On a universal framework of the homogenization problems for infinite dimensional diffusions

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Abstract

By restricting the universal framework work of the homogenization problem of infinite dimensional diffusions posed in [AY] to the case where the state space of the ergodic process, that corresponds to the original infinite dimensional diffusion, for which the homogenization problem is considered, a sufficient condition for the mapping between these processes under which the ergodic process is a unique Markov process that corresponds to a unique Markovian extension of a closable symmetric bilinear form is considered.

1 Introduction

In this note, by restricting the universal framework work of the homogenization problem of infinite dimensional diffusions posed in [AY] to the case where the state space of the ergodic process denoted by \((Y_\theta(t))_{t\geq 0}\), that corresponds to \((X_\theta(t))_{t\geq 0}\), the original infinite dimensional diffusion, for which the homogenization problem is considered, we discuss a sufficient condition for the mapping between these processes (denoted by \(T_x(\theta)\)) under which the ergodic process is the one that corresponds to a unique Markovian extension of a closable symmetric bilinear form. Since, the present announcement plays a part of introduction of our subsequent researches on this subject, we give here a statement in a rough style without proof. All the exact and new results on this concern will be found in forthcoming papers.

2 Probability space \((\Theta, \overline{B}, \overline{\mu})\), the ergodic flow and the core

Suppose that we are given the following:

\[ \{(\Theta_k, B_k, \lambda_k)\}_{k \in \mathbb{Z}^d}; \] a system of complete probability (resp. measure) spaces,

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where $d$ is a given natural number. (resp. for each $k$, $\lambda_k$ is a $\sigma$-finite measure.)

$(\Theta, \overline{\mathcal{B}}, \overline{\lambda})$: the probability (resp. complete measure) space that is the completion of $(\prod_k \Theta_k, \otimes_k \mathcal{B}_k, \prod_k \lambda_k)$, i.e., the completion of the direct product probability (resp. complete measure) space.

$(\Theta, \overline{\mathcal{B}}, \mu)$: a complete probability space (corresponding to a Gibbs state) defined as follows:

for $\forall D \subset \subset \mathbb{Z}^d$ and for any bounded measurable function $\varphi$ defined on $\prod_{k \in D} \Theta_k$ with some $\forall D' \subset \subset \mathbb{Z}^d$, $\mu$ satisfies

$$(\mathbb{E}^D \varphi, \mu) = (\varphi, \mu), \quad (2.1)$$

where

$$\mathbb{E}^D \varphi(\theta) \equiv \int_{\Theta} \varphi(\theta' \cdot \theta_{D^c}) \mathbb{E}^D(d\theta' | \theta_{D^c}) \quad (2.2)$$

is the natural projection,

$\theta' \cdot \theta_{D^c}$ is the element $\theta'' \in \Theta$ such that

$\theta'' = \theta', \quad \theta''_{D^c} = \theta_{D^c},$

$D^+ = \{k' | \text{support of } U_{k'} \cap D \neq \emptyset\},$

also, for each $k \in \mathbb{Z}^d$, $U_k$ is a given bounded measurable function of which support is in $\prod_{|k' - k| \leq L} \Theta_{k'}$, where the number $L$ (the range of interactions) does not depend on $k$, and $Z_D(\theta_{D^c})$ is the normalizing constant.

On $(\Theta, \overline{\mathcal{B}}, \lambda)$ we are given a measure preserving map $T_x$ (which is also a map on $(\Theta, \overline{\mathcal{B}}, \mu)$, but is not a measure preserving map on it an ergodic flow) as follows:

Suppose that

$$\exists M_1 < \infty \quad \text{and} \quad \forall k \in \mathbb{Z}^d \quad \text{there exists a } d_k \text{ such that } d_k \leq M_1. \quad (2.4)$$
For each \( x \in \prod_{k} \mathbb{R}^{d_{k}} \) such that \( x = (x^{k})_{k \in \mathbb{Z}^{d}} \) with \( x^{k} = (x_{1}^{k}, \ldots, x_{d_{k}}^{k}) \), the map \( T_{x} \) on \((\Theta, \overline{B}, \overline{\lambda})\) is defined by

i) \( T_{x} : \Theta \rightarrow \Theta \)

that is a measure preserving transformation with respect to the measure \( \overline{\lambda} \); ii) \( T_{0} = \) the identity, 

for \( x, x' \in x \in \prod_{k \in \mathbb{Z}^{d}} \mathbb{R}^{d_{k}} \) \( T_{x+x'} = T_{x} \circ T_{x'}, \)

where 

\[
x + x' \equiv (x^{k} + x'^{k})_{k \in \mathbb{Z}^{d}},
\]

with 

\[
x^{k} + x'^{k} = (x_{1}^{k} + x_{1}'^{k}, \ldots, x_{d_{k}}^{k} + x_{d_{k}}'^{k}),
\]

for 

\[
x = (x^{k})_{k \in \mathbb{Z}^{d}}, \quad x^{k} = (x_{1}^{k}, \ldots, x_{d_{k}}^{k}),
\]

\[
x' = (x'^{k})_{k \in \mathbb{Z}^{d}}, \quad x'^{k} = (x_{1}'^{k}, \ldots, x_{d_{k}}'^{k}),
\]

and 

\[
0 \equiv (0^{k})_{k \in \mathbb{Z}^{d}}, \quad 0^{k} = (0, \ldots, 0) \in \mathbb{R}^{d_{k}};
\]

iii) 

\[
(x, \theta) \in (\prod_{k \in \mathbb{Z}^{d}} \mathbb{R}^{d_{k}}) \times \Theta \rightarrow T_{x}(\theta) \in \Theta
\]

is \( B(\prod_{k \in \mathbb{Z}^{d}} \mathbb{R}^{d_{k}}) \times \overline{B}/\overline{B} \)-measurable, where \( \prod_{k \in \mathbb{Z}^{d}} \mathbb{R}^{d_{k}} \) is assumed to be the topological space with the direct product topology;

iv) A function which is \( T_{x} \) invariant for all \( x \in \prod_{k \in \mathbb{Z}^{d}} \mathbb{R}^{d_{k}} \) is a constant function on \((\Theta, \overline{B}, \mu)\);

v) For \( D \subset \mathbb{Z}^{d} \), let 

\[
\prod_{k \in \mathbb{Z}^{d}} \mathbb{R}^{d_{k}} \ni x \mapsto x_{D} \in \prod_{k \in D} \mathbb{R}^{d_{k}}
\]

be the natural projection. If \( x_{D^{c}} = 0_{D^{c}} \), then 

\[
(T_{x}(\theta))_{D^{c}} = \theta_{D^{c}}, \quad \forall \theta \in \Theta, \quad \forall D \subset \subset \mathbb{Z}^{d}.
\]
We assume that an existence of a core \( \mathcal{D}^\Theta \). Namely, there exists \( \mathcal{D}^\Theta \) which is a dense subset of both \( L^2(\mu) \) and \( L^1(\mu) \), and \( \forall \varphi \in \mathcal{D}^\Theta \) satisfies

(D-1) \( \varphi \) is a bounded measurable function having only a finite number of variables \( \theta_D \) for some \( D \subset \subset \mathbb{Z}^d \),

(D-2) \( \varphi(T_{x_D}(\theta)) \in C^\infty(\prod_{k \in D} \mathbb{R}^{d_k} \to \mathbb{R}), \ \forall \theta \in \Theta \),

(cf. v) in the previous section) where we identify \( x_D \in \prod_{k \in D} \mathbb{R}^{d_k} \) with an \( x \in (\prod_{k \in \mathbb{Z}^d} \mathbb{R}^{d_k}) \) of which projection to \( \prod_{k \in \dot{D}} \mathbb{R}^{d_k} \) is \( x_D \).

(D-3) in (D-2) for each \( \theta \in \Theta \), all the partial derivatives of all orders of the function \( \varphi(T(\theta)) \) (with the variables \( x_D \)) are bounded and

\[ \forall \varphi \in \mathcal{D}, \exists M < \infty; \quad |\nabla_k \varphi(T_{x}(\theta))| < M, \ \forall \theta \in \Theta, \forall x, \forall k \in \mathbb{Z}^d, \] (2.5)

where

\[ \nabla_k = (\frac{\partial}{x_1^k}, \ldots, \frac{\partial}{x_{d_k}^k}). \]

\[ \square \]

3 Probability space \((\Omega, \mathcal{F}, P; \mathcal{F}_t)\) and the processes

Suppose that we are given a system of family of functions \( a_{ij}^k \), \( k \in \mathbb{Z}^d \), \( 1 \leq i, j \leq d_k \) on \((\Theta, \overline{B}, \overline{\mu})\) such that for each \( k \in \mathbb{Z}^d \) and each \( 1 \leq i, j \leq d_k \), \( a_{ij}^k \) is a measurable function on \( \Theta_k \) and there exists \( M_2 \in (0, \infty) \) and

\[ M_2^{-1} \|x\|^2 \leq \sum_{1 \leq i, j \leq d_k} a_{ij}^k(\theta_k)x_i x_j \leq M_2 \|x\|^2, \ \forall k \in \mathbb{Z}^d, \forall \theta_k \in \Theta_k, \] (3.1)

also

\[ a_{ij}^k(\cdot) = a_{ji}^k(\cdot). \]

We assume that

\[ U_k, \ a_{ij}^k \in \mathcal{D}^\Theta, \ \ k \in \mathbb{Z}^d, \ 1 \leq i, j \leq d_k. \]
Also, we assume that there exists a common $M < \infty$ by which the evaluation (2.5) holds for all $a_{i,j}^k$ and $U_k$.

Finally, suppose that we are given a complete probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$, $(t \in \mathbb{R}_+)$ with a filtration $\mathcal{F}_t$. On $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ suppose that there exists a system of independent $1$-dimensional $\mathcal{F}_t$-adapted Brownian motion processes

$$\{(B^k_i(t))_{t \geq 0}\}_{k \in \mathbb{Z}^d, 1 \leq i \leq d_k}.$$ 

Now, for each $\theta \in \Theta$, let

$$X^\theta \equiv \{(X^\theta,k,i(t))_{t \geq 0}\}_{k \in \mathbb{Z}^d, 1 \leq i \leq d_k}$$

be the unique solution of

$$X^{\theta,k,i}(t) = X^{\theta,k,i}(0) + \int_0^t \sum_{1 \leq j \leq d_k} \left\{ \frac{\partial}{\partial x_j^k} a_{ij}^k(T_{X^\theta,k}(s)(\theta)) - a_{ij}^k(T_{X^\theta,k}(s)(\theta)) \frac{\partial}{\partial x_j^k} \left( \sum_{k' \in \{k\}^+} U_{k'}(T_{X^\theta}(s)(\theta)) \right) \right\} ds$$

$$+ \int_0^t \sum_{1 \leq j \leq d_k} \sigma_{ij}^k(T_{X^\theta,k}(s)(\theta)) dB^k_j(s), \quad t \geq 0, \quad (3.2)$$

where, as the matrix sense,

$$(\sigma_{ij}^k) = \left(2a_{ij}^k\right)^{\frac{1}{2}},$$

and

$$X^{\theta,k}(t) = (X^{\theta,k,1}(t), \ldots, X^{\theta,k,d_k}(t)), \quad \{k\}^+ = \{k' \mid \text{support of } U_{k'} \cap \{k\} \neq \emptyset\},$$

also, by $X^\theta(t)$ we denote the vector

$$(X^{\theta,k}(t))_{k \in \mathbb{Z}^d} \in \prod_{k \in \mathbb{Z}^d} \mathbb{R}^{d_k}.$$

To get the unique solution for (3.2) we assume the following:

**Assumption 1.** All the coefficients appeared in (3.2) are uniformly bounded and equi-continuous for all $1 \leq i, j \leq d_k$ and $k \in \mathbb{Z}^d.$

□
Proposition 3.1 Under Assumption 1, for each \( \theta \in \Theta \) the SDE (3.2) has a unique solution, and the random variable \( X^\theta \) on \( (\Omega, \mathcal{F}, P; \mathcal{F}_t) \) is the one taking values in
\[
C([0, \infty) \to \prod_{k \in \mathbb{Z}^d} \mathbb{R}^{d_k}).
\]

Definition 3.1 For \( \theta \in \Theta \), let \( (X^\theta_0(t))_{t \geq 0} \) be the stochastic process defined by (3.2) with the initial condition \( X^\theta_0(0) = 0 \). By using \( (X^\theta_0(t))_{t \geq 0} \) and the map \( T_x(\cdot) \) we define a \( \Theta \)-valued process \( (Y_\theta(t))_{t \geq 0} \) on \( (\Omega, \mathcal{F}, P; \mathcal{F}_t) \) as follows:
\[
(Y_\theta(t))_{t \geq 0} = (X^\theta_0(t))_{t \geq 0}.
\]

4 A homeomorhism

The problem of homogenization of the process \( (X^\theta_0(t))_{t \geq 0} \) is described as follows:

Problem. For each \( \theta \in \Theta \), \( \mu - a.s. \), we are concerning the scaling limit of \( (X^\theta_0(t))_{t \geq 0} \) such that
\[
\lim_{\epsilon \downarrow 0} \{ \epsilon X^\theta_0(\frac{t}{\epsilon^2}) \}_{t \geq 0} \tag{4.1}
\]
More precisely, we consider the weak convergence of (4.1), where the sequence of the processes \( \{ \epsilon X^\theta_0(\frac{t}{\epsilon^2}) \}_{t \geq 0} \) is understood as the sequence of random variables on \( (\Omega \times \Theta, \mathcal{F} \times \mathcal{B}, P \times \mu; \mathcal{F}_t \times \{ \Theta, \emptyset \}) \) taking values in the direct product space
\[
\prod_{k \in \mathbb{Z}^d} C([0, \infty) \to \mathbb{R}^{d_k})
\]
equipped with the direct product topology.

In order to prove the weak convergence of (4.1), the ergodicity of the process \( (Y_\theta(t))_{t \geq 0} \) plays a crucial role (cf. [ABRY 1,2,3] and [AY]). Hence, for a concrete analysis on this problem, in any lal, we have to characterize both the probabilistic and analytic properties of \( (Y_\theta(t))_{t \geq 0} \). In this report, assuming in particular that \( \Theta_k, k \in \mathbb{Z}^d \), are topological spaces, and then we consider a sufficient condition under which \( (Y_\theta(t))_{t \geq 0} \) is a process corresponding to a unique Markovian extension of a symmetric quadratic form.
Definition 4.1 For each $k \in \mathbb{Z}^d$ and $i = 1, \ldots, d_k$, define an operator $D^{k,i} : \mathcal{D}^\Theta \to \mathcal{D}^\Theta$ such that

$$(D^{k,i} \varphi)(\theta) \equiv \frac{\partial}{\partial x^k_i} \varphi(T_x(\theta))|_{x=0}, \quad \varphi \in \mathcal{D}^\Theta, \quad \theta \in \Theta.$$ 

Also, define a quadratic form $\mathcal{E}$ on $L^2(\mu)$ such that

$$\mathcal{E}(\varphi, \psi) \equiv \sum_{k \in \mathbb{Z}^d} \sum_{1 \leq i, j \leq d_k} \int_\Theta (D^{k,i} \varphi)(\theta) a^{k}_{i,j}(\theta) (D^{k,j} \psi)(\theta) \mu(d\theta), \quad \varphi, \psi \in \mathcal{D}^\Theta.$$

Theorem 4.1 Let $\prod_{k \in \mathbb{Z}^d} \mathbb{R}^{d_k}$ be the topological space with the direct product topology, and for each $M > 0$ let $C^{X,M}$ be the space of continuous functions with the uniform convergence topology such that

$$C^{X,M} \equiv \{x(\cdot) | x(\cdot) \in C([0, M] \to \prod_{k \in \mathbb{Z}^d} \mathbb{R}^{d_k}) \text{ with } x(0) = 0\}.$$

Suppose that for each $k \in \mathbb{Z}^d$, $\Theta_k$ is a topological space and let $B_k$ be its Borel $\sigma$-field, also $\Theta = \prod_k \Theta_k$ be the direct product space with the direct product topology. For each $\theta \in \Theta$ and $M > 0$ let $C^{\theta,Y,M}$ be the space of continuous functions with the uniform convergence topology such that

$$C^{\theta,Y,M} \equiv \{y(\cdot) | y(\cdot) \in C([0, M] \to \Theta) \text{ with } y(0) = \theta\}.$$

For any $\theta \in \Theta$ and $M > 0$ if the map $f$ defined by

$$f : C^{X,M} \ni x(\cdot) \longmapsto T_{x(\cdot)}(\theta) \in C^{\theta,Y,M}$$

is a continuous onto one to one mapping of which inverse map $f^{-1}$ is also continuous (i.e. $C^{X,M}$ and $C^{\theta,Y,M}$ are homeomorphic), then the probability law of the process $(Y_\theta(t))_{t \geq 0}$ is identical with the probability law of the Markov process which corresponds to a unique Markovian extension of the quadratic form $\mathcal{E}(\varphi, \psi)$ defined by Definition 4.1.

References

