数理解析研究所講究録
第 1705 巻 2010 年 95-102



# Lifting and Its Application to a Non-Kolmogorovian Model

Masanari Asano, Masanori Ohya
Department of Information Sciences, Tokyo University of Science

## 1 Introduction

Recently a few authors pointed to a possibility to apply the mathematical formalism of quantum mechanics to cognitive psychology, in particular, to games of the Prisoners Dilemma (PD) type. It was found that statistical data obtained in some experiments of cognitive psychology cannot be described by classical probability model (Kolmogorov's model). These experiments play an important role in behavioral economics; these are tests of rationality of behavior of agents acting at the market (including the financial market). Quantum probability is one of the most advanced mathematical models for non-classical probability. Therefore it was natural to try to apply quantum probability to, e.g., PD-type games.

Recently we proposed a quantum-like model of decision making by using the generalized quantum formalism based on lifting of density operators [10]. In [9] we presented a toy model of quantum-like decision making based on a simple system of differential equations for the equilibrium quantum state. This system was a quantum version of the standard system of equation for chemical equilibrium.

## 2 Channels and Liftings

In quantum information theory, a certain map is important for describing an information transition such as a measurement process or a signal transmission. This map is called a channel $\Lambda^* : \mathcal{S}(\mathcal{A}) \mapsto \mathcal{S}(\mathcal{B})$ ($\mathcal{S}(\mathcal{A})$ and $\mathcal{S}(\mathcal{B})$ are state spaces of $C^*$-algebras $\mathcal{A}$ and $\mathcal{B}$.) For example, a set of all bounded linear operators $\mathcal{B}(\mathcal{H})$ on Hilbert space realizes $C^*$-algebras. If a channel is affine, i.e. $\Lambda^*(\sum \lambda_n \rho_n) = \sum \lambda_n \Lambda^*(\rho_n), \forall \rho_n \in \mathcal{S}(\mathcal{A}), \forall \lambda_n \in [0,1], \sum \lambda_n = 1$, it is called linear channel. A completely positive (CP) channel is a linear channel $\Lambda^*$ that its dual $\Lambda : \mathcal{B} \mapsto \mathcal{A}$ (i.e. $\mathrm{tr}(\Lambda^*(\rho)A) = \mathrm{tr}(\rho\Lambda(A))$ for any $A \in \mathcal{A}$) satisfies

$$\sum_{i,j=1}^{n} A_i^* \Lambda(B_i^* B_j) A_j \geq 0,$$

for any $\{A_j\} \subset \mathcal{A}, \{B_j\} \subset \mathcal{B}$ and $n \in \mathbb{N}$.

Liftings are a class of channels from $\mathcal{S}(\mathcal{A})$ to $\mathcal{S}(\mathcal{A} \otimes \mathcal{B})$;

$$\mathcal{E}^* : \mathcal{S}(\mathcal{A}) \mapsto \mathcal{S}(\mathcal{A} \otimes \mathcal{B})$$

The following liftings are often used in physics.

1. **Linear lifting**: A linear lifting is affine and its dual is CP map.

2. **Pure lifting**: A pure lifting maps a pure state into pure state.

3. **Non-demolition lifting**: A non-demolition lifting satisfies

$$(\mathcal{E}^*\rho)(A \otimes 1) = \rho(A).$$

   Here, $\rho(A) \equiv \mathrm{tr}(\rho A), A \in \mathcal{A}$.

4. **Compound state lifting**: A compound state lifting is a non-linear and non-demolition lifting such that for a density matrix $\rho = \sum \lambda_k E_k$, $E_k \in \mathcal{S}(\mathcal{A})$,

$$\mathcal{E}^*(\rho) = \sum_k \lambda_k E_k \otimes \Lambda^* E_k.$$

5. **Transition lifting**: A transition expectation is a CP and linear map given by $\mathcal{E} : \mathcal{A} \otimes \mathcal{B} \mapsto \mathcal{A}$, and it satisfies

$$\mathcal{E}(1 \otimes 1) = 1.$$

   Transition expectations play a crucial role in the construction of quantum Markov chains and they appear in the framework of measurement theory. The dual of a transition expectation is a transition lifting.

6. **Isometric lifting**: An isometric lifting is defined as

$$\mathcal{E}^*\rho = V\rho V^*,$$

   where the operator $V : \mathcal{H}_A \mapsto \mathcal{H}_A \otimes \mathcal{H}_B$ satisfies $V^*V = 1$. It is useful to describe open system dynamics.

# 3 Quantum-like Model for Decision-making in Two-player Game

In the paper of [9], we designed a quantum-like model for decision-making process in two-player games. We explain briefly how a player in our model decides his own action.

## 3.1 Pay-off Table of Two-player Game

Firstly, let us consider a two-player game with two strategies. We name two players "A" and "B". Two strategies A and B choose are denoted by "0" and "1". The following table shows pay-offs assigned to possible four consequences of "$0_A 0_B$", "$0_A 1_B$", "$1_A 0_B$" and "$1_A 1_B$".

| $A\backslash B$ | $0_B$ | $1_B$ |
|---|---|---|
| $0_A$ | $(a\backslash a)$ | $(b\backslash c)$ |
| $1_A$ | $(c\backslash b)$ | $(d\backslash d)$ |

Here, $a$, $b$, $c$ and $d$ mean values of pay-offs.

For example, a game of prisoner's dilemma (PD) type gives the relation of $c > a > d > b$. For the player A, his pay-off will be $a$ or $c$ if the player B chose "0" and $b$ or $d$ if the player B chose "1". In the both cases, from the relations of $c > a$ and $d > b$, he can obtain larger pay-offs if he choose 1. Such the condition is same for the player B. Conventional game theory concludes, in PD game, a "rational" player, who wants to maximize his own payoff, always chooses "1".

However, the above discussion does not explain process of decision-making in real player's mind, completely. Actually, as seen in statistical data in some experiments, real players frequently behave "irrational". Our model is an attempt to explain such real player's behaviours in "a quantum-like model" which is derived from basic concepts of quantum mechanics.

### 3.2 Decision-making Process in Player's Mind

Let us explain our model for decision-making process in two-players games. We focus on player A's mind. In principle, the player A is not informed of which action the player B chose. The player A will be conscious of two potentials of B's action simultaneously, and then he can not deny either of these potentials. In our model, this indeterminacy the player A holds is described with using the following quantum superposition.

$$|\phi_B\rangle = \alpha\,|0_B\rangle + \beta\,|1_B\rangle \ \in \mathbb{C}^2. \tag{1}$$

The values of $\alpha$ and $\beta$ relate with degrees of consciousness to B's actions. With using the state We call this $|\phi_B\rangle$ prediction state vector. (In accordance with the formalism of quantum mechanics, we assume $|\alpha|^2 + |\beta|^2 = 1$.)

The player A who is getting to choose the action "0" will be conscious of two consequences of "$0_A 0_B$" and "$0_A 1_B$" with weight of $\alpha$ and $\beta$. This situation is described with a vector on $\mathbb{C}^2 \otimes \mathbb{C}^2$ as

$$\begin{aligned} |\Phi_{0_A}\rangle &= \alpha\,|0_A 0_B\rangle + \beta\,|0_A 1_B\rangle \\ &= |0_A\rangle \otimes |\phi_B\rangle \end{aligned} \tag{2}$$

Similarly,

$$|\Phi_{1_A}\rangle = |1_A\rangle \otimes |\phi_B\rangle\,, \tag{3}$$

is given for the situation that A is getting to choose "1".

The player A in our model chooses his own action probabilistically in general. This situation is represented by a state with the form of

$$\begin{aligned} & P_{0_A} \left|\Phi_{0_A}\right\rangle \left\langle\Phi_{0_A}\right| + P_{0_B} \left|\Phi_{1_A}\right\rangle \left\langle\Phi_{1_A}\right| \\ = \ & \left(P_{0_A} \left|0_A\right\rangle \left\langle 0_A\right| + P_{0_B} \left|1_A\right\rangle \left\langle 1_A\right|\right) \otimes \left|\phi_B\right\rangle \left\langle\phi_B\right| \\ \equiv \ & \rho_A \otimes \sigma_B \equiv \Theta \in \mathcal{S}(\mathbb{C}^2 \otimes \mathbb{C}^2). \end{aligned} \tag{4}$$

This state vector is mental state. $P_{0_A}$ and $P_{1_A}$ denote probabilities of choices of "0" and "1". The decision-making process is described as a dynamics changing $P_{0_A}$ and $P_{1_A}$, and its dynamics has a stability solution. Such the stabilization of mental state explains the following psychological activity in the player's mind: The player has two psychological tendencies, the one to choose 0 and the one to choose 1. Degrees of these two opposing tendencies change in his mind, and they become stable with balancing. As a most simple dynamics of the stabilization, the equations like chemical equilibration are introduced;

$$\begin{aligned} \frac{d}{dt} P_{0_A} &= -k P_{0_A} + \tilde{k} P_{1_A}, \\ \frac{d}{dt} P_{1_A} &= k P_{0_A} - \tilde{k} P_{1_A}. \end{aligned} \tag{5}$$

The parameter of $k(\tilde{k})$ corresponds to velocity of reaction from $0_A$ to $1_A$ (from $1_A$ to $0_A$), and in the stable state, the probabilities $P_{0_A}$ and $P_{1_A}$ are given as

$$P^E_{0_A} = \frac{\tilde{k}}{k+\tilde{k}}, \ P^E_{1_A} = \frac{k}{k+\tilde{k}}. \tag{6}$$

The stable state of $\rho_A$ in Eq. (4) is denoted by $\rho^E_A$.

How to define $k$ and $\tilde{k}$ is a very important point in our model. We assume, these values are given by "comparison" of possible consequences, $0_A 0_B$, $0_A 1_B$, $1_A 0_B$ and $1_A 1_B$. The player in our model will consider the following four kinds of comparisons;

$$\begin{aligned} & 0_A 0_B \underset{\tilde{k}_1}{\overset{k_1}{\rightleftharpoons}} 1_A 0_B, \ 0_A 1_B \underset{\tilde{k}_2}{\overset{k_2}{\rightleftharpoons}} 1_A 1_B, \\ & 0_A 1_B \underset{\tilde{k}_3}{\overset{k_3}{\rightleftharpoons}} 1_A 0_B, \ 0_A 0_B \underset{\tilde{k}_4}{\overset{k_4}{\rightleftharpoons}} 1_A 1_B. \end{aligned} \tag{7}$$

These comparisons are represented like chemical equilibrations, each of which is specified by reaction velocities, $k_i$ and $\tilde{k}_i$. (In the next subsection, we mention the relation between $k_i$ ($\tilde{k}_i$) and pay-off table of game.) The player A in our model holds indeterminacy about B's action, so his concerns about the consequence of "$0_A 0_B$" and "$0_A 1_B$" ( or "$1_A 0_B$" and "$1_A 1_B$") are always "fluctuated". Under such the situation, the four kinds of comparisons affect the player's tendency to choose 0 (or 1) simultaneously, and these are crrelated each other. The velocities of $k$ and $\tilde{k}$ should have the forms reflecting effects of the four comparisons and the correlations. In order to define such the form, we introduce complex numbers $\mu$ and $\tilde{\mu}$, which decide $k$ and $\tilde{k}$ by

$$k = |\mu|^2, \ \tilde{k} = |\tilde{\mu}|^2,$$

and define these $\mu$ and $\tilde{\mu}$ as

$$\begin{aligned} \mu &\equiv |\alpha|^2\mu_1 + |\beta|^2\mu_2 + \alpha\beta^*\mu_3 + \alpha^*\beta\mu_4 = \sum_i c_i\mu_i, \\ \tilde{\mu} &\equiv |\alpha|^2\tilde{\mu}_1 + |\beta|^2\tilde{\mu}_2 + \alpha^*\beta\tilde{\mu}_3 + \alpha\beta^*\tilde{\mu}_4 = \sum_i c_i^*\tilde{\mu}_i. \end{aligned} \tag{8}$$

Here $\mu_{i=1,2,3,4}$ and $\tilde{\mu}_{i=1,2,3,4}$ are complex numbers satisfying $|\mu_i|^2 = k_i \ |\tilde{\mu}_i|^2 = \tilde{k}_i$ for given $k_i$ and $\tilde{k}_i$. As results, $k$ and $\tilde{k}$ are defined as

$$\begin{aligned} k &= \sum_{i=1,2,3,4} |c_i|^2 k_i + \sum_{i\neq j} c_i c_j^* \mu_i \mu_j^*, \\ \tilde{k} &= \sum_{i=1,2,3,4} |c_i|^2 k_i + \sum_{i\neq j} c_i^* c_j \tilde{\mu}_i \tilde{\mu}_j^*. \end{aligned} \tag{9}$$

In this definition, a most important point is to introduce the complex numbers $\mu_i$ and $\tilde{\mu}_i$. These parameters will have no meanings as physical quantities, rather, they have meanings similar to "amplitudes" introduced in the quantum mechanical sense. The "correlation terms" as $\sum_{i\neq j} c_i c_j^* \mu_i \mu_j^*$ will provide effects similar to "quantum interference" to the value of $k$.

### 3.3 Decision-making in PD-type Game and Irrational Choice

The parameters $(k_i, \tilde{k}_i)$ introduced in the previous subsection specify the player's four kinds of comparisons, see Eq. (7). It is natural that these comparisons depend on a given game, namely its pay-off table, see the subsection. 3.1. The most simple relation of pay-offs and parameters $(k_i, \tilde{k}_i)$ is decided with depending on magnitude relation between values of pay-off. In the case of prisoner's dilemma (PD) type game, the relation of pay-offs is $c > a > d > b$, and then, $k_i$ and $\tilde{k}_i$ are given as

$$\begin{aligned} &k_1 = f_1(|a-c|), \ k_2 = f_2(|b-d|), \ k_3 = f_3(|b-c|), \ k_4 = 0 \\ &\tilde{k}_1 = 0, \ \tilde{k}_2 = 0, \ \tilde{k}_3 = 0, \ \tilde{k}_4 = \tilde{f}_4(|a-d|). \end{aligned} \tag{10}$$

The functions $f_i(x)$ are assumed to be monotone increasing functions.

Under the settings of $k_i$ and $\tilde{k}_i$ of Eq. (10), the probability $P_{0A}^E$ of Eq. (6) is not zero as a result. Thus, our model explains the player A generally has potential to choose the "irrational" choice of 0 in PD game. The reason of this result is that the parameter of $\tilde{k}_4$ is not zero. $\tilde{k}_4$ represents the degree of tendency to choose 0 which occur from the comparison between consequences of $0_A 0_B$ and $1_A 1_B$. It should be noted that such the comparison is not considered in classical game theory.

### 3.4 Representation of Dynamics with Lifting

As seen in Eq. (6), the stable state $\rho_A^E$ depends on the reaction velocities of $k$, $\tilde{k}$. The values of $k$ and $\tilde{k}$ are decided by Eqs. (8) and(9). In Eq. (8), $\alpha$, $\beta$ are given as coeficients of the prediction state $\sigma_B$, and the parameters $\mu_i$, $\tilde{\mu}_i$ are given for four kinds of comparisons of Eq. (7). Thus, the stable mental $\Theta^E$ can be defined as a lifting from the prediction state $\sigma_B$;

$$\Theta^E \equiv \mathcal{E}^*(\sigma_B). \tag{11}$$

The lifting $\mathcal{E}^*$ is a map from a state in $\mathcal{S}(\mathcal{B})$ to a compound state in $\mathcal{S}(\mathcal{A}\otimes\mathcal{B})$. In general, the prediction state $\sigma_B$ is a mixed state. For Schatten decomposition of $\sigma_B = \sum_j \lambda_j |\phi_j\rangle\langle\phi_j| = \sum_j \lambda_j \sigma_j$,

$$\mathcal{E}^*(\sigma_B) = \sum \lambda_j \mathcal{E}_j^*(\sigma_j)$$

is assumed. In order to define the lifting $\mathcal{E}_j^*$, we introduce operators as

$$\mathcal{T}_j \equiv (I\otimes\sigma_j)T(I\otimes\sigma_j), \tag{12}$$

where, $T$ is a matrix given by the form of

$$T = \begin{pmatrix} 0 & 0 & \tilde{\mu}_1 & \tilde{\mu}_3 \\ 0 & 0 & \tilde{\mu}_4 & \tilde{\mu}_2 \\ \mu_1 & \mu_4 & 0 & 0 \\ \mu_3 & \mu_2 & 0 & 0 \end{pmatrix}, \tag{13}$$

on the basis $|m_A, n_B\rangle$ $(m,n = 0,1)$. We call this $\mathcal{T}_i$, "a comparison operator for $\sigma_i$". With using $\mathcal{T}_i$, the lifting $\mathcal{E}_j^*$ is defined in the following form;

$$\mathcal{E}_j^*(\sigma_j) \equiv \frac{T_j\rho_0\otimes\sigma_j T_j^*}{\mathrm{tr}(T_j\rho_0\otimes\sigma_j T_j^*)}, \tag{14}$$

where $\rho_0 \equiv \frac{1}{2}|0_A\rangle\langle 0_A| + \frac{1}{2}|1_A\rangle\langle 1_A|$. From $\Theta^E = \sum\lambda_j\mathcal{E}_j^*(\sigma_j)$, the probabilities $P_{m_A}$ $(m = 0,1)$ of Eq. (6) are generalized as

$$P_{m_A} = \sum_j \lambda_j \mathrm{tr}(\mathcal{E}_j^*(\sigma_j)|m_A\rangle\langle m_A|\otimes I). \tag{15}$$

Actually, when $\sigma_B = |\phi\rangle\langle\phi|$, one obtains the comparison operator with the form of

$$\mathcal{T}_{\sigma_B} = \mu|\Phi_{1_A}\rangle\langle\Phi_{0A}| + \tilde{\mu}|\Phi_{0_A}\rangle\langle\Phi_{1A}|, \tag{16}$$

and then, the parameters $k$ and $\tilde{k}$ are written as

$$k = \langle\Phi_{0A}|\mathcal{T}_\sigma^*\mathcal{T}_\sigma|\Phi_{0A}\rangle \equiv \langle\Phi_{0A}|K_\sigma|\Phi_{0A}\rangle,$$
$$\tilde{k} = \langle\Phi_{1A}|\mathcal{T}_\sigma^*\mathcal{T}_\sigma|\Phi_{1A}\rangle \equiv \langle\Phi_{1A}|K_\sigma|\Phi_{1A}\rangle. \tag{17}$$

These coform to the values in defintion of Eq. (9). (The operator $K_\sigma = \mathcal{T}_\sigma^*\mathcal{T}_\sigma$ is Hermitian, and in a sense, it means an operator of velocity. This operator is different from conventional operators of physical quantities defined in quantum mechanics, in that its form is decided with depending on the prediction state $\sigma$. This property indicates that the dynamics in our model has "state adaptivity" which is an important concept in the adaptive dynamics theory proposed in the paper of [11].)

### 3.5 Non-Kolmogorovian Structure of Our Model

The result of Eq. (15) indicates our model is a non-Kolmogorovian model. Let us consider the following probabilities relating with the player A's decision.

$P_{\sigma_B}(m_A)$ ($m = 0$ **or** $1$) : Probability that the player A with the prediction $\sigma_B$ chooses the action $m$.

$P_{\sigma_B}(n_B)$ ($n = 0$ **or** $1$) : Probability that the player A with the prediction $\sigma_B$ decides "the player B will choose $n$" in a definitive way.

$P(m_A|n_B)$ ($m, n = 0$ **or** $1$) : Conditional probability that the player A chooses $m$ under the condition that he decided "the player B will choose $n$".

In our model, these are given by $P_{\sigma_B}(m_A) = P_{m_A}$, $P_{\sigma_B}(n_B) = \mathrm{tr}(\sigma_B |n_B\rangle\langle n_B|)$ and $P(m_A|n_B) = \mathrm{tr}(\mathcal{E}^*(|n_B\rangle\langle n_B|)\,|m_A\rangle\langle m_A| \otimes I)$. One can check that

$$P(0_A|n_B) + P(1_A|n_B) = 1,$$
$$P(m_A|0_B) + P(m_A|1_B) \neq 1, \tag{18}$$

which are properties as seen in the classical probability theory. However, one can find that

$$P_{\sigma_B}(m_A) \neq P(m_A|0_B)P_{\sigma_B}(0_B) + P(m_A|1_B)P_{\sigma_B}(1_B) \tag{19}$$

in general. This is the violation of total probability low, which is known as a property of non-Kolmogorovian model.

## References


[1] Shafir, E. and Tversky, A.: Thinking through uncertainty: nonconsequential reasoning and choice. Cognitive Psychology 24, 449-474 (1992)

[2] Tversky, A. and Shafir, E.: The disjunction effect in choice under uncertainty. Psychological Science, 3, 305-309 (1992)

[3] Croson, R.: The disjunction effect and reasoning-based choice in games. Organizational Behavior and Human Decision Processes 80, 118-133 ( 1999)

[4] Hofstader, D. R.: Dilemmas for superrational thinkers, leading up to a luring lottery. Scientific American, 6 (1983)

[5] Hofstader, D. R.: Metamagical themes: Questing for the essence of mind and pattern. Basic Books, New York (1985)

[6] A. Khrennikov, *BioSystems* **84**, 225–241 (2006).

[7] L. Accardi, A. Khrennikov, M. Ohya, Quantum Markov Model for Data from. Shafir-Tversky Experiments in Cognitive Psychology. *Open Systems and Information Dynamics,* **16**, 371 • 85 (2009).

[8] A. Khrennikov, *Ubiquitous quantum structure: from psychology to finance,* Springer, Heidelberg- Berlin-New York, 2010.

[9] M. Asano, A. Khrennikov, M. Ohya, *Foundations of Physics*, to be published 2010.

[10] Accardi L., Ohya M.: Compound channels, transition expectations and liftings, *Applied Mathematics Optimization* **39** (1999) 33-59 Volterra preprint N. 75 (1991)

[11] M. Ohya.: Adaptive dynamics and its applications to chaos and NPC problem. QP-PQ:Quantum Prob. White Noise Anal. **21**, 186-216 (2008)