数理解析研究所講究録
第 1705 巻 2010 年 241-252

Kähler geometric structures in generalized coherent state systems

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Abstract

Generalized coherent state systems associated to Lie groups have complex geometric structures in a natural way [8, 10]. These geometric structures of coherent homogeneous phase spaces play essential roles for physical interpretations. Families of distribution functions on these spaces, containing Wigner functions, Husimi functions and so on [7], can be systematically defined in terms of geometric structure of phase spaces.

1 Generalized coherent state systems

The discussion in this section is mainly based on Perelomov's formulation of generalized coherent state systems [10].

1.1 Basic settings

Let G be a real Lie group and T be its unitary irreducible representation on a Hilbert space $\mathcal{H}$. Fix an arbitrary vector $\psi_0 \in \mathcal{H}$. Then we can consider a set of states whose definition is as follows:

$$\{|\psi_g\rangle := T(g)|\psi_0\rangle; g \in G\}.$$

In order to clarify characteristics of this set of states, we consider an important subgroup $H < G$, called an isotropy subgroup with respect to $\psi_0 \in \mathcal{H}$. The isotropy subgroup H is defined as

$$H := \{h \in G; \exists \alpha : H \to \mathbb{R}, T(h)|\psi_0\rangle = \exp(i\alpha(h))|\psi_0\rangle\}.$$

Then we can find a 1-to-1 correspondence between the set $\{|\psi_g\rangle | g \in G\}$ and the homogeneous space $G/H =: X$. X allows some geometric structure, and it is natural to denote the state $|\psi_{g(x)}\rangle$ corresponding to $x \in X$ by $|x\rangle$ [1]:

$$\begin{array}{ccc}
\{|\psi_g\rangle; g \in G\} & \ni & |\psi_{g(x)}\rangle = |x\rangle \\
{\scriptstyle 1\text{ to }1}\updownarrow & & \updownarrow{\scriptstyle 1\text{ to }1} \\
G/H = X & \ni & x
\end{array}$$

It is convenient to set $|\psi_0\rangle = |0\rangle$ as the base point of the manifold X. In this way, the set of states $\{|\psi_g\rangle\}$ can be interpreted as a geometric object X.

As an additional remark, we obtain the formula of the action $T(g)|\psi_0\rangle = \exp(i\tilde{\alpha}(g))|\psi_{g\cdot 0}\rangle$ for $\forall g \in G$ with some phase factor $\tilde{\alpha}$ as an extension of α [i.e., $\tilde{\alpha}\restriction_H = \alpha$]. This observation leads us to a viewpoint of bundle structures as shown below:

$$\begin{tikzcd}
g(x) \in G \arrow[d, shift left, "H"] \arrow[dr, "\pi"] & \\
x \in G/H \arrow[u, shift left, "\text{local sections}"] & \tilde{M} \arrow[l, "S^1"']
\end{tikzcd}$$

$\tilde{M} := \{(\exp(i\alpha(g)), x) \in S^1 \times X\}$ forms a transformation groupoid $X \rtimes S^1$ via the natural action of $S^1 \curvearrowright X$ as the phase factor.

Now we are ready to introduce our definition and notation of generalized coherent state (GCS, in short) systems using the objects shown above.

Definition 1 (GCS systems)

- $\{|\psi_g\rangle; g \in G\}$ *is called a coherent state system associated with* G *with respect to a unitary irreducible representation* T *and a vector* $\psi_0 \in \mathcal{H}$, *and is denoted by* $\{G, T, \psi_0\}$.
- $|\psi_0\rangle$ *and* ψ_0 *are, respectively, called a standard state and a standard vector of* $\{G, T, \psi_0\}$.
- $G/H = X$ *is called a coherent phase space of* $\{G, T, \psi_0\}$, *while* $\tilde{M}$ *is called a coherent state manifold of* $\{G, T, \psi_0\}$.
- $|\psi_{g(x)}\rangle = |\psi_{g\cdot 0}\rangle = |x\rangle = |x\{g\}\rangle$ *; in rightmost side of this equation,* g *means the representative of equivalence class of* x.
- $|\psi_0\rangle = |0\rangle$ *as the base point of* X.

This formulation of GCS systems is introduced and studied in 1960-70s.[2] Every well-known coherent state[3] can be understood in this framework. In

[1] This is equivalent to considering some local sections $g(\cdot)$ of the fibre bundle $G \underset{H}{\to} G/H$.

[2] Historical details are seen in [8, 10].

[3] Photon CS, Bloch CS, and so on.

some typical cases, a coherent phase space X actually has a meaning of physical phase space.[4] The criterion for such a physical reality of coherent phase space is explained in the next subsection.

Now we see some general properties of coherent state systems.

1) Symmetries of phase spaces Since G is a Lie group, $X = G/H$ forms a homogeneous space with a differential structure, and there is a natural action $G \curvearrowright X$. Then we can find a measure dx which is quasi-invariant under this action owing to the group symmetry of G. Thus $(X, d\nu(x))$ is a measurable space, and simultaneously, X is a metric space with an quasi-invariant Riemannian metric ds^2 which is compatible with $d\nu(x)$.

2) Resolution of unity Let $d\nu(x)$ be an quasi-invariant measure introduced in 1). Then the following is valid.

Proposition 2

$$\exists C > 0 \ s.t., \int_X d\mu(x)|x\rangle\langle x| = \hat{I}, \ \textit{where } d\mu(x) := \frac{d\nu(x)}{C}.$$

Remark 3 *The positive constant C is determined in the following way: Let $|y\rangle$ be a coherent state and $\hat{B} := \int_X d\nu(x)|x\rangle\langle x|$. Then $\hat{B} = C\hat{I}$, so we obtain*

$$\langle y|\hat{B}|y\rangle = \int_{x\in X} d\nu(x)\,|\langle y|x\rangle|^2 = \int_{x\in X} d\nu(x)\,|\langle 0|x\rangle|^2\,(< \infty).$$

The last step is from the quasi-invariance of $d\nu(x)$ under the action of G. On the other hand, this quantity is obviously equal to C, so C is determined from the quantity $|\langle 0|x\rangle|^2$, which can be said a "distribution around the base point" of X.

3) Description for operational quantum physics On the basis of the resolution of unity derived above, we can define the corresponding POVM on X associated to the coherent state system: Let $\mathcal{B}(X)$ be a Borel set of X. Then the POVM $M : \mathcal{B}(X) \to \mathcal{L}(\mathcal{H})$ is defined as

$$M(A) := \int_{x\in A} d\mu(x)|x\rangle\langle x| \quad for\ \forall A \in \mathcal{B}(X).$$

A triplet (X, ds^2, M) consisting of a homogeneous space $X = G/H$, a metric ds^2, and a POVM M (defined above) is also called coherent phase space.

[4] as represented by the $2n$-dimensional space $\Gamma = \{(q, p)\}$ equipped with canonical coordinates

4) Expansion on "CS base" and symbols of states Every state $|\psi\rangle \in \mathcal{R}(\mathcal{H})$(: rays of $\mathcal{H}$) can be expanded over coherent state system in the following way:

$$\begin{aligned} |\psi\rangle &= \int_X d\mu(x)|x\rangle\langle x|\psi\rangle \\ &= \int_X d\mu(x)\psi(x)|x\rangle, \end{aligned}$$

where $\psi(x) := \langle x|\psi\rangle$ ($\psi : X \to \mathbb{C}$), which is called a symbol of a state $|\psi\rangle$. This symbol satisfies the following formula for inner products:

$$\langle\phi|\psi\rangle = \int_X d\mu(x)\overline{\phi(x)}\psi(x).$$

5) Reproducing kernels Two symbols $\psi(x)$, $\psi(y)$ for $\forall x, y \in X$ depend on each other:

$$\begin{aligned} \psi(x) = \langle x|\psi\rangle &= \int_{y\in X} d\mu(y)\langle x|y\rangle\langle y|\psi\rangle \\ &= \int_{y\in X} d\mu(y)K(x,y)\psi(y), \end{aligned}$$

where $K(x,y) := \langle x|y\rangle$. $K : X \times X \to \mathbb{C}$ is called a reproducing kernel associated to a GCS system $\{G, T, \psi_0\}$ because it satisfies such property as $K(x,z) = \int_{y\in X} K(x,y)d\mu(y)K(y,z)$.

6) "Overcomplete" linear dependence Any two coherent states $|x\rangle$, $|y\rangle$ have a mutual relation via the reproducing kernel K introduced in 5);

$$|x\rangle = \int_{y\in X} d\mu(y)K(y,x)|y\rangle.$$

This formula can be seen as an linear expansion of the state $|x\rangle$ over the GCS system.[5]

1.2 Semiclassical systems

In the previous subsection we discussed general properties of coherent state systems. Here we consider the criterion for *semiclassical systems*: in other words, "how to know whether a coherent phase space X has some coordinates which allows physical interpretations". The criterion is, in short, maximality of isotropy subalgebras of the corresponding Lie algebra defined as below.

[5]The set of coherent states $\{|x\rangle; x \in X\}$ is, of course, not a base of the state space. Actually, it contains some base as its proper subset.

Let $\mathcal{G}$ be the Lie algebra corresponding to the Lie group G. Since G is real, $\mathcal{G}$ is a real Lie algebra and we can construct its complex extension $\mathcal{G}_{\mathbb{C}} := \mathcal{G} \oplus_{\mathbb{R}} i\mathcal{G}$ and a representation $\mathcal{T}$ of $\mathcal{G}_{\mathbb{C}}$ induced from the unitary irreducible representation $T : G \to GL(\mathcal{H})$. Then we can define an algebra $\mathcal{B} \subset \mathcal{G}_{\mathbb{C}}$ called isotropy subalgebra with respect to a fixed vector $\psi_0 \in \mathcal{H}$ as follows: $\forall b \in \mathcal{B}, \exists \lambda_b \in \mathbb{C}$ s.t., $\mathcal{T}(b)|\psi_0\rangle = \lambda_b|\psi_0\rangle$. $\mathcal{B}$ is necessarily a complex subalgebra, and its Hermite conjugate $\overline{\mathcal{B}}$ is also a subalgebra of $\mathcal{G}_{\mathbb{C}}$.

Now we can give the definition of maximality of isotropy subalgebras, which is nothing but the necessary condition for physical interpretation of coherent phase spaces.

Definition 4 (Maximality condition) *Fix an arbitrary vector $\psi_0 \in \mathcal{H}$, and let $\mathcal{B}$ is maximal in $\mathcal{G}_{\mathbb{C}}$ iff $\mathcal{B} \oplus \overline{\mathcal{B}} = \mathcal{G}_{\mathbb{C}}$, where the direct sum is in the sense of Lie algebras.*

Take an isotropy subalgebra $\mathcal{B}$ with respect to ψ_0 which is maximal in $\mathcal{G}_{\mathbb{C}}$. Then a remarkable proposition holds: The corresponding coherent phase space $X = G/H$, which is a real homogeneous space, is identified with complex homogeneous space $G_{\mathbb{C}}/B$ or $\overline{B}/D$, where B, $\overline{B}$, and D are, respectively, the Lie groups corresponding to $\mathcal{B}$, $\overline{\mathcal{B}}$, and $\mathcal{D}$, and $\mathcal{D} = \mathcal{B} \cap \overline{\mathcal{B}}$. It is remarkable that a complex structure is induced in X from $G_{\mathbb{C}}/B$ via the relation $X = G/H \simeq G_{\mathbb{C}}/B \simeq \overline{B}/D$.[6]

Such a case can be understood as a typical one that G is a compact semisimple Lie group. Let T be a maximal torus group of G and $\mathcal{G}$ and $\mathcal{T}$ be, respectively, the Lie algebra of G and T. Let us fix a Cartan base $\{T_j, E_\alpha\}_{j,\alpha}$ such that $\{T_j\}$ spans T. We can take the following fundamental objects:

$B_\pm$: Borel subgroups, $\mathcal{B}_\pm := Lie(B_\pm)$; spanned by $\{T_j, E_\alpha\}_{j, \substack{\alpha>0 \\ \alpha<0}}$.

$Z_\pm$: Nilpotent subgroups, $\mathcal{Z}_\pm := Lie(Z_\pm)$; spanned by $\{E_\alpha\}_{\substack{\alpha>0 \\ \alpha<0}}$.

$T_{\mathbb{C}}$: Complexified group of T; $\mathcal{T}_{\mathbb{C}} := Lie(T_{\mathbb{C}})$; spanned by $\{T_j\}_j$;

$r := dimT = dim(\mathcal{T})$: rank of G.

Then $X = G/T$ admits a complex structure in a similar way;

$$\begin{array}{ccccc} B_+ \backslash G_{\mathbb{C}} & \simeq & G/T & \simeq & G_{\mathbb{C}}/B- \\ \| & & \| & & \| \\ X_- & & X & & X_+ \end{array} .$$

These homogeneous spaces $X, X_\pm$ are called flag manifolds. An essential structure of this isomorphic relation comes from the canonical decomposition in $G_{\mathbb{C}}$: $\exists G_0$: dense in $G_{\mathbb{C}}; \forall g \in G_0, \exists! \zeta_\pm \in Z_\pm, \exists! h \in T_{\mathbb{C}}, \exists! \eta_\pm \in B_\pm s.t., g = \zeta_+ h \zeta_- = \eta_+ \zeta_- = \zeta_+ \eta_-$. In addition, both $X_\pm$ admit a Hermitian G-invariant

[6] The details are seen in [3].

structure (metric and 1-form) in common:

$$ds^2_\omega = h_{j\bar{k}} d\xi_j d\overline{\xi_k},$$
$$\omega = \frac{i}{2} h_{j\bar{k}} d\xi_j \wedge d\overline{\xi_k},$$
$$\text{where } h_{j\bar{k}} := \frac{\partial}{\partial \xi_j \partial \overline{\xi_k}} F(\xi, \bar{\xi})$$

with respect to the corresponding unitary irreducible representation T of G, where F is the Kähler potential of ω. The corresponding Kähler potential F is determined by the Lie-algebraic structure of $\mathcal{G}_{\mathbb{C}}$ [9]. The reason why the Kähler structure is essential is that any Kähler manifold can be seen as real symplectic manifold so the coordinate variables play roles of canonically conjugate variables. Especially, for our proposal of formulation of Wigner-type functions via geometric structures of GCS system, these symplectic manifolds are seen as physical phase spaces and nothing but the domains of distribution functions defined in the following §.2. The discussions there reveal more precisely the deeper aspects of geometric structures of coherent phase spaces.

2 Kernel operators and distribution functions

In the previous section we surveyed general aspects of GCS systems, especially their (Kähler) geometric structures of coherent phase spaces. Then we can find the corresponding symplectic structures as real manifolds. In this section, we deal with phase spaces with real symplectic forms, and construct families of distribution functions on these spaces as generalized Wigner or Husimi functions, which are well-known in the context of CCR algebras. Before the main discussion, let us review the case of CCR.

2.1 The case of CCR, or systems of photons

Let $\mathcal{W}_1$ be the 1-degrees-of-freedom CCR algebra (3-dimensional Heisenberg algebra) generated by $\hat{q}$, $\hat{p}$ and $\hat{I}$ which satisfy the canonical commutation relation $[\hat{q}, \hat{p}] = i\hbar\hat{I}, [\hat{q}, \hat{I}] = [\hat{p}, \hat{I}] = 0$, or using the creation/annihilation operator, $[\hat{a}, \hat{a}^\dagger] = \hat{I}, [\hat{a}, \hat{I}] = [\hat{a}^\dagger, \hat{I}] = 0$ (where $\hat{a} = \frac{1}{\sqrt{2\hbar}}(\hat{q} + i\hat{p}), \hat{a}^\dagger = \frac{1}{\sqrt{2\hbar}}(\hat{q} - i\hat{p})$). A conventional description of GCS system for the Lie group $H_1 = \mathrm{Exp}(\mathcal{W}_1)$ is written down in the following way: $\forall g \in H_1$ can be represented as $g = [s; t_1; t_2]$ with the parameters $s, t_1, t_2 \in \mathbb{R}$, and let T_r be a representation on some Hilbert space $\mathcal{H}_r$ whose action on the ray $\mathcal{R}(\mathcal{H}_r)$ is as follows:

$$T_r(g)|\phi_0\rangle = \exp[i(s\hat{I} + t_1\hat{a} + t_2\hat{a}^\dagger)]|\phi_0\rangle \quad for\ |\phi_0\rangle \in \mathcal{R}(\mathcal{H}_r).$$

Then the arbitrary coherent state of $\{H_1, T_r, \phi_0\}$ is

$$\begin{aligned}|\phi_g\rangle &= T_r(g)|\phi_0\rangle = \exp[i(s\hat{I} + t_1\hat{a} + t_2\hat{a}^\dagger)]|\phi_0\rangle \\ &= \exp(\alpha\hat{a}^\dagger - \overline{\alpha}\hat{a})\exp(is'\hat{I})|\phi_0\rangle\end{aligned}$$

with 2 parameters $\alpha \in \mathbb{C}, s' \in \mathbb{R}$ which are some functions of s, t_1, t_2. Since the isotropy subalgebra is $(\hat{I})_{\mathbb{R}} = \{\exp(is\hat{I})|s \in \mathbb{R}\} \simeq S^1$, this GCS system is equivalent to $\{|\phi_g\rangle = D(\alpha)|\phi_0\rangle\}$ with the standard state $|\phi_0\rangle \in \mathcal{R}(\mathcal{H}_r)$, where $D(\alpha) := \exp(\alpha\hat{a}^\dagger - \overline{\alpha}\hat{a})$, and the arbitrary coherent state is parametrized by $\alpha \in \mathbb{C} \simeq X := H_1/S^1$.

Remark 5 *Let us take the Fock vacuum $|0\rangle$ as the standard state, then we obtain the well-known representation of photon coherent states:*

$$\begin{aligned}|\alpha\rangle &= D(\alpha)|0\rangle \\ &= \exp\left(-\tfrac{|\alpha|^2}{2}\hat{I}\right)\exp(\alpha\hat{a}^\dagger)\exp(-\overline{\alpha}\hat{a})|0\rangle \\ &= \exp\left(-\tfrac{|\alpha|^2}{2}\hat{I}\right)\exp(\alpha\hat{a}^\dagger)|0\rangle \\ &= e^{-\frac{|\alpha|^2}{2}}\sum_{n\in\mathbb{N}}\frac{\alpha^n}{\sqrt{n!}}|n\rangle.\end{aligned}$$

More generally, we should take an eigenstate of $\hat{a}$ as the standard state. Then the isotropy subalgebra is generated by $\hat{a}, \hat{I}$, and the subalgebras $\mathcal{B} = \underset{\mathbb{C}}{span}\{\hat{a}, \hat{I}\}$ and $\overline{\mathcal{B}} = \underset{\mathbb{C}}{span}\{\hat{a}^\dagger, \hat{I}\}$ satisfy the maximality condition $\mathcal{B} \oplus \overline{\mathcal{B}} = \mathcal{W}_{1\mathbb{C}}$. Thus $H_{1\mathbb{C}}/B \simeq H_1/S^1 \simeq \mathbb{C} = X$, which has the Kähler structure $\omega_{H_1} = \frac{i}{\pi}d\alpha \wedge d\overline{\alpha}$, corresponds to a classical phase space $\Gamma = \mathbb{R}^2 = \{(q,p)\}$. Actually, this Kähler form can be seen as the symplectic form coordinatized by (q,p): $\omega_{H_1} = \frac{i}{\pi}d\alpha \wedge d\overline{\alpha} = \frac{\hbar}{\pi}dq \wedge dp$ where $q = \frac{1}{\sqrt{2\hbar}}(\alpha + \overline{\alpha}), p = \frac{1}{i\sqrt{2\hbar}}(\alpha - \overline{\alpha})$. The coherent states are also parametrized by $(q,p) \in \Gamma$: $|\alpha\rangle = \exp(\alpha\hat{a}^\dagger - \overline{\alpha}\hat{a})|0\rangle = \exp\left(\frac{i}{\hbar}(p\hat{q} - q\hat{p})\right)|0\rangle =: |q,p\rangle$.

Instead of formulating Wigner-type functions in a direct way, we first define so-called Δ-operators for CCR CS systems defined as

$$\Delta_s(z) := \int_{\alpha\in X}\frac{d^2\alpha}{\pi}D(\alpha)e^{\frac{s}{2}|\alpha|^2 - \alpha\overline{z} + \overline{\alpha}z} \quad for\ z \in \mathbb{C} \tag{1}$$

for each $s \in \mathbb{R}$, where $X = \mathbb{C}, d^2\alpha = d(\mathrm{Re}\alpha)d(\mathrm{Im}\alpha) = \frac{1}{2i}d\alpha d\overline{\alpha}$ and $D(\alpha)$ is the coherent shift operator defined above [1].[7] These Δ-operators satisfy the following properties[8]:

[7] introduced for the purpose of discussing relaxation of coherence

[8] The Δ-operators $\Delta_s(z)$ contains the shift operator $D(\alpha)$, so the discussion here depends on the choice of representations of H_1 or $\mathcal{W}_1$.

Proposition 6 *For* $\forall s \in \mathbb{R}$,
1) $Tr\Delta_s(z) = 1$ (normalization),
2) $\displaystyle\int_{z\in\mathbb{C}} \frac{d^2z}{\pi}\Delta_s(z) = 1$ (completeness),
3) $Tr[\Delta_s(z)\Delta_{-s}(z')] = \pi\delta^{(2)}(z - z')$ (orthogonality)
holds.

In the next step we define a family of Wigner-type functions as a generalization of well-known Wigner functions and Husimi functions.

Definition 7 (Wigner-type functions for CCR) *Let* $\rho \in \mathcal{T}(\mathcal{H})$ *be an arbitrary positive trace-class operator (or a density operator) on* $\mathcal{H}$. *The functions* $\{F_s(z)\}_{s\in\mathbb{R}}$ *with the parameter* $s \in \mathbb{R}$ *defined by*

$$F_s(z) := Tr[\rho\Delta_{-s}(z)]$$

is called Wigner-type functions of the state ρ.

$F_s : \mathbb{C} \to \mathbb{C}$ is a function on the phase space $X = \mathbb{C}$, and on the basis of the correspondence of $X = \{z \in \mathbb{C}\} \simeq \mathbb{R}^2 = \{(q,p)\}$ $(q = \frac{1}{\sqrt{2\hbar}}(z + \bar{z}), p = \frac{1}{i\sqrt{2\hbar}}(z - \bar{z}))$, $F_s(z)$ can be seen as $F_s(q,p)$ (for ease, we use the same notation F_s) with real symplectic variables $(q,p) \in \mathbb{R}^2$.

Remark 8 *Some of the special cases which correspond to named quasi-probability distributions*[9] *are shown below:*
($s = 0$*)* $F_0(z)$*: Wigner function*
($s = 1$*)* $F_1(z)$*: Husimi function (Q-function)*
($s = -1$*)* $F_{-1}(z)$*: Glauber-Sudarshan function (P-function) [5]*

We can check they satisfy the following properties, which characterize these quasi-distributions by primary objects and operations [2].

Proposition 9

- *For the Husimi function* $F_1(z) = F_1(p,q)$,

$$F_1(q,p) = \langle q,p|\rho|q,p\rangle \quad for \quad \forall\rho \in \mathcal{T}(\mathcal{H})$$

holds.

- *For the Glauber-Sudarshan function* $F_{-1}(z) = F_{-1}(q,p)$,

$$\rho = \int\int_{\mathbb{R}^2} \frac{dqdp}{2\pi}|q,p\rangle F_{-1}(q,p)\langle q,p| \quad for \quad \forall\rho \in \mathcal{T}(\mathcal{H})$$

holds.

[9] The prefix "quasi-" means that "the value is not always positive".

It is remarkable that the Δ-operators are determined by the shift operator $D(\alpha)$ and the measure $d^2\alpha$, which are elementary objects of the GCS system $\{H_1, T, \psi_0\}$ for the CCR algebra, and the Wigner-type functions are defined almost directly from $\Delta_s(z)$ as shown in definition 7. Thus we can construct the quasi-probability distributions only with the knowledge of GCS systems. This leads us to the question how to understand these quasi-probability distributions from geometric structures of coherent phase spaces.

2.2 General case

Let us generalize the discussions in the previous section to the case of arbitrary Lie groups. Such a procedure has already given in [4], on the basis of the idea of harmonic functions on manifolds, but our method has rather geometric aspects and tells how to formulate quasi-probability distributions more directly. In the general theory discussed later, we also utilize complex geometric structures associated to GCS systems.

Let G be a Lie group and $T : G \to \mathcal{L}(\mathcal{H})$ be a unitary irreducible representation, and consider the case that we can construct the corresponding GCS system $\{G, T, \psi_0\}$ with the Kähler geometric phase space $X = \{\eta \in \mathbb{C}^N\}$ ($N = dim_{\mathbb{C}} X$) and the shift operator $D(\eta)$.[10] In order to generalize Δ-operators introduced in the case of CCR algebras, we first define kernel operators $K_s(\eta)$ for $\{G, T, \psi_0\}$ (or the phase space X) which satisfy the following axioms, and in the next step define symbols of operators using $K_s(\eta)$. In the final step, generalized quasi-distributions are defined as symbols of density operators.

Definition 10 (Kernel operators) *Components of a 1-parameter family of operators $\{K_s(\eta)\}_{s\in\mathbb{R}}$ on the Hilbert space $\mathcal{H}$ is called kernel operators associated to the coherent phase space X iff they satisfy the following* (K.1)-(K.5)*:*

(K.1) $K_s(\eta) = K_s(\eta)^*$ *for* $\forall \eta \in X$ *(self-adjointness),*

(K.2) $K_s(g \cdot \eta) = T(g) K_s(\eta) T(g^{-1})$ *for* $\forall \eta \in X, \forall g \in G$
(covariance with $G \curvearrowright X$),

(K.3) $Tr[K_s(\eta)] = 1$ *(normalization),*

(K.4) $\int_X d\mu(\eta) K_s(\eta) = 1$ *(completeness),*

(K.5) $Tr[K_s(\eta) K_{-s}(\eta')] = C\delta^{(N)}(\eta - \eta')$ *(orthogonality relation),*

where C is the positive constant appearing in the formula of resolution of unity (proposition 2): $1 = \frac{1}{C} \int d\mu(g) |g\rangle\langle g|$.

[10] Any GCS system has some Kähler structure, but it is not always easy to find appropriate coordinates as physical phase spaces like the case discussed in §1.2.

Definition 11 (Symbols of operators) *Components of a 1-parameter family of mappings* $\{\Sigma^{(s)} : \mathcal{L}(\mathcal{H}) \to \mathbb{C}\}_{s\in\mathbb{R}}$ *defined by* $\Sigma^{(s)}(\hat{A}) := Tr[\hat{A}K_{-s}(\eta)]$ $(\hat{A} \in \mathcal{L}(\mathcal{H}))$ *are called symbols of operators in* $\mathcal{L}(\mathcal{H})$.[11]

Definition 12 (Generalized distribution functions) *Symbols of density operators* $\rho \in \mathcal{T}(\mathcal{H})$ *are called generalized distribution functions (on the phase space* X*). They also form a 1-parameter family* $\{F_\rho^{(s)}(\eta) := \Sigma^{(s)}(\rho) = Tr[\rho K_{-s}(\eta)]\}_{s\in\mathbb{R}}$.

It is remarkable that these distribution functions and kernel operators are Fourier duals via the density operator ρ.

Our definition of kernel operators is compatible with Brif and Mann's formulation of distribution functions on (real) differentiable manifolds [4]. They presented that their distribution functions have properties as quasi-probability distributions shown in proposition 9, i.e., the estimation formula of Husimi functions by coherent states $F_1(x) = \langle x|\rho|x\rangle$ $(x \in X)$ and the reconstruction formula of states with Glauber-Sudarshan functions $\rho = \int_{x\in X} d\mu(x)|x\rangle F_{-1}(x)\langle x|$.

The remaining problem is how to realize our distribution functions $F_\rho^{(s)}$ on coherent phase spaces. For this, we need to find an explicit form of the kernel operator K_s. It is difficult to obtain the general solution of this problem, but we can propose a typical and conceptual example which satisfies the axiom in definition 10 with ingredients of GCS system. Let us show the concrete formula of $K_s(\eta)$.

Proposition 13 *Set some GCS system* $\{G, T, \psi_0\}$ *with phase space* $(X, d\sigma^2)$. *Let us consider a 1-parameter family of operators* $\{K_s(\eta)\}_{s\in\mathbb{R}}$ *with an index* $\eta \in X$ *defined below:*

$$K_s(\eta) := \int_{\xi\in X} d\mu(\xi)D(\xi)e^{\frac{s}{2}d_\sigma(0,\xi)^2}e^{\omega_\sigma(\xi,\eta)} \tag{2}$$

where

$d\mu$*: the measure which is compatible with the metric* $d\sigma^2$,

$D(\cdot)$*: the shift operator,*

d_σ*: the distance induced from* $d\sigma^2$,

ω_σ*: the symplectic form of coherent phase space as a symplectic manifold.*

Then $\{K_s(\eta)\}_{s\in\mathbb{R}}$ *satisfies* (K.1)-(K.5) *introduced in definition 10.*

[11] From the viewpoint of algebraic probability theory, these symbols of operators are rather the (normal) states of the system described by $\mathcal{G}$.

The proof needs a few properties of shift operators symplectic forms [6].[12] This proposition means that $\{K_s(\eta)\}_{s\in\mathbb{R}}$ gives a family of distribution functions via the procedure in definition 11-12, containing a generalized version of Wigner, Husimi and Glauber-Sudarshan functions. The formula (2), actually, recovers eq.(1) as its specialized form. We emphasize that this construction of K_s is canonical and related to some algebraic viewpoints. Especially, it can be naturally understood with the method of GNS construction, and the Gaussian-like factor $e^{\frac{s}{2}|\alpha|^2}$ appearing in eq.(1) is generalized as a mollifier of kernel operators (if $s < 0$). The complete proof of proposition 13 and the details of algebraic aspects mentioned above will be given in a paper [6] in preparation.

We have constructed a scheme how to define generalized distribution functions associated to GCS systems on the basis of their geometric structures. Our distribution functions have the same properties as Brif and Mann's version, i.e., the estimation formula of Husimi functions, the reconstruction formula of states and so on. Moreover, our scheme enables some quantitative analyses for quantum systems via geometries of GCS systems; for example, we can concretely calculate the Wehrl entropy $S_W[F_\rho^{(1)}] := -\int_X d\mu(\eta)F_\rho^{(1)}logF_\rho^{(1)}$, which measures degrees of incoherence of quantum states in some sense [2].

3 Summary

In this article we have reviewed general structures of GCS systems mainly based on Perelomov's theory, and discussed its application to formulation of distribution functions on physical phase spaces. As the main result, we obtain concrete forms of kernel operators or quantum Fourier transforms of distribution functions. It is remarkable that (Kähler) geometric structures of coherent phase spaces play essential roles there. We believe that our distribution functions are powerful tools for concrete and conceptual investigation of quantum systems in various contexts, especially in information geometric ones.

Both of the authors are very grateful to Mr. H. Ando, Mr. T. Hasebe, Mr. K. Okamura, Mr. H. Saigo and Prof. S. Tanimura for their valuable discussions and comments.

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