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Weak Harnack inequality for fully nonlinear PDEs with superlinear growth terms in Du

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1 Introduction

In this note, we present the weak Harnack inequality for L^p-viscosity nonnegative supersolutions of fully nonlinear elliptic PDEs with unbounded coefficients and inhomogeneous terms. Moreover, we discuss the case when PDEs have superlinear growth terms in Du.

Throughout this paper, we suppose at least

$$p > \frac{n}{2}.$$

For measurable sets $U \subset \mathbf{R}^n$, we use the standard L^p-norm and $W^{2,p}$-norm, $\|\cdot\|_{L^p(U)}$ and $\|\cdot\|_{W^{2,p}(U)}$, respectively. We will write $\|\cdot\|_p$ and $\|\cdot\|_{2,p}$ for them if there is no confusion. We also use the following notation:

$$L^p_+(U) = \{u \in L^p(U) \mid u \geq 0 \ \text{ a.e. in } U\}.$$

Let S^n be the set of $n \times n$ symmetric matrices with the standard order. For fixed uniform ellipticity constants $0 < \lambda \leq \Lambda$, we denote by $S^n_{\lambda,\Lambda}$ the set of all $A \in S^n$ such that $\lambda I \leq A \leq \Lambda I$. We then define the Pucci operators $\mathcal{P}^{\pm}$: for $X \in S^n$,

$$\mathcal{P}^+(X) = \max\{-\text{trace}(AX) \mid A \in S^n_{\lambda,\Lambda}\},$$

$$\mathcal{P}^-(X) = \min\{-\text{trace}(AX) \mid A \in S^n_{\lambda,\Lambda}\}.$$

Note that $X \to \mathcal{P}^+(X)$ (resp., $\mathcal{P}^-(X)$) is convex (resp., concave).

Let us consider the most general PDEs of second-order:

$$F(x, u, Du, D^2u) = f(x) \tag{1}$$

in Ω, where $\Omega \subset \mathbf{R}^n$ is an open set. Here, we suppose that $F : \Omega \times \mathbf{R} \times \mathbf{R}^n \times S^n \to \mathbf{R}$ and $f : \Omega \to \mathbf{R}$ are given measurable functions, and that F is continuous in the last three variables.

Definition 1.1 We call $u \in C(\Omega)$ an L^p-viscosity subsolution (resp., supersolution) of (1) in Ω if

$$\operatorname*{ess\,lim\,inf}_{y \to x} \{F(y, u(y), D\phi(y), D^2\phi(y)) - f(y)\} \leq 0$$

$$\left(\text{resp.,} \quad \operatorname*{ess\,lim\,sup}_{y \to x} \{F(y, u(y), D\phi(y), D^2\phi(y)) - f(y)\} \geq 0\right)$$

whenever $\phi \in W^{2,p}_{\mathrm{loc}}(\Omega)$ and $x \in \Omega$ is a local maximum (resp., minimum) point of $u - \phi$. Finally, we call $u \in C(\Omega)$ an L^p-viscosity solution of (1) in Ω if it is an L^p-viscosity subsolution and an L^p-viscosity supersolution of (1) in Ω.

In order to memorize the right inequality, we will often say that u is an L^p-viscosity (sub)solution of

$$F(x, u, Du, D^2u) \leq f(x)$$

when it is an L^p-viscosity subsolution of (1) for instance.

We also recall the notion of strong solutions.

Definition 1.2 We call $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ an L^p-strong subsolution (resp., supersolution) of (1) in Ω if u satisfies

$$F(x, u(x), Du(x), D^2u(x)) - f(x) \leq 0 \quad (\text{resp.,} \ \geq 0) \quad \text{a.e. in } \Omega.$$

Finally, we call $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ an L^p-strong solution of (1) in Ω if the equality holds in the above.

Remark 1.3 Suppose that $p > p' > n/2$. It is trivial to see that u is an L^p-strong subsolution (resp., supersolution) of (1) in Ω, then it is an $L^{p'}$-strong subsolution (resp., supersolution) of (1) in Ω. However, for L^p-viscosity solutions, the opposite implication holds true; if u is an $L^{p'}$-viscosity subsolution (resp., supersolution) of (1) in Ω, then it is also an L^p-viscosity subsolution (resp., supersolution) of (1) a.e. in Ω.

2 Known results

Since the weak Harnack inequality is derived from the Aleksandrov-Bakelman-Pucci (ABP for short) maximum principle, we recall it from [8]. Thus, in what follows, we only consider the case when F is independent of u-variable.

Now we suppose the uniform ellipticity for F:

$$\mathcal{P}^-(X-Y) \leq F(x,\xi,X) - F(x,\xi,Y) \leq \mathcal{P}^+(X-Y)$$

for $x \in \Omega$, $\xi \in \mathbf{R}^n$, and $X, Y \in S^n$. A typical example of F is given by

$$F(x,\xi,X) := \max_{1\leq i\leq M} \min_{1\leq j\leq N} \{-\mathrm{trace}(A(x;i,j)X) + \langle b(x;i,j),\xi\rangle\},$$

where for $M, N > 1$, functions $x \in \Omega \to A(x;i,j) \in S^n_{\lambda,\Lambda}$ and $x \in \Omega \to b(x;i,j) \in \mathbf{R}^n$ are measurable ($1 \leq i \leq M$, $1 \leq j \leq N$). Notice that the above F is non-convex and non-concave in general.

Under the uniform ellipticity assumption, if u is an L^p-viscosity solution of (1) in Ω, then it is also an L^p-viscosity solution of

$$\mathcal{P}^-(D^2u) + F(x,Du,O) \leq f(x), \text{ and } \mathcal{P}^+(D^2u) + F(x,Du,O) \geq f(x)$$

in Ω. Therefore, for the sake of simplicity, instead of (1), we shall study the following extremal PDEs: for $m \geq 1$,

$$\mathcal{P}^\pm(D^2u) \pm \mu(x)|Du|^m = \mp f(x), \tag{$2)_{m,\pm}$}$$

where μ, f are often supposed to be nonnegative.

We recall the ABP maximum principle for L^n-strong solutions of $(2)_{1,-}$.

Proposition 2.1 (cf. [6]) There exist $C_k = C_k(n,\lambda,\Lambda) > 0$ ($k = 1,2$) such that if $f, \mu \in L^n_+(\Omega)$, and $u \in C(\overline{\Omega}) \cap W^{2,n}_{\mathrm{loc}}(\Omega)$ is an L^n-strong subsolution of $(2)_{1,-}$ in Ω, then it follows that

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+ + C_1 \exp(C_2\|\mu\|^n_n)\|f\|_{L^n(\{u>0\})},$$

where $\{u > 0\} := \{x \in \Omega \mid u(x) > 0\}$.

Remark 2.2 In the above statement, we can replace $\|f\|_{L^n(\{u>0\})}$ by $\|f\|_{L^n(\Gamma[u])}$, where $\Gamma[u]$ is the upper contact set of u in Ω. See Gilbarg-Trudinger's book for the definition of $\Gamma[u]$.

From Proposition 2.1, it is trivial to obtain the corresponding result for L^p-strong supersolutions of $(2)_{1,+}$ by taking $v = -u$.

Now, we recall an L^p-viscosity version of the ABP maximum principle. We will use a constant $p_0 = p_0(n, \lambda, \Lambda) \in [\frac{n}{2}, n)$, which was introduced in [4]. We note that p_0 does not depend on Ω because we only need to solve extremal PDEs in balls. See [8] (also [5]) for the details.

Theorem 2.3 (cf. Proposition 2.8 and Theorem 2.9 in [8]) Assume that

$$q \geq p > p_0 \quad \text{and} \quad q > n \quad \text{hold.} \tag{3}$$

For $\mu \in L^q_+(\Omega)$, there exists $C_3 = C_3(n, \lambda, \Lambda, \|\mu\|_q) > 0$ such that if $f \in L^p_+(\Omega)$, and $u \in C(\overline{\Omega})$ is an L^p-viscosity subsolution of $(2)_{1,-}$ in Ω, then it follows that

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+ + C_3 \|f\|_{L^p(\{u>0\})}.$$

Remark 2.4 For more precise dependence of C_3 with respect to $\|\mu\|_q$, we refer to [8].

We next consider the case when $m > 1$ for $(2)_{m,-}$. In general, when $m > 1$, the ABP maximum principle for $(2)_{m,-}$ fails even for classical solutions (see [7, 8]).

Theorem 2.5 (Theorems 2.11 and 2.12 in [8]) Assume that (3) and

$$mq(n-p) < n(q-p) \tag{4}$$

holds. For $m > 1$, there exists $\delta_1 = \delta_1(n, \lambda, \Lambda, m, p, q) > 0$ satisfying the following property: for $\mu \in L^q_+(\Omega)$, there is $C_4 = C_4(n, \lambda, \Lambda, m, p, q, \|\mu\|_q) > 0$ such that if $f \in L^p_+(\Omega)$ satisfies

$$\|f\|_p^{m-1} \|\mu\|_q < \delta_1,$$

and $u \in C(\overline{\Omega})$ is an L^p-viscosity subsolution of $(2)_{m,-}$ in Ω, then it follows that

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+ + C_4 \|f\|_{L^p(\{u>0\})}.$$

Remark 2.6 We note that under (3), the relation (4) holds true when $p \geq n$. Thus, when $p \geq n$, we may choose arbitrary $m > 1$.

3 Weak Harnack inequality ($m = 1$)

From now on, we consider PDEs in cubes although it is possible to replace them by balls. We denote by Q_R the open cube with its center at the origin and with its length $R > 0$; $Q_R = (-\frac{R}{2}, \frac{R}{2}) \times \cdots \times (-\frac{R}{2}, \frac{R}{2})$.

Theorem 3.1 (Theorems 4.5 and 4.7 in [9]) Assume that (3) holds. There exists $r = r(n, \lambda, \Lambda) > 0$ satisfying the following property: for $\mu \in L^q_+(Q_2)$, there is $C_5 = C_5(n, \lambda, \Lambda, p, q, \|\mu\|_q) > 0$ such that if $f \in L^p_+(Q_2)$ and $u \in C(\overline{Q}_2)$ is a nonnegative L^p-viscosity supersolution of $(2)_{1,+}$ in Q_2, then it follows that

$$\left(\fint_{Q_1} u^r dx \right)^{\frac{1}{r}} \leq C_5 \left\{ \inf_{Q_1} u + \|f\|_{L^p(Q_2)} \right\}.$$

Idea of proof: We first reduce the assertion to the case when $f \equiv 0$. For this purpose, due to our strong solvability (Theorem 2.3 in [9]), we find an L^p-strong supersolution $v \in C(\overline{Q}_2) \cap W^{2,p}_{\mathrm{loc}}(Q_2)$ of

$$\mathcal{P}^-(D^2 v) - \mu(x)|Dv| \geq f(x) \quad \text{in } Q_2$$

such that $0 \leq v \leq C_6 \|f\|_p$ in Q_2 for some $C_6 = C_6(n, \lambda, \Lambda, p, \|\mu\|_q) > 0$. Setting $w := u + v$, we see that w is an L^p-viscosity supersolution of $(2)_{1,+}$ in Q_2 with $f \equiv 0$. Thus, if we verify the assertion when $f \equiv 0$, then we find $C_7 = C_7(n, \lambda, \Lambda, p, q, \|\mu\|_q) > 0$ such that

$$\left(\fint_{Q_1} u^r dx \right)^{\frac{1}{r}} \leq \left(\fint_{Q_1} w^r dx \right)^{\frac{1}{r}} \leq C_7 \inf_{Q_1} w \leq C_7 \inf_{Q_1} u + C_7 C_6 \|f\|_p,$$

which concludes our proof.

Next, by considering $U := u/(\inf_{Q_1} u + \varepsilon)$ ($\forall \varepsilon > 0$), it is enough to show that $\inf_{Q_1} u \leq 1$ implies that $\fint_{Q_1} u^r dx \leq C_0$ for some $r, C_0 > 0$, which are independent of u and $\varepsilon > 0$. (In fact, we can prove a weaker fact that $\inf_{Q_3} u \leq 1$ implies $\fint_{Q_1} u^r dx \leq C_0$. However, we skip this because we will not go into the details of "cube-decomposition lemma".)

By the strong solvability (Theorem 2.3 in [8]) again, we then choose an L^p-strong solution $\phi \in C(\overline{Q}_2) \cap W^{2,p}_{\mathrm{loc}}(Q_2)$ of

$$\mathcal{P}^-(D^2 \phi) + \mu(x)|D\phi| = \xi(x) \quad \text{in } Q_2$$

such that $0 \geq \phi$ in Q_2, $-2 \geq \phi$ in Q_1, and $\xi \in C(Q_2)$ with supp $\xi \subset Q_1$. Setting $V := -u - \phi$, we see that V is an L^p-viscosity subsolution of

$$\mathcal{P}^-(D^2V) - \mu(x)|DV| \leq -\xi(x) \quad \text{in } Q_2.$$

Hence, the ABP maximum principle (Theorem 2.3) implies

$$1 \leq \sup_{Q_1} V \leq C_3\|\xi\|_{L^n(\{V>0\})} \leq C_3\|\xi\|_\infty|\{x \in Q_1 \mid u(x) < M_1\}|,$$

where $M_1 = \sup(-\phi) > 1$. Therefore, we have

$$|\{x \in Q_1 \mid u(x) \geq M_1\}| \leq \theta$$

for some $\theta \in (0,1)$. It is now enough to obtain

$$|\{x \in Q_1 \mid u(x) \geq M_1^k\}| \leq \theta^k \tag{5}$$

because this yields $\int_{Q_1} u^r dx \leq C_0$ for some $r, C_0 > 0$. To prove (5), we need a "cube-decomposition" lemma (e.g. in [1, 2]) but we omit this here.

4 Weak Harnack inequality ($m > 1$)

To follow the argument in section 3, we need to establish the existence of L^p-strong solutions of the associated extremal PDEs:

$$\mathcal{P}^+(D^2u) + \mu(x)|Du|^m = f(x).$$

In order to show the strong solvability of the above PDEs, we will apply the Schauder fixed point theorem. To this end, we use a recent result by Winter in [14] on the global $W^{2,p}$-estimate of L^p-viscosity solutions of extremal PDEs:

$$\mathcal{P}^+(D^2u) = f(x) \quad \text{in } B_1$$

under "smooth" Dirichlet condition.

Our strong solvability resut is as follows:

Theorem 4.1 (Theorem 3.1 in [10]) Assume that $\partial\Omega \in C^{1,1}$, $f \in L^p(\Omega)$, $\mu \in L^q(\Omega)$ and $\psi \in W^{2,p}(\Omega)$ hold. Assume also that one of the following conditions holds:

$$\begin{cases} (a) & q = \infty, p_0 < p, m(n-p) < n, \\ (b) & n < p \leq q < \infty, \\ (c) & p_0 < p \leq n < q < \infty, mq(n-p) < n(q-p). \end{cases} \tag{6}$$

There exists $\delta_2 = \delta_2(n, \lambda, \Lambda, p, q, m, \Omega) > 0$ such that if

$$\|\mu\|_q(\|f\|_p + \|\psi\|_{2,p})^{m-1} < \delta_2,$$

then there exists L^p-strong solutions $u \in W^{2,p}(\Omega)$ of

$$\begin{cases} \mathcal{P}^+(D^2u) + \mu(x)|Du|^m = f(x) & \text{in } \Omega, \\ u = \psi & \text{on } \partial\Omega. \end{cases}$$

Moreover, there is $C_8 = C_8(n, \lambda, \Lambda, p, q, m, \Omega) > 0$ such that

$$\|u\|_{2,p} \leq C_8(\|f\|_p + \|\psi\|_{2,p}).$$

Idea of proof: It is enough to verify that we can apply the Schauder fixed point theorem to the mapping $T : v \in W^{1,r}(\Omega) \to Tv \in W^{2,p}(\Omega)$ (for some $r > 1$), where $w := Tv$ is an L^p-strong solution of

$$\begin{cases} \mathcal{P}^+(D^2w) + \mu(x)|Dv|^m = f(x) & \text{in } \Omega, \\ w = \psi & \text{on } \partial\Omega. \end{cases}$$

See [10] for the details.

Since we do not know if the weak Harnack inequality holds true even when μ is bounded, we will also consider this case. We refer to [13] for related results.

Theorem 4.2 (Theorem 4.2 in [10]) Assume that one of (6) holds. Assume also that

$$1 < m < 2 - \frac{n}{q}. \tag{7}$$

For $M > 0$, there exist $\delta_3 = \delta_3(n, \lambda, \Lambda, p, m, M) > 0$, $C_9 = C_9(n, \lambda, \Lambda, p, q, m) > 0$ and $r = r(n, \lambda, \Lambda, p, q, m) > 0$ such that if $f \in L^p_+(Q_2)$ and $\mu \in L^q_+(Q_2)$ satisfy

$$\|\mu\|_q(1 + \|f\|_p^{m-1}) < \delta_3,$$

and an L^p-viscosity supersolution $u \in C(Q_2)$ of $(2)_{m,+}$ in Q_2 satisfies $0 \leq u \leq M$ in Q_2, then it follows that

$$\left(\int_{Q_1} u^r dx\right)^{\frac{1}{r}} \leq C_9 \left\{\inf_{Q_1} u + \|f\|_p\right\}.$$

Remark 4.3 The hypothesis (7) is necessary when we use a scaling argument to apply the cube-decomposition lemma.

Idea of proof: In section 3, we used strong solvability of extremal PDEs $(2)_{1,\pm}$ twice in the idea of proof of Theorem 3.1. Instead, we need to utilize Theorem 4.1 here. In order to obtain (5), we have to modify the scaling argument in [1] (also [2]) as in [11].

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