(\{M_t, \Delta_t\})$ of the SDE with reflecting barrier conditions:

- $dM_t = (1 - \gamma)(dt - \frac{\sigma^2 \gamma}{2} M_t dt + \sigma M_t dB_t) - d\Delta_t, \quad M_0 = x^{1-\gamma} - \Delta_0 > 0$,
- $\Delta_t = (x^{1-\gamma} - (x^*)^{1-\gamma})^+ + \int_0^t 1_{\{M_s \in \partial S\}} d\Delta_s$,
- $\Delta_t$ is continuous a.s.,
- $M_t \in S, \quad \forall t \geq 0, \quad \text{a.s.,}$
- $\int_0^t 1_{\{M_s \in \partial S\}} ds = 0, \quad \forall t \geq 0, \quad \text{a.s.,}$

where $S = [0, (x^*)^{1-\gamma}]$ and $\{\Delta_t\}$ is a bounded variation process. Define

$$K_t^* = M_t^\beta, \quad D_t^* = \Delta_0^\beta + \int_0^t \beta M_s^{-1} 1_{\{M_s > 0\}} d\Delta_s, \quad \beta := 1/(1 - \gamma).$$

Then, Ito's formula completes the proof.
Proof of Theorem 6.1. Let $D \in \mathcal{A}$ be arbitrary. By the variational inequality and the continuity of $\{D_t\}$, we can apply the generalized Itô formula to $\{K_t\}$ for convex functions (cf. [5]). Then

\[
e^{-\alpha s}v(K_s) - v(x_D) = \int_0^s e^{-\alpha t}\left\{-\alpha v + \frac{1}{2}\sigma^2 x^2 v'' + x^\gamma v'\right\}_{x=K_t} dt + \int_0^s e^{-\alpha t} v'(K_t) \sigma K_t dB_t - \int_0^s e^{-\alpha t} v'(K_t) K_t dD_t
\]

and

\[
\leq \int_0^s e^{-\alpha t} v'(K_t) \sigma K_t dB_t - \int_0^s e^{-\alpha t} v'(K_t) K_t dD_t, \quad a.s. \quad s \geq 0.
\]

Hence

\[
E[\int_0^{\tau_R} e^{-\alpha t} K_t dD_t] \leq v(x_D) \leq v(x).
\]

where $\tau_R := R \wedge \inf\{t \geq 0 : K_t \geq R \text{ or } K_t \leq 1/R\}$ for $R > 0$. Letting $R \to \infty$,

\[
J(D) = E[\int_0^\infty e^{-\alpha t} K_t dD_t] \leq v(x).
\]

By the same argument as above, we get

\[
v(x) = E[\int_0^\infty e^{-\alpha t} v'(K^*_t) K^*_t dD^*_t].
\]

Since $D^*_t$ increases only when $K^*_t = x^*$ and $v'(x^*) = 1$,

\[
v(x) = E[\int_0^\infty e^{-\alpha t} v'(K^*_t) 1_{K^*_t = x^*} K^*_t dD^*_t] = E[\int_0^\infty e^{-\alpha t} K^*_t dD^*_t] = J(D^*),
\]

which completes the proof.

References


