数理解析研究所講究録
第 1697 巻 2010 年 185-194



# Clebsch parameterization —*theory and applications*

Zensho Yoshida

*The University of Tokyo, Graduate School of Frontier Sciences*

**Abstract**

The Clebsch parameterization ($\boldsymbol{u}=\nabla\varphi+\alpha\nabla\beta$ ) has advantages in elucidating structural properties of vector fields; for example, it helps formulating the Hamiltonian form of ideal fluid mechanics, representing topological constraints (Casimir invariants), integrating the Cauchy characteristics of vortex fields, etc. Because of its "nonlinear" formulation, however, there are some difficulties which must be carefully overcome: (1) It is not complete, i.e., for an arbitrary vector field $\boldsymbol{u}$, we may fail to find three scalar fields (Clebsch parameters) $\varphi, \alpha, \beta$ that satisfy $\boldsymbol{u}=\nabla\varphi+\alpha\nabla\beta$ globally in space. (2) It is not uniquely determined, i.e., the map $(u_1, u_2, u_3)\mapsto(\varphi, \alpha, \beta)$ is not injective. A generalized form such that $\boldsymbol{u}=\nabla\varphi+\sum_{j=1}^{\nu}\alpha_j\nabla\beta^j$ can be complete if $\nu=n-1$ ($n$ is the space dimension). However, when we need to control the boundary values of $\varphi$, $\alpha_j$ and $\beta^j$ (for example to determine them uniquely), we have to set $\nu=n$.

## 1 Introduction

The method of Clebsch parameterization [1] proffers to represent a three-dimensional vector field in the form

$$\boldsymbol{u}(\boldsymbol{x})=\nabla\phi(\boldsymbol{x})+\alpha(\boldsymbol{x})\nabla\beta(\boldsymbol{x}), \tag{1}$$

where $\boldsymbol{x}\in\mathbf{R}^3$, $\boldsymbol{u}(\boldsymbol{x})$ is a three-dimensional vector field, $\phi(\boldsymbol{x}), \alpha(\boldsymbol{x})$ and $\beta(\boldsymbol{x})$ are scalar fields, and $\nabla$ denotes the gradient of a scalar field (exterior derivative of a 0-form). The first component $\nabla\phi$ is an *irrotational* field, viz., $\nabla\times(\nabla\phi)\equiv 0$ ($\nabla\times$ denotes the curl of a vector field, that is the exterior derivative of a 1-form). Adding the second component $\alpha\nabla\beta$, $\boldsymbol{u}$ may have a finite *vorticity*:

$$\nabla\times\boldsymbol{u}(\boldsymbol{x})=\nabla\alpha(\boldsymbol{x})\times\nabla\beta(\boldsymbol{x}). \tag{2}$$

We call (1) and (2) a Clebsch 1-form and a Clebsch 2-form, respectively. The Clebsch form has often advantages in elucidating the structure of vector fields; for example, it helps representing the vortex-line equations in a Hamiltonian form [2, 3, 4], integrating the Cauchy characteristics of vortex fields [5], formulating the Hamiltonian form of ideal fluid mechanics [6, 7], representing topological constraints

(Casimir invariants) [8, 9], casting a non-Abelian Chern-Simons 3-form into an exact 3-form [10], etc.

Here we pose a question of the "completeness" of such parameterization: For an "arbitrary" vector field $\boldsymbol{u}(\boldsymbol{x})$, can we find three scalar functions $\phi(\boldsymbol{x}), \alpha(\boldsymbol{x}), \beta(\boldsymbol{x})$ to represent it in the form of (1)? Or, for an "arbitrary" vorticity $\boldsymbol{\omega}(\boldsymbol{x}) = \nabla \times \boldsymbol{u}(\boldsymbol{x})$, can we find two scalar functions $\alpha(\boldsymbol{x}), \beta(\boldsymbol{x})$ to represent it in the form of (2)?

Obviously, the number of field variables on the both sides of (1) is the same (= 3). Hence, a very naive expectation might be that the Clebsch 1-form (1) is "complete". The curled relation (2) is also seemingly complete; the left-hand-side $\nabla \times \boldsymbol{u}$ is divergence-free, so only two components of this vector field are independent, which may be represented by the two scalar fields $\alpha$ and $\beta$.

As we shall show (and as has been noticed by many different practical examples), these expectations are not true. The number of field variables is not the degree of freedom of a vector *field* —a field is a member of an infinite-dimensional function space. Hence, the counting of field variables is irrelevant to the argument of the completeness. Our question is, then, how we can generalize (1) or (2) for complete parameterization. In this paper, we propose the form of

$$\boldsymbol{u}(\boldsymbol{x}) = \nabla\varphi(\boldsymbol{x}) + \sum_{j=1}^{\nu} \alpha_j(\boldsymbol{x})\nabla\beta^j(\boldsymbol{x}). \tag{3}$$

We shall show that $\nu$ must be $n-1$ ($n$ is the dimension of the coordinate space).

In Sec. 2, we shall show that the original Clebsch form (1) or (2) is incomplete —in fact, the totality of the Clebsch forms is "measure-zero" in the function space of general 1-forms or 2-forms. In Sec. 3, we shall prove the completeness of the generalized Clebsch form (3) with $\nu = n-1$. In Sec. 4 we add some remarks on the practical applications of the (generalized) Clebsch forms.

For a more detailed mathematical theory and some explicit examples of applications, the reader is referred to Ref. [11].

## 2 Incompleteness of the Clebsch Parameterization

### 2.1 Vorticity and helicity

Let $\Omega \subset \mathbf{R}^3$ be a bounded domain. We assume that the boundary $\partial\Omega$ is a union of $\ell$ (a finite number) surfaces $\Gamma_k$ ($k = 1, \cdots, \ell$), where every $\Gamma_k$ is a connected $(n-1)$-dimensional smooth ($C^2$-class) manifold such that $\partial\Gamma_k = \emptyset$. Let $\Sigma_1 \cdots \Sigma_m$ ($m \geq 0$, $\Sigma_i \cap \Sigma_j = \emptyset$ $(i \neq j)$) be the cuts of $\Omega$ such that $\Omega \setminus (\cup_{i=1}^m \Sigma_i)$ becomes a simply connected domain. The number $m$ of such cuts is the *genus* of $\Omega$. When $m > 0$, we define the flux through each cut by

$$\Phi(\Sigma_i, \boldsymbol{u}) = \int_{\Sigma_i} \boldsymbol{n} \cdot \boldsymbol{u} ds, \quad (i = 1, \cdots, m),$$

where $\boldsymbol{n}$ is the unit normal vector on $\Sigma_i$ with an appropriate orientation. By Gauss's formula $\Phi(\Sigma_i, \ )$ is independent of the place of the cut $\Sigma_i$, if $\nabla \cdot \boldsymbol{u} = 0$ in $\Omega$ and $\boldsymbol{n} \cdot \boldsymbol{u} = 0$ on $\partial\Omega$.

We denote by $L^2(\Omega)$ the totality of the square-integrable Lebesgue-measurable functions on $\Omega$, which we topologize by the standard inner product $(u, v) = \int_\Omega u \cdot v \, dx$ and the norm $\|u\| = (u, u)^{1/2}$. For vector-valued functions, we write $\boldsymbol{u} \cdot \boldsymbol{v} = \sum_j u_j v^j$, and define the Hilbert space in the same way —we denote it by the same symbol $L^2(\Omega)$. We define a subspace of the vector-valued $L^2(\Omega)$:

$$L^2_\Sigma(\Omega) = \{\boldsymbol{w};\ \nabla \cdot \boldsymbol{w} = 0,\ \boldsymbol{n} \cdot \boldsymbol{w} = 0,\ \Phi(\Sigma_i, \boldsymbol{w}) = 0 \, (\forall j)\},$$

where $\boldsymbol{n}\cdot$ is the "trace operator" that evaluates the normal component of the vector at the boundary. In terms of the *vector potential*, we may write

$$L^2_\Sigma(\Omega) = \{\nabla \times \boldsymbol{w};\ \boldsymbol{w} \in H^1(\Omega),\ \boldsymbol{n} \times \boldsymbol{w} = 0\}, \tag{4}$$

where $\boldsymbol{n}\times$ is the trace of the tangential components on the boundary. We find that

$$L^2(\Omega) = L^2_\Sigma(\Omega) \oplus \mathrm{Ker(curl)}. \tag{5}$$

We call $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ the *vorticity* of a vector field $\boldsymbol{u}$. Conversely, we call $\boldsymbol{u}$ the *vector potential* of a given $\boldsymbol{\omega}$. The vector potential has the gauge freedom, i.e., with an arbitrary *gauge field* $\boldsymbol{g} \in$ Ker(curl), we may write $\boldsymbol{\omega} = \nabla \times \boldsymbol{u} = \nabla \times (\boldsymbol{u} + \boldsymbol{g})$.

The *helicity* of $\boldsymbol{\omega}$ is the quadratic form

$$K = \int_\Omega \boldsymbol{u} \cdot \boldsymbol{\omega} \, dx. \tag{6}$$

As $K$ is a quantity defined by $\boldsymbol{\omega}$, $K$ is gauge-dependent; transforming $\boldsymbol{u} \to \boldsymbol{u}' = \boldsymbol{u} + \boldsymbol{g}$ with some $\boldsymbol{g} \in$ Ker(curl), we obtain

$$K' = \int_\Omega \boldsymbol{u}' \cdot \boldsymbol{\omega} \, dx = K + \int_\Omega \boldsymbol{g} \cdot \boldsymbol{\omega} \, dx.$$

By (5), we find that the helicity of $\boldsymbol{\omega} \in L^2_\Sigma(\Omega)$ is gauge-invariant.

The following relations are obvious:

**Proposition 1.** *Let $\boldsymbol{u}$ be a smooth vector field defined on a smoothly bounded domain $\Omega \subset \boldsymbol{R}^3$.*

*(1) If $\boldsymbol{u}$ is written as $\boldsymbol{u} = \nabla\phi$, $\boldsymbol{u}$ has zero vorticity, i.e., $\nabla \times (\nabla\phi) = 0$.*

*(2) If $\boldsymbol{u}$ is written as $\boldsymbol{u} = \nabla\phi + \alpha\nabla\beta$, $\boldsymbol{u}$ may have a non-zero vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{u} = \nabla\alpha \times \nabla\beta$, and $\boldsymbol{\omega}$ has a helicity*

$$\int_\Omega \boldsymbol{u} \cdot \boldsymbol{\omega}\ dx = \int_\Omega \nabla\phi \cdot \nabla\alpha \times \nabla\beta\ dx = \int_{\partial\Omega} \boldsymbol{n} \cdot (\phi\nabla\alpha \times \nabla\beta)\ ds.$$

*(3) Suppose that $\boldsymbol{\omega} = \nabla \times \boldsymbol{u} \in L^2_\Sigma(\Omega)$ (cf. (4)). If such $\boldsymbol{u}$ is written as $\boldsymbol{u} = \nabla\phi + \alpha\nabla\beta$, $\boldsymbol{\omega}$ has zero helicity.*

By Proposition 1-(3), we find that the Clebsch parameterization $\boldsymbol{u} = \nabla\phi + \alpha\nabla\beta$ falls short to represent a general vector:

**Theorem 1.** *Almost every member $\boldsymbol{\omega} \in L^2_\Sigma(\Omega)$ has a finite helicity, thus it cannot be written in the Clebsch 2-form.*

We start with the following lemma that clarifies some basic properties of the space $L_\Sigma(\Omega)$.

**Lemma 1.** *Let us consider a vector field $\boldsymbol{\omega} \in L^2_\Sigma(\Omega)$.*
*(1) We may decompose $\boldsymbol{\omega}$ as*

$$\boldsymbol{\omega} = \sum_{j=1}^{\infty} \omega_j \boldsymbol{\varphi}_j \tag{7}$$

*with the* Beltrami eigenfunctions *$\boldsymbol{\varphi}_j$ such that*

$$\nabla \times \boldsymbol{\varphi}_j = \lambda_j \boldsymbol{\varphi}_j \quad (j = 1, 2, \cdots). \tag{8}$$

*All eigenvalues $\lambda_j$ ($j = 1, 2, \cdots$) are non-zero real numbers, and eigenfunctions are mutually orthogonal, i.e., $(\boldsymbol{\varphi}_j, \boldsymbol{\varphi}_k) = \delta_{jk}$.*
*(2) We may de-curl $\boldsymbol{\varphi}_j$ to define the vector potential*

$$\boldsymbol{u}_j = \lambda_j^{-1} \boldsymbol{\varphi}_j + \nabla\chi_j \quad (j = 1, 2, \cdots), \tag{9}$$

*where $\chi_j$ can be chosen to satisfy the boundary condition $\boldsymbol{n} \times \boldsymbol{u}_j = 0$. The vector potential of $\boldsymbol{\omega}$ is given by*

$$\boldsymbol{u} = \sum_{j=1}^{\infty} \omega_j \left( \lambda_j^{-1} \boldsymbol{\varphi}_j + \nabla\chi_j \right). \tag{10}$$

*(3) Each $\boldsymbol{\varphi}_j$ has a non-zero (gauge-invariant) helicity:*

$$\int_\Omega \boldsymbol{u}_j \cdot \boldsymbol{\omega}_j\ dx = \lambda_j^{-1}. \tag{11}$$

*The helicity of $\boldsymbol{\omega}$ is given by*

$$\int_\Omega \boldsymbol{u} \cdot \boldsymbol{\omega}\ dx = \sum_{j=1}^{\infty} \lambda_j^{-1} \omega_j^2. \tag{12}$$

(proof) The complete spectral resolution of the "self-adjoint curl operator" in $L^2_\Sigma(\Omega)$ allows us to decompose $\omega \in L^2_\Sigma(\Omega)$ as (7); see Yoshida & Giga [12]. The boundary value of a Beltrami eigenfunction can be written as $\boldsymbol{n} \times \boldsymbol{\varphi}_j = \boldsymbol{n} \times \nabla \eta_j$. By smoothly extending the function $\eta_j(\boldsymbol{x})$ $(\boldsymbol{x} \in \partial\Omega)$ into $\Omega$, we can define $-\chi_j$. Then $\boldsymbol{u}_j$ of (9) satisfies $\nabla \times \boldsymbol{u}_j = \boldsymbol{\varphi}_j$ and $\boldsymbol{n} \times \boldsymbol{u}_j = 0$. By (5), (11) and (12) are obvious.

(QED)

(proof of Theorem 1) By (12), we find that a general member of $L_\Sigma(\Omega)$ has a non-zero helicity. As shown in Proposition 1-(3), however, for an $\omega \in L_\Sigma(\Omega)$ to be written in the Clebsch 2-form, it must have zero helicity: this condition reads as

$$\sum_j \lambda_j^{-1} \omega_j^2 = 0, \tag{13}$$

which restricts the expansion coefficients $\omega_j$. Thus almost every $\omega \in L_\Sigma(\Omega)$, having a non-zero helicity, cannot be cast in the Clebsch 2-form.

(QED)

From the zero-helicity condition (13), which describes a quadratic relation among infinite-dimensional vector components $\omega_j$, it is obvious that the set of the zero-helicity vorticities is not a linear subspace (we note that the zero-helicity condition is a necessary condition for $\omega \in L^2_\Sigma(\Omega)$ to be written in the Clebsch 2-form, but not a sufficient condition). About the "nonlinearity" of the Clebsch forms, we have the following direct observation.

**Proposition 2.** *The set of all Clebsch 2-forms (or Clebsch 1-forms) is not a linear space.*

(proof) We denote by $\mathcal{W}^{(1)}(\Omega)$ the totality of smooth Clebsch 2-forms:

$$\mathcal{W}^{(1)}(\Omega) = \{\nabla\alpha \times \nabla\beta;\ \alpha, \beta \in C^\infty(\Omega)\}.$$

If $\mathcal{W}^{(1)}(\Omega)$ is a linear space, the sum of any pair 2-forms

$$(\nabla\alpha_1) \times (\nabla\beta^1) + (\nabla\alpha_2) \times (\nabla\beta^2)$$

is a member of $\mathcal{W}^{(1)}(\Omega)$, and hence, it can be rewritten as $(\nabla\alpha) \times (\nabla\beta)$ with some $\alpha$ and $\beta$. Let $x^1, x^2, x^3$ be the Cartesian coordinates of $\mathbf{R}^3$, and $\alpha_1(\boldsymbol{x}), \alpha_2(\boldsymbol{x}), \alpha_3(\boldsymbol{x})$ be three independent scalar functions. Each $\alpha_j \nabla x^j$ $(j = 1, 2, 3)$ is a member of $\mathcal{W}^{(1)}(\Omega)$. If we assume that $\mathcal{W}^{(1)}(\Omega)$ is a linear space, the linear combination

$$\omega = \sum_{j=1}^{3} (\nabla\alpha_j) \times (\nabla x^j) \tag{14}$$

must be a member of $\mathcal{W}^{(1)}(\Omega)$. We notice that the right-hand side of (14) is nothing but the curl of a general covariant vector $(\sum \alpha_j \nabla x^j)$. Thus the claim that $\boldsymbol{\omega} \in \mathcal{W}^{(1)}(\Omega)$ implies that every exact contravariant vector (2-form) can be cast in a Clebsch 2-form, contradicting Theorem 1. Therefore, $\mathcal{W}^{(1)}(\Omega)$ cannot be a linear space. From the foregoing argument, it is obvious that the set of all Clebsch 1-forms also cannot be a linear space.

(QED)

In this subsection, we have revealed the "incompleteness" of the Clebsch parameterization pertaining to the helicity, and have shown that the Clebsch parameterization does not apply *almost everywhere* in the function space (i.e., the totality of the Clebsch 2-forms is included in the zero-helicity subset that is an *hyper-surface* in the space $L_\Sigma(\Omega)$). In the next subsection, we shall point out another aspect of the specialty of Clebsch forms.

## 2.2 Remarks on the integrability

A Clebsch 2-form can describe only "integrable" dynamics. Let us consider a three-dimensional autonomous dynamical system governed by an ordinary differential equation

$$\frac{d}{dt}\boldsymbol{x} = \boldsymbol{\omega}(\boldsymbol{x}), \tag{15}$$

where $\boldsymbol{\omega}$ is a Lipschitz continuous vector field in $\Omega$. Here we assume that $\boldsymbol{\omega} \in L_\sigma(\Omega)$ so that the dynamics is conservative $(\nabla \cdot \boldsymbol{\omega} = 0)$ and the orbits are confined in the domain $\Omega$ $(\boldsymbol{n} \cdot \boldsymbol{\omega} = 0)$.

We say that a dynamical system on an $n$-dimensional system is *integrable* if there are $n-1$ independent constants of motion. Let us show how such constants of motion are related to the Clebsch parameters.

We assume that $\boldsymbol{\omega}$ can be cast in a Clebsch 2-form

$$\boldsymbol{\omega}(\boldsymbol{x}) = \nabla\alpha(\boldsymbol{x}) \times \nabla\beta(\boldsymbol{x}). \tag{16}$$

Here we do not have to restrict the functions $\alpha(\boldsymbol{x})$ and $\beta(\boldsymbol{x})$ to be single-valued. For every solution $\boldsymbol{x}(t)$ of (15), with an arbitrary initial condition, we observe

$$\begin{aligned} \frac{d}{dt}\alpha(\boldsymbol{x}(t)) &= \nabla\alpha \cdot \left(\frac{d\boldsymbol{x}(t)}{dt}\right) = \nabla\alpha \cdot \boldsymbol{\omega} = 0, \\ \frac{d}{dt}\beta(\boldsymbol{x}(t)) &= \nabla\beta \cdot \left(\frac{d\boldsymbol{x}(t)}{dt}\right) = \nabla\beta \cdot \boldsymbol{\omega} = 0. \end{aligned}$$

Thus the Clebsch parameters $\alpha(\boldsymbol{x})$ and $\beta(\boldsymbol{x})$ are the constants of motion (they are independent in the sense that $\nabla\alpha \times \nabla\beta \neq 0$). The orbit $\boldsymbol{x}(t)$ is contained in the level-sets of both functions $\alpha(\boldsymbol{x})$ and $\beta(\boldsymbol{x})$, implying that the orbit is the intersection of these two "integral surfaces".

The foregoing argument reveals the geometrical specialty of the Clebsch 2-forms —they can describe only integrable dynamics. Needless to say, a general three-dimensional dynamics is not integrable; the limitation of the Clebsch parameterization of the generator $\omega$ is directly related to the nonintegrability of the orbits.

### 2.3 Superfluity

The Clebsch forms fall short to represent general vector fields, and yet they are "superfluous" in the sense that the map $\mathcal{C}: \ (u_1, u_2, u_3) \mapsto (\varphi, \alpha, \beta)$, when it is definable, is not unique. In fact, the Clebsch 1-form $\nabla\varphi + \alpha\nabla\beta$ is invariant under the transformation such that

$$\alpha \to \alpha + f(\beta), \quad \varphi \to \varphi - F(\beta) \tag{17}$$

with an arbitrary $f(\beta)$ and $F(\beta) = \int f(\beta)d\beta$, or

$$\beta \to \beta + g(\alpha), \quad \varphi \to \varphi - [G(\alpha) - \alpha g(\alpha)] \tag{18}$$

with an arbitrary $g(\alpha)$ and $G(\alpha) = \int g(\alpha)d\alpha$.

Imposing an appropriate boundary conditions on variables $(\varphi, \alpha, \beta)$ may suppress the superfluity of the parameterization, but they may reduce the possibility of the parameterization (i.e. the domain of the map $\mathcal{C}$).

## 3 Generalized Clebsch Form

### 3.1 Formulation of generalized Clebsch forms

Our next mission is to generalize the Clebsch form and formulate a complete parameterization of general vector fields. We proffer a generalization of (1) such as

$$\boldsymbol{u} = \nabla\varphi + \sum_{j=1}^{\nu} \alpha_j \nabla\beta^j. \tag{19}$$

If we take $\nu = n$ (the dimension of the coordinate space), (19) is complete even if we eliminate the first term $\nabla\varphi$. In fact, if we put $\varphi = 0$ and $\beta^j = x^j$ (the coordinates of the Euclidean frame) (19) is nothing but the covariant form of a general vector field ($\boldsymbol{u} = \alpha_j dx^j$).

In this section, we consider vector fields on a bounded domain $\Omega$ of an Affine space of a general dimension $n$ ($2 \leq n < \infty$). We shall show that the generalized

Clebsch 1-form (19) is "complete" when $\nu = n-1$. This is equivalent to the fact that the generalized Clebsch 2-form

$$\omega = \sum_{j=1}^{\nu} \nabla\alpha_j \times \nabla\beta^j. \tag{20}$$

is complete to represent general exact 2-forms on $\Omega$.

### 3.2 Classical construction of the Clebsch parameters

Let $\{e^1, \cdots, e^n\}$ be the orthonormal system of unit vectors spanning an $n$-dimensional Euclidean space $\mathbf{R}^n$, and $\{x^1, \cdots, x^n\}$ be the corresponding coordinates ($e^j = dx^j$). We consider a bounded open set $\Omega \subset \mathbf{R}^n$ whose boundary $\partial\Omega$ is sufficiently smooth ($C^1$-class).

First we assume that $\Omega$ is a sphere $\mathcal{S}_R$ with a finite radius $R$ (centered at $x = 0$). Let $u$ be a 1-form on $\mathcal{S}_R$ such that

$$u = \sum_{j=1}^{n} u_j dx^j, \tag{21}$$

where $u_j(x^1, \cdots, x^n) \in C^1(\mathcal{S}_R)$. Let us choose one $dx^j$, say $dx^n$. We define a "Clebsch potential" $\varphi(x^1, \cdots, x^n)$ by

$$\varphi(x^1, \cdots, x^n) = \int_0^{x^n} u_n(x^1, \cdots, x^{n-1}, y)dy. \tag{22}$$

Since the path of the integral on the right-hand side is included in $\mathcal{S}_R$, this integral is well defined. Putting $u'_j = u_j - \partial_{x^j}\varphi$ ($j = 1, \cdots, n-1$), we may rewrite (21) as

$$u = \sum_{j=1}^{n-1} u'_j dx^j + d\varphi. \tag{23}$$

We thus find that any smooth ($C^1$-class) vector in a spherical domain $\mathcal{S}_R$ can be represented as a generalized Clebsch 1-form (23) (which is an realization of (19) with $\nu = n-1$ and $\beta^j = x^j$).

From the foregoing construction of (22), it is evident that the domain $\Omega$ may be a general set such that the path of the integral (22) stays always in $\Omega$ for every $x^1, \cdots, x^{n-1}$.

We may consider a more general $\Omega$. Suppose that $\Omega$ is a $C^1$-class bounded domain, and $u_j \in C^1(\overline{\Omega})$ ($j = 1, \cdots, n$). Then, we can extend each $u_j$ into some larger sphere $\mathcal{S}_R$ ($\overline{\Omega} \subset \mathcal{S}_R$), i.e., there is an extension $\tilde{u}_j \in C^1(\mathcal{S}_R)$ such that $\tilde{u}_j = u_j$ in $\Omega$. Defining $\tilde{\varphi}$ by (22) in $\mathcal{S}_R$, and restricting $\tilde{\varphi}$ and $\tilde{u'}_j = \tilde{u}_j - \partial_{x^j}\tilde{\varphi}$ ($j = 1, \cdots, n-1$) in $\Omega$, we obtain the generalized Clebsch 1-form (23).

We thus have the following theorem:

**Theorem 2.** *Let $\Omega$ be a $C^1$-class bounded open set in $\boldsymbol{R}^n$.*
*(1) Every 1-form (or covariant vector field) $u \in C^1(\overline{\Omega})$ may be rewritten as a Clebsch 1-form such that*

$$u = \sum_{j=1}^{n-1} u'_j dx^j + d\varphi. \tag{24}$$

*(2) Every exact 2-form $\omega\,(= du) \in C^1(\overline{\Omega})$ may be written as a Clebsch 2-form such that*

$$\omega = \sum_{j=1}^{n-1} du'_j \wedge dx^j. \tag{25}$$

Here we have given a "classical" construction of the Clebsch parameters. We may prove the *completeness* under a more general assumptions, i.e., for more general "coordinate variables" $\beta^1, \cdots \beta^n$ and distributions $\alpha_1, \cdots, \alpha_n \in L^2(\Omega)$; see Ref. [11].

## 4 Concluding Remarks

There are a variety of methods to represent (parameterize) vector fields, and each of them has some particular advantage in developing theories. The Clebsch representation features a "nonlinear" formulation —the term $\alpha\nabla\beta$ involves two variables nonlinearly, and this nonlinearity may simplify the nonlinear term of some equation that governs the vector field.

In the discussions of Sec. 3, the parameters $\beta^j$ are dealt as "coordinates", so the forms (19) and (20) are, in principle, linear with respect to the fields $\alpha_j$ and $\varphi$ that are related with a given vector field (and by them, the number of the field variables balances the dimension of the vector field). If we consider that $\beta^1, \cdots, \beta^\nu$ are "dynamical variables", however, some obstacles may emerge. For instance, in (19), we do not have to restrict $\beta^j$ to be a single-valued function —it may be an "angular coordinate" (modulo $2\pi$). But when we deal with $\beta^j$ as a dynamical variable, this field must be usually a single-valued function.

The boundary conditions also pose constraints on the parameterization when we consider each parameter to represent independent degree of freedom. From the construction (22) of the potential $\varphi$, we find that the boundary value of $\varphi$ cannot be related to those of $u_j$ (it depends on the internal distribution of $u_n$) and through $\partial_{x^j}\varphi$ the boundary values of $u'_1, \cdots, u'_{n-1}$ are also modified. We thus encounter difficulty when we have to control the boundary value of each Clebsch parameter in accordance with the physical conditions on the boundary value of $\boldsymbol{u}$. For example, if we assume, in (19) or (20), that $\boldsymbol{n} \cdot \nabla\beta^j = 0$ $(j = 1, \cdots, n-1)$ and $\beta^n =$ constant on the boundary, all of $\beta^j$ ($j = 1, \cdots, n-1$) cannot be single-valued. Thus, we may not assume $\nu = n - 1$ to represent general vector fields.

## References


[1] A. Clebsch, J. Reine Angew. Math. **56**, 1 (1859).

[2] J.R. Cary and R.G. Littlejohn, Ann. Phys. **151**, 1 (1983).

[3] A.H. Boozer, Phys. Fluids **26**, 1288 (1983).

[4] Z. Yoshida, Phys. Plasmas **1**, 208 (1994).

[5] H. Grad and H. Rubin, *Proc. of Second United Nation Conference on Peaceful Uses of Atomic Energy*, Vol. 31, Geneva, (1958) 190.

[6] C.C. Lin, in *Proc. Int. Sch. Phys. "Enrico Fermi" XXI*, (Academic Press, 1963), 93.

[7] P.J. Morrison and J.M. Greene, Phys. Rev. Lett. **45**, 790 (1980).

[8] P.J. Morrison, Rev. Mod. Phys. **70**, 467 (1998).

[9] J. Marsden and A. Weinstein, Physica **7D**, 305 (1983).

[10] R. Jackiw, *Lectures on Fluid Dynamics –a particle theorist's view of supersymmetric, non-Abelian, noncommutative fluid mechanics and d-branes* (Springer, New York, 2002).

[11] Z. Yoshida, *Clebsch parameterization: basic properties and its applications*, J. Math. Phys. (in print).

[12] Z. Yoshida and Y. Giga, Math. Z. **204**, 235 (1990).