数理解析研究所講究録
第 1699 巻 2010 年 25-29

Siegel disks and wandering domains of transcendental entire functions

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Abstract

We give a brief survey of results on the dynamics of transcendental entire functions with Siegel disks whose singular values are just two points. One of the two singular values is not only a superattracting fixed point with multiplicity more than two but also an asymptotic value. Another one is a critical value with free dynamics under iterations. If the multiplicity of the superattracting fixed point is large enough, then the restriction of the transcendental entire function near the Siegel point is a quadratic-like map. As its applications, the logarithmic lift of the above transcendental entire function has a wandering domain whose shape looks like a Siegel disk of a quadratic polynomial.

1 Results

In this article, we consider the transcendental entire function

$$f(z) = z^n \exp\{(\lambda - n)(z-1)\},$$

where $n \geq 2$, $\lambda = e^{2\pi i\alpha}$ and $0 < \alpha < 1$ is a Bryuno number. The origin is a superattracting fixed point and 1 is the center of a Siegel disk Δ with rotation number α. The function f has the following properties:

- The origin is a superattracting fixed point with multiplicity n.
- 1 is a fixed point with multiplier $f'(1) = \lambda$.
- $n/(n-\lambda)$ is a critical point and there is no other critical point.

- The origin is an asymptotic value.
- There is no singular value other than the origin and $f(n/(n-\lambda))$.
- f is of finite order.

Main Result. *If n is large enough, then there exist bounded simply connected domains U and V satisfying $1 \in U \Subset V$ such that $f : U \to V$ is a quadratic-like map.*

By the Main Result, properties of the Siegel disk Δ of f centered at 1 and its boundary $\partial\Delta$ correspond to those of the Siegel disk $\mathcal{D}$ of Q centered at the origin and its boundary $\partial\mathcal{D}$, where $Q(z) = e^{2\pi i\alpha}z + z^2$. For examples:

1. *The function f is locally linearizable at 1 if and only if α is a Bryuno number.*
2. *If α is of bounded type, then the boundary $\partial\Delta$ is a quasicircle containing the critical point $n/(n-\lambda)$.*
3. *The boundary $\partial\Delta$ is a quasicircle but the critical point $n/(n-\lambda)$ is not on $\partial\Delta$ for some α.*

Topological and quasiconformal statements which hold for the Siegel disk of the quadratic polynomial Q centered at the origin hold for the Siegel disk of the transcendental entire function f centered at 1.

2 Applications

We consider the logarithmic lift of f,

$$\tilde{f}(z) = nz + (\lambda - n)(e^z - 1).$$

Then the functional equation $\exp \circ \tilde{f} = f \circ \exp$ holds. The Fatou component $B = \tilde{f}(B)$ such that $\exp B$ is the immediate basin of f at the origin is an invariant Baker domain of $\tilde{f}$. The origin is a fixed point of $\tilde{f}$ with multiplier $\lambda = e^{2\pi i\alpha}$. Let $\tilde{\Delta}$ be the Siegel disk of $\tilde{f}$ centered at the origin and $\tilde{\Delta}_k$ the Fatou component containing $2\pi ki$, where k is a non-zero integer. Since the functional equation $\exp \circ \tilde{f} = f \circ \exp$ holds, the exponential map projects $\tilde{\Delta}$ and $\tilde{\Delta}_k$ down to the Siegel disk Δ of f centered at 1. The behavior of $2\pi ki$ is

$$2\pi ki \overset{\tilde{f}}{\mapsto} 2\pi kni \overset{\tilde{f}}{\mapsto} 2\pi kn^2 i \overset{\tilde{f}}{\mapsto} \cdots \overset{\tilde{f}}{\mapsto} 2\pi kn^m i \overset{\tilde{f}}{\mapsto} \cdots$$

or $\tilde{f}^m(2\pi ki) = 2\pi kn^m i$. Therefore $\{\tilde{\Delta}_{\pm k}\}_{k:\text{a prime number}}$ is a family of infinitely many wandering domains having distinct orbits.

The logarithmic lift $\tilde{f}$ has the following properties:

- The Fatou component $B = \tilde{f}(B)$ such that $\exp B$ is the immediate basin of f at the origin is an invariant Baker domain.
- The Siegel disk $\tilde{\Delta}$ of $\tilde{f}$ centered at the origin projects down to the Siegel disk Δ of f centered at 1 via the exponential map.
- Any Fatou component $\tilde{\Delta}_k$ containing $2\pi ki$ is a wandering domain and projects down to the Siegel disk Δ of f centered at 1 via the exponential map.
- $\{\tilde{\Delta}_{\pm k}\}_{k:\text{a prime number}}$ is a family of infinitely many wandering domains having distinct orbits.
- $\omega_k = \log|n/(n-\lambda)| + i\cdot[\arg\{n/(n-\lambda)\} + 2\pi k]$ are critical points.

The following is an application of the Main Result:

Corollary. *If n is large enough, then the following statements hold*:

(1) *The Siegel disk $\tilde{\Delta}$ of $\tilde{f}$ centered at the origin is a "quadratic-polynomial-type" Siegel disk.*

(2) *The shape of Wandering domains $\tilde{\Delta}_k$ containing $2\pi ki$ looks like the Siegel disk Δ of f centered at* 1.

The Corollary indicates that if the irrational number α is of bounded type, then

- the boundary $\partial\tilde{\Delta}$ is a quasicircle containing the critical point ω_0,
- the boundary $\partial\tilde{\Delta}_k$ is also a quasicircle containing the critical point ω_k.

Acknowledgment. This work was supported by Grant-in-Aid for Young Scientists (B) No. 21740121 of Japan Society for the Promotion of Science.

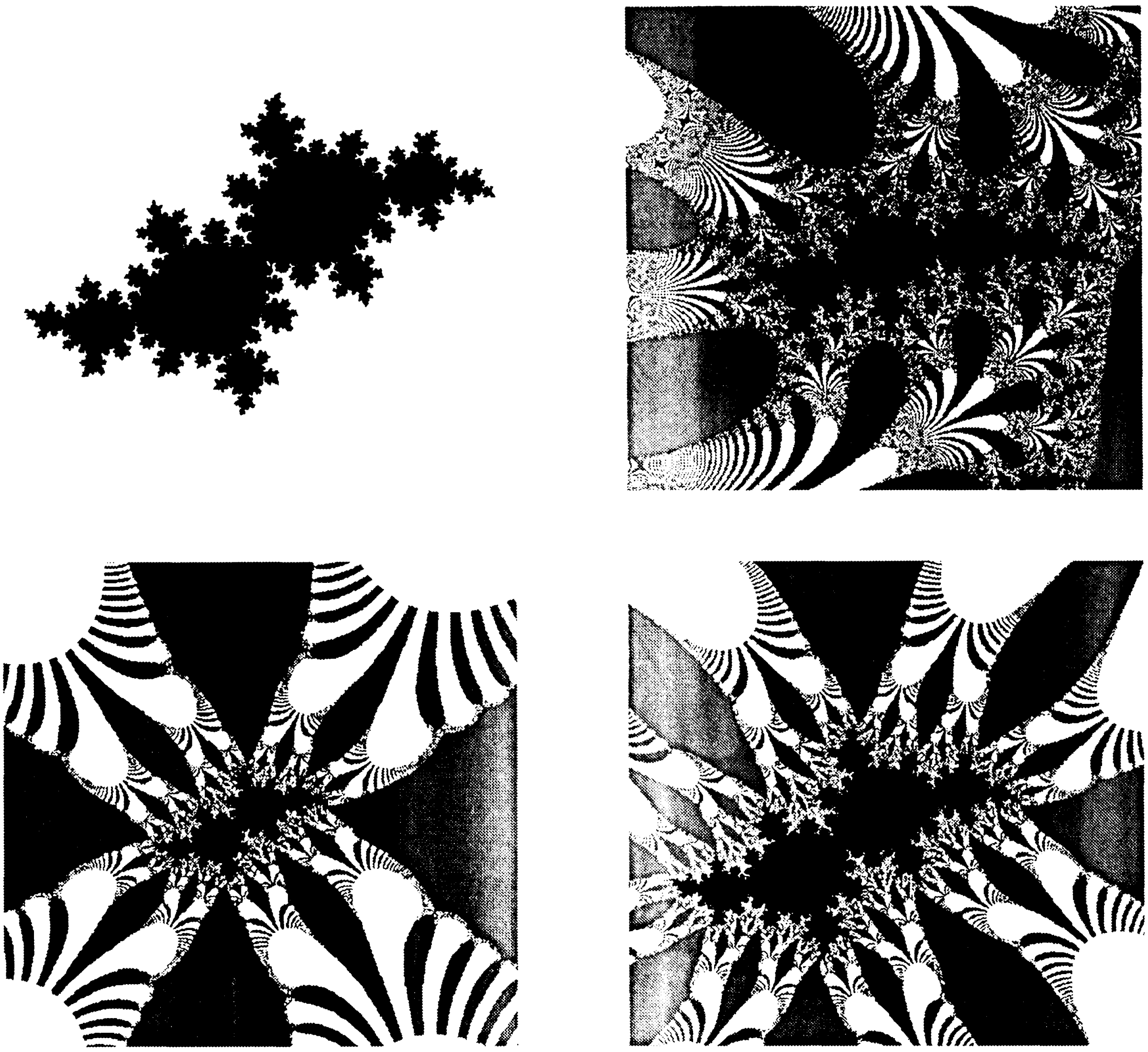

Figure 1: The upper left-hand figure is the Siegel disk $\mathcal{D}$ of the quadratic polynomial $Q(z) = e^{2\pi i\alpha}z + z^2$ centered at the origin, where the rotation number $\alpha = (\sqrt{5} - 1)/2 = [1, 1, 1, \dots]$. Another all figures are the Siegel disk Δ of the transcendental entire function f centered at 1. Black regions are the Siegel disk Δ and its preimages. Gray ones are the attracting basin at the origin. The upper right-hand one is in the case that $n = 2$. The lower left-hand one is in the case that $n = 32$. It would be a "quadratic-polynomial-type" Siegel disk. The lower right-hand one is the enlargement of the lower left-hand one.

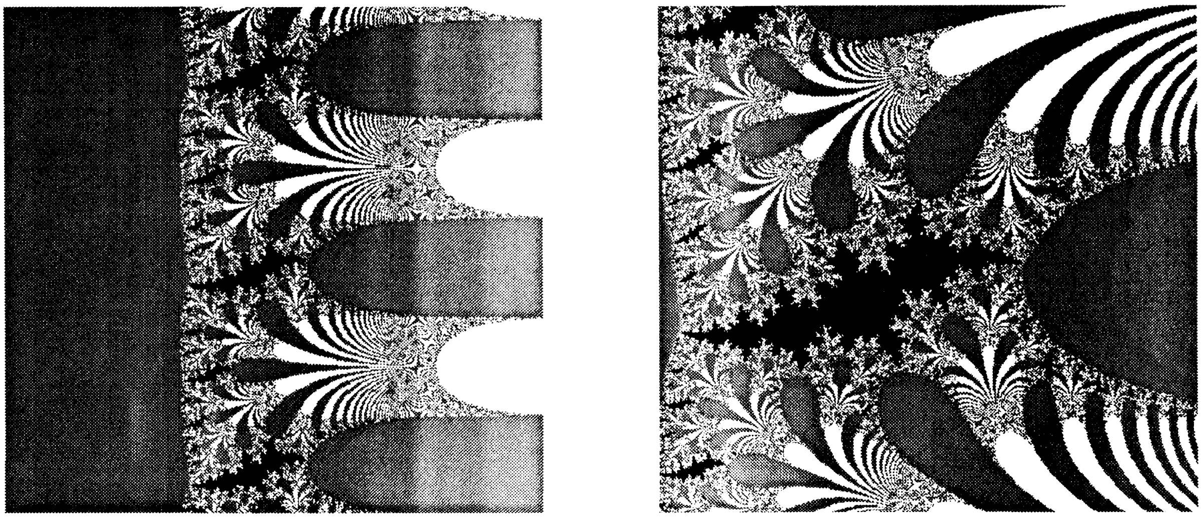

Figure 2: The two figures are in the case that $n = 2$ and the rotation number $\alpha = (\sqrt{5} - 1)/2 = [1, 1, 1, \dots]$. The left-hand one is the Siegel disk $\tilde{\Delta}$ centered at the origin and wandering domains $\tilde{\Delta}_1$ and $\tilde{\Delta}_{-1}$. The right-hand one is the enlargement of the left-hand one. Black regions are the Siegel disk Δ or wandering domains $\tilde{\Delta}_k$ and their preimages. Gray ones are the Baker domain B and its preimages.

References

[1] R. Berenguel and N. Fagella, An entire transcendental family with a persistent Siegel disc, `http://www.gsd.uab.es/personal/nfagella`.

[2] N. Fagella and C. Henriksen, The Teichmüller space of an entire function, *Complex dynamics*, 297–330, A K Peters, Wellesley, MA, 2009.

[3] J. Milnor, *Dynamics in One Complex Variable*, Vieweg, 2nd edition, 2000.

[4] S. Morosawa, Y. Nishimura, M. Taniguchi and T. Ueda, *Holomorphic Dynamics*, Cambridge Studies in Advanced Mathematics 66, 2000.

[5] S. Zakeri, On Siegel Disks of a Class of Entire Maps, `http://www.math.qc.edu/~zakeri/papers/papers.html`.