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Caustics and wave front propagations

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1 Introduction

The study of sigularities of caustics and wave fronts was the starting point of the theory of Lagrangian and Legendrian singularities developed by several mathematicans and phyists [1, 2, 3, 5, 6, 7, 9, 19, 20, 21] etc.

This is a survey on the theory of Lagrangian and Legendrian singularities. Especially, we consider a relationship between caustics and wave front propagations, see [14]. This is also the announcement of results obtained in [15]. Refer [14, 15] for detailed proofs.

In §2, we give a brief review on the theory of Lagrangian singularities. We also give a brief review on the theory of big Legendrian submanifolds in §3. The big wave front consists of a one-parameter family of wave fronts which given by the projection of a big Legendrian submanifold of the contact fibering onto the basis of this fibering. We define the $S.P^+$-Legendrian equivalence among big Legendrian submanifold germs in §3. The $S.P^+$-Legendrian equivalence has been introduced in [8, 21] which preserves both the diffeomorphism types of bifurcations for families of small fronts (i.e., wave front propagations) and the caustics. We consider a special class of the big Legendrian submanifold which is called a graphlike Legendrian unfolding in §4. The graphlike Legendrian unfolding can be always induced by Lagrangian submanifolds. Moreover, we modify the theory of graphlike Legendrian unfoldings a little in §5. By definition, a Lagrangian equivalence preserves the caustics. However, in general the converse does not hold even if Lagrangian submanifold germs are Lagrange stable (cf. [1, 20]). We give a sufficient condition that the converse hold in §6. As an application of these equivalence relations and modified graphlike Legendrian unfoldings, we consider a relationship between caustics of submanifolds and of the canal hypersurfaces of the submanifolds in Euclidean space in §7.

We shall assume throughout the whole paper that all maps and manifolds are C^∞ unless the contrary is explicitly stated.

2 Lagrange submanifolds and caustics

We consider the cotangent bundle $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$ over $\mathbb{R}^n$. Let $(x,p) = (x_1,\dots,x_n,p_1,\dots,p_n)$ be the canonical coordinate on $T^*\mathbb{R}^n$. Then the canonical symplectic structure on $T^*\mathbb{R}^n$ is given by the *canonical two form* $\omega = \sum_{i=1}^n dp_i \wedge dx_i$. A submanifold $i : L \subset T^*\mathbb{R}^n$ is a *Lagrangian submanifold* if $\dim L = n$ and $i^*\omega = 0$. In this case, the critical value of $\pi \circ i$ is called the *caustic* of $i : L \subset T^*\mathbb{R}^n$ and it is denoted by C_L. One of the main results in the theory of Lagrangian singularities is the description of Lagrangian submanifold germs by using families of function germs. Let $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be an n-parameter unfolding of function germs. We say that F is a *Morse family of functions* if the map germ

$$\Delta F = \left(\frac{\partial F}{\partial q_1}, \dots, \frac{\partial F}{\partial q_k}\right) : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$$

is a non-singular, where $(q, x) = (q_1, \dots, q_k, x_1, \dots, x_n) \in (\mathbb{R}^k \times \mathbb{R}^n, 0)$. In this case, we have a smooth n-dimensional submanifold germ $C(F) = (\Delta F)^{-1}(0) \subset (\mathbb{R}^k \times \mathbb{R}^n, 0)$ and a map germ $L(F) : (C(F), 0) \to T^*\mathbb{R}^n$ defined by

$$L(F)(q, x) = \left(x, \frac{\partial F}{\partial x_1}(q, x), \dots, \frac{\partial F}{\partial x_n}(q, x)\right).$$

We can show that $L(F)(C(F))$ is a Lagrangian submanifold germ. Then we have the following fundamental result ([1], page 300).

Proposition 2.1 *All Lagrangian submanifold germs in $T^*\mathbb{R}^n$ are constructed by the above method.*

For an n-parameter unfolding of function germs $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$, we call

$$C(F) = \left\{(q, x) \in (\mathbb{R}^k \times \mathbb{R}^n, 0) \;\middle|\; \frac{\partial F}{\partial q_1}(q, x) = \cdots = \frac{\partial F}{\partial q_k}(q, x) = 0\right\},$$

the *catastrophe set* of F and

$$\mathcal{B}_F = \left\{x \in (\mathbb{R}^n, 0) \;\middle|\; \text{there exist } q \in (\mathbb{R}^k, 0) \text{ such that } (q, x) \in C(F),\ \operatorname{rank}\left(\frac{\partial^2 F}{\partial q_i \partial q_j}(q, x)\right) < k\right\}$$

the *bifurcation set* of F.

Let $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a Morse family of functions. We call F a *generating family* of $L(F)$. Let $\pi_n : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be the canonical projection, then we can easily show that the bifurcation set of F is the critical value set of $\pi_n|_{C(F)}$. Hence, the caustic of $L(F)$ coinsides with the bifurcation set of F, namely, $C_{C(F)} = \mathcal{B}_F$.

We now define an equivalence relation among Lagrangian submanifold germs. Let $i : (L, x) \subset (T^*\mathbb{R}^n, p)$ and $i' : (L', x') \subset (T^*\mathbb{R}^n, p')$ be Lagrangian submanifold germs. Then we say that i and i' are *Lagrangian equivalent* if there exist a diffeomorphism germ $\sigma : (L, x) \to (L', x')$, a symplectic diffeomorphism germ $\hat{\tau} : (T^*\mathbb{R}^n, p) \to (T^*\mathbb{R}^n, p')$ and a diffeomorphism germ $\tau : (\mathbb{R}^n, \pi(p)) \to (\mathbb{R}^n, \pi(p'))$ such that $\hat{\tau} \circ i = i' \circ \sigma$ and $\pi \circ \hat{\tau} = \tau \circ \pi$, where $\pi : (T^*\mathbb{R}^n, p) \to (\mathbb{R}^n, \pi(p))$ is the canonical projection and a symplectic diffeomorphism germ is a diffeomorphism germ which preserves symplectic structure on $T^*\mathbb{R}^n$. Then the caustic C_L is diffeomorphic to the caustic $C_{L'}$ by the diffeomorphism germ τ.

A Lagrangian submanifold germ into $T^*\mathbb{R}^n$ at a point is said to be *Lagrange stable* if for every map with the given germ there is a neighborhood in the space of Lagrangian immersions (in the Whitney C^∞-topology) and a neighborhood of the original point such that each Lagrangian immersion belonging to the first neighborhood has in the second neighborhood a point at which its germ is Lagrangian equivalent to the original germ.

We can interpret the Lagrangian equivalence by using the notion of generating families. Let $\mathcal{E}_x$ be the ring of function germs of $x = (x_1, \dots, x_n)$ variables at the origin. Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be function germs. We say that F and G are *P-$\mathcal{R}^+$-equivalent* if there exist a diffeomorphism germ $\Phi : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}^k \times \mathbb{R}^n, 0)$ of the form $\Phi(q, x) = (\phi_1(q, x), \phi_2(x))$ and a function germ $h : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ such that $G(q, x) = F(\Phi(q, x)) + h(x)$. For any $F_1 : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ and $F_2 : (\mathbb{R}^{k'} \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$, F_1 and F_2 are said to be *stably P-$\mathcal{R}^+$-equivalent* if they become P-$\mathcal{R}^+$-equivalent after the addition to the arguments to q_i of new arguments q'_i and to the functions F_i of nondegenerate quadratic forms Q_i in the new arguments, i.e., $F_1 + Q_1$ and $F_2 + Q_2$ are P-$\mathcal{R}^+$-equivalent.

Let $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a function germ. We say that F is a *$\mathcal{R}^+$-versal deformation* of $f = F|_{\mathbb{R}^k \times \{0\}}$ if

$$\mathcal{E}_q = J_f + \left\langle \frac{\partial F}{\partial x_1}|\mathbb{R}^k \times \{0\}, \dots, \frac{\partial F}{\partial x_n}|\mathbb{R}^k \times \{0\} \right\rangle_{\mathbb{R}} + \langle 1 \rangle_{\mathbb{R}},$$

where

$$J_f = \left\langle \frac{\partial f}{\partial q_1}(q), \dots, \frac{\partial f}{\partial q_k}(q) \right\rangle_{\mathcal{E}_q}.$$

Then we have the following theorem:

Theorem 2.2 *Let $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ and $G : (\mathbb{R}^{k'} \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be Morse families of functions. Then we have the following:*
(1) *$L(F)(C(F))$ and $L(G)(C(G))$ are Lagrangian equivalent if and only if F*

and G are stably P-$\mathcal{R}^+$-equivalent.
(2) *$L(F)(C(F))$ is a Lagrange stable if and only if F is a $\mathcal{R}^+$-versal deformation of f.*

For the proof of the above theorem, see [1, page 304 and 325]. The following proposition describes the well-known relationship between bifurcation sets and equivalence among unfoldings of function germs:

Proposition 2.3 *Let $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ and $G : (\mathbb{R}^{k'} \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be function germs. If F and G are stably P-$\mathcal{R}^+$-equivalent, then there exists a diffeomorphism germ $\phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $\phi(\mathcal{B}_F) = \mathcal{B}_G$.*

3 Big Legendrian submanifolds and wave front propagations

In this section, we give a brief review on the theory of big Legendrian submanifolds and wave front propagations.

We consider the projective cotangent bundle $\overline{\pi} : PT^*(\mathbb{R}^n \times \mathbb{R}) \to \mathbb{R}^n \times \mathbb{R}$ over $\mathbb{R}^n \times \mathbb{R}$. Let $\Pi : TPT^*(\mathbb{R}^n \times \mathbb{R}) \to PT^*(\mathbb{R}^n \times \mathbb{R})$ be the tangent bundle over $PT^*(\mathbb{R}^n \times \mathbb{R})$ and $d\overline{\pi} : TPT^*(\mathbb{R}^n \times \mathbb{R}) \to T(\mathbb{R}^n \times \mathbb{R})$ the differential map of $\overline{\pi}$.

For any $X \in TPT^*(\mathbb{R}^n \times \mathbb{R})$, there exists an element $\alpha \in T^*_{(x,t)}(\mathbb{R}^n \times \mathbb{R})$ such that $\Pi(X) = [\alpha]$. For an element $V \in T_{(x,t)}(\mathbb{R}^n \times \mathbb{R})$, the property $\alpha(V) = 0$ does not depend on the choice of representative of the class $[\alpha]$. Thus we can define *the canonical contact structure* on $PT^*(\mathbb{R}^n \times \mathbb{R})$ by

$$K = \{X \in TPT^*(\mathbb{R}^n \times \mathbb{R}) \mid \Pi(X)(d\overline{\pi}(X)) = 0\}.$$

Because of the trivialization $PT^*(\mathbb{R}^n \times \mathbb{R}) \cong (\mathbb{R}^n \times \mathbb{R}) \times P(\mathbb{R}^n \times \mathbb{R})^*$, we call

$$((x_1, \ldots, x_n, t), [\xi_1 : \cdots : \xi_n : \tau])$$

a homogeneous coordinate, where $[\xi_1 : \cdots : \xi_n : \tau]$ is the homogeneous coordinate of the dual projective space $P(\mathbb{R}^n \times \mathbb{R})^*$. It is easy to show that $X \in K_{((x,t),[\xi:\tau])}$ if and only if $\sum_{i=1}^n \mu_i \xi_i + \lambda\tau = 0$, where $d\overline{\pi}(X) = \sum_{i=1}^n \mu_i(\partial/\partial x_i) + \lambda(\partial/\partial t)$.

We remark that $PT^*(\mathbb{R}^n \times \mathbb{R})$ is a fiberwise compactification of the 1-jet space $J^1(\mathbb{R}^n, \mathbb{R})$ as follows: We consider an affine open subset $U_\tau = \{((x,t),[\xi : \tau]) | \tau \neq 0\}$ of $PT^*(\mathbb{R}^n \times \mathbb{R})$. For any $((x,t),[\xi : \tau]) \in U_\tau$, we have

$$((x,t),[\xi : \tau]) = ((x_1, \ldots, x_n, t), [-(\xi_1/\tau) : \cdots : -(\xi_n/\tau) : -1]),$$

so that we may adopt the corresponding *affine coordinates*

$$(x_1, \ldots, x_n, t, p_1, \ldots, p_n),$$

where $p_i = -\xi_i/\tau$. On U_τ we can easily show that $\theta^{-1}(0) = K|U_\tau$, where $\theta = dt - \sum_{i=1}^n p_i dx_i$. This means that U_τ may be identified with the 1-jet space $J^1(\mathbb{R}^n, \mathbb{R})$. We call the above coordinate *a system of canonical coordinates.* Throughout this paper, we use this identification so that we have $J^1(\mathbb{R}^n, \mathbb{R}) \subset PT^*(\mathbb{R}^n \times \mathbb{R})$.

A submanifold $i : L \subset PT^*(\mathbb{R}^n \times \mathbb{R})$ is *a Legendrian submanifold* if $\dim L = n$ and $di_p(T_pL) \subset K_{i(p)}$ for any $p \in L$. We say that a point $p \in L$ is *a Legendrian singular point* if $\operatorname{rank} d(\overline{\pi} \circ i)_p < n$.

For a Legendrian submanifold $i : L \subset PT^*(\mathbb{R}^n \times \mathbb{R})$, $\overline{\pi} \circ i(L) = W(L)$ is called *a big wave front.* We have a family of *small fronts*:

$$W_t(L) = \pi_1(\pi_2^{-1}(t) \cap W(L)) \quad (t \in \mathbb{R}),$$

where $\pi_1 : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $\pi_2 : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ are the canonical projections which gives $\pi_1(x,t) = x$ and $\pi_2(x,t) = t$ respectively. In this sense, we call L a *big Legendrian submanifold.* The *discriminant of the family* $W_t(L)$ is defined as the image of singular points of $\pi_1|W(L)$. In the general case, the discriminant consists of three components: *the caustics* C_L, the projection of the set of singular points of $W(L)$, *the Maxwell stratum* M_L, the projection of self intersection points of $W(L)$; and also of the *envelope of the family of small fronts* Δ (for more detail, see [13, 21]).

For any Legendrian submanifold germ $i : (L, p_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0)$, there exists a generating family of i by the theory of Legendrian singularity. Let $\mathcal{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ be a function germ such that $(\mathcal{F}, d_2\mathcal{F}) : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R} \times \mathbb{R}^k, 0)$ is a non-singular, where

$$d_2\mathcal{F}(q,x,t) = \left(\frac{\partial \mathcal{F}}{\partial q_1}(q,x,t), \ldots, \frac{\partial \mathcal{F}}{\partial q_k}(q,x,t)\right).$$

In this case, we call $\mathcal{F}$ a *big Morse family of hypersurfaces.* Then $\Sigma_*(\mathcal{F}) = (\mathcal{F}, d_2\mathcal{F})^{-1}(0)$ is a smooth n-dimensional submanifold germ. Define

$$\mathcal{L}_\mathcal{F} : (\Sigma_*(F), 0) \to PT^*(\mathbb{R}^n \times \mathbb{R})$$

by

$$\mathcal{L}_\mathcal{F}(q,x,t) = \left(x, t, \left[\frac{\partial \mathcal{F}}{\partial x}(q,x,t) : \frac{\partial \mathcal{F}}{\partial t}(q,x,t)\right]\right),$$

where

$$\left[\frac{\partial \mathcal{F}}{\partial x}(q,x,t) : \frac{\partial \mathcal{F}}{\partial t}(q,x,t)\right] = \left[\frac{\partial \mathcal{F}}{\partial x_1}(q,x,t) : \cdots : \frac{\partial \mathcal{F}}{\partial x_n}(q,x,t) : \frac{\partial \mathcal{F}}{\partial t}(q,x,t)\right].$$

It is easy to show that $\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ is a Legendrian submanifold germ. One of main result in the theory of Legendrian singularity (cf. [1]), we can show the following proposition:

Proposition 3.1 *All big Legendrian submanifold germs are constructed by the above method.*

For a function germ $\mathcal{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$, we call

$$D(\mathcal{F}) = \Big\{(x,t) \in (\mathbb{R}^n \times \mathbb{R}, 0) \;\Big|\; \text{there exists } q \in (\mathbb{R}^k, 0) \text{ such that } (q,x,t) \in \Sigma_*(\mathcal{F})\Big\},$$

the *discriminant set* of $\mathcal{F}$.

Let $\mathcal{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ be a big Morse family of hypersurfaces. We call $\mathcal{F}$ a *generating family of* $\mathcal{L}_{\mathcal{F}}$. In this case, the big wave front coincides with the discriminant set of $\mathcal{F}$, namely, $W(\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))) = D(\mathcal{F})$.

We now consider an equivalence relation among Legendrian submanifolds which preserves both the qualitative pictures of bifurcations and the discriminant of families of small fronts.

Let $i : (L, p_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0)$ and $i' : (L', p'_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p'_0)$ be Legendrian submanifold germs. We say that i and i' are *strictly parametrized*$^+$ *Legendrian equivalent* (or, briefly *$S.P^+$-Legendrian equivalent*) if there exist diffeomorphism germs $\Phi : (\mathbb{R}^n \times \mathbb{R}, \overline{\pi}(p_0)) \to (\mathbb{R}^n \times \mathbb{R}, \overline{\pi}(p'_0))$ of the form $\Phi(x,t) = (\phi_1(x), t + \alpha(x))$ and $\Psi : (L, p_0) \to (L', p'_0)$ such that $\widehat{\Phi} \circ i = i \circ \Psi$, where $\widehat{\Phi} : (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0) \to (PT^*(\mathbb{R}^n \times \mathbb{R}), p'_0)$ is the unique contact lift of Φ.

We also consider the notion of stability of Legendrian submanifold germs with respect to $S.P^+$-Legendrian equivalence is analogous to the stability of Lagrangian submanifold germs with respect to Lagrangian equivalence in section 2 (see, [1, Part III]).

We study the $S.P^+$-Legendrian equivalence by using the notion of generating families of Legendrian submanifold germs.

Let $f, g : (\mathbb{R}^k \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ be function germs. We say that f and g are *$S.P$-$\mathcal{K}$-equivalent* (or, *strictly P-$\mathcal{K}$-equivalent*) if there exists a diffeomorphism germ $\Phi : (\mathbb{R}^k \times \mathbb{R}, 0) \to (\mathbb{R}^k \times \mathbb{R}, 0)$ of the form $\Phi(q,t) = (\phi(q,t), t)$ such that $\langle f \circ \Phi \rangle_{\mathcal{E}_{(q,t)}} = \langle g \rangle_{\mathcal{E}_{(q,t)}}$.

Let $\mathcal{F}, \mathcal{G} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ be function germs. We say that $\mathcal{F}$ and $\mathcal{G}$ are *x-$S.P^+$-$\mathcal{K}$-equivalent* if there exists a diffeomorphism germ $\Phi : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0)$ of the form $\Phi(q,x,t) = (\phi(q,x,t), \phi_1(x), t + \alpha(x))$ such that $\langle \mathcal{F} \circ \Phi \rangle_{\mathcal{E}_{(q,x,t)}} = \langle \mathcal{G} \rangle_{\mathcal{E}_{(q,x,t)}}$. Let $\mathcal{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ and $\mathcal{G} : (\mathbb{R}^{k'} \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ be function germs. We say that $\mathcal{F}$ and $\mathcal{G}$ are *stably x-$S.P^+$-$\mathcal{K}$-equivalent* if they become x-$S.P^+$-$\mathcal{K}$-equivalent after the addition of non-degenerate quadratic forms in additional variables q' like stably P-$\mathcal{R}^+$-equivalence relations.

The notion of $S.P^+$-$\mathcal{K}$-versal deformation plays an important role for our purpose. We define the extended tangent space of $f : (\mathbb{R}^k \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ relative to $S.P^+$-$\mathcal{K}$ by

$$T_e(S.P^+\text{-}\mathcal{K})(f) = \left\langle \frac{\partial f}{\partial q_1}, \ldots, \frac{\partial f}{\partial q_k}, f \right\rangle_{\mathcal{E}_{(q,t)}} + \left\langle \frac{\partial f}{\partial t} \right\rangle_{\mathbb{R}}.$$

Then we say that $\mathcal{F}$ is *$S.P^+$-$\mathcal{K}$-versal* deformation of $f = \mathcal{F}|_{\mathbb{R}^k \times \{0\} \times \mathbb{R}}$ if it satisfies

$$\mathcal{E}_{(q,t)} = T_e(S.P^+\text{-}\mathcal{K})(f) + \left\langle \frac{\partial \mathcal{F}}{\partial x_1} | \mathbb{R}^k \times \{0\} \times \mathbb{R}, \ldots, \frac{\partial \mathcal{F}}{\partial x_n} | \mathbb{R}^k \times \{0\} \times \mathbb{R} \right\rangle_{\mathbb{R}}.$$

Theorem 3.2 *Let $\mathcal{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ and $\mathcal{G} : (\mathbb{R}^{k'} \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ be big Morse families of hypersurfaces. Then*
(1) $\mathcal{L}_{\mathcal{F}}(C(\mathcal{F}))$ and $\mathcal{L}_{\mathcal{G}}(C(\mathcal{G}))$ are $S.P^+$-Legendrian equivalent if and only if $\mathcal{F}$ and $\mathcal{G}$ are stably x-$S.P^+$-$\mathcal{K}$-equivalent.
(2) $\mathcal{L}_{\mathcal{F}}(C(\mathcal{F}))$ is $S.P^+$-Legendre stable if and only if $\mathcal{F}$ is a $S.P^+$-$\mathcal{K}$-versal deformation of $f = \mathcal{F}|_{\mathbb{R}^k \times \{0\} \times \mathbb{R}}$.

Since the big Legendrian submanifold germ $i : (L, p_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0)$ is uniquely determined on the regular part of the big wave front $W(L)$, we have the following simple but significant property of Legendrian submanifold germs:

Proposition 3.3 *Let $i : (L, p_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0)$ and $i' : (L', p_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0)$ be big Legendrian submanifold germs such that regular sets of $\overline{\pi} \circ i, \overline{\pi} \circ i'$ are dense respectively. Then $(L, p_0) = (L', p_0)$ if and only if $(W(L), \overline{\pi}(p_0)) = (W(L'), \overline{\pi}(p_0))$.*

This result has been firstly pointed out by Zakalyukin [20]. Also see [16]. The assumption in the above proposition is a generic condition for i, i'. Specially, if i and i' are $S.P^+$-Legendre stable, then these satisfy the assumption. Concerning the discriminant and the bifurcation of small fronts, we define the following equivalence relation among big wave front germs. Let $i : (L, p_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0)$ and $i' : (L', p'_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p'_0)$ be big Legendrian submanifold germs. We say that $W(L)$ and $W(L')$ are *$S.P^+$-diffeomorphic* if there exists a diffeomorphism germ $\Phi : (\mathbb{R}^n \times \mathbb{R}, \overline{\pi}(p_0)) \to (\mathbb{R}^n \times \mathbb{R}, \overline{\pi}(p'_0))$ of the form $\Phi(x, t) = (\phi_1(x), t + \alpha(x))$ such that $\Phi(W(L)) = W(L')$. Remark that the SP^+-diffeomorphism among big wave front germs preserves both the diffeomorphism types of bifurcations for families of small fronts and caustics [13, 21].

By proposition 3.3, we have the following proposition.

Proposition 3.4 *Let $i : (L, p_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0)$ and $i' : (L', p'_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p'_0)$ be big Legendrian submanifold germs such that regular sets of $\overline{\pi} \circ i, \overline{\pi} \circ i'$ are dense respectively. Then i and i' are $S.P^+$-Legendrian equivalent if and only if $(W(L), \overline{\pi}(p_0))$ and $(W(L'), \overline{\pi}(p'_0))$ are $S.P^+$-diffeomorphic.*

4 Graphlike Legendrian unfoldings

A big Legendrian submanifold $i : L \subset PT^*(\mathbb{R}^n \times \mathbb{R})$ is a *graphlike Legendrian unfolding* if $L \subset J^1(\mathbb{R}^n, \mathbb{R})$. We use notations in section 3. Since L is a big Legendrian submanifold in $PT^*(\mathbb{R}^n \times \mathbb{R})$, it has a generating family at least locally. In this case, it has a special form as follows: Let $\mathcal{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ be a big Morse family of hypersurfaces. We say that $\mathcal{F}$ is a *graphlike Morse family of hypersurfaces* if $(\partial \mathcal{F}/\partial t)(0) \neq 0$. It is easy to show that the corresponding big Legendrian submanifold germ is a graphlike Legendrian unfolding. Of course, all graphlike Legendrian unfolding germs can be constructed by the above way. We say that $\mathcal{F}$ is a *graphlike generating family* of $\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$.

We remark that the notion of graphlike Legendrian unfoldings and corresponding generating families have been introduced in [8] to describe the perestroikas of wave fronts given as the level surfaces of the solution for the eikonal equation given by a general Hamiltonian function. In this case, there is an additional condition, that is, $\mathcal{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ is a *generating family* if $\mathcal{F}$ satisfies the conditions $(\partial \mathcal{F}/\partial t)(0) \neq 0$ and $(\mathcal{F}, d_2\mathcal{F})|_{\mathbb{R}^k \times \mathbb{R}^n \times \{0\}}$ is a submersion germ, where

$$d_2\mathcal{F}(q, x, t) = \left(\frac{\partial \mathcal{F}}{\partial q_1}(q, x, t), \ldots, \frac{\partial \mathcal{F}}{\partial q_k}(q, x, t) \right).$$

We call such a generating family $\mathcal{F}$ a *non-degenerate graphlike generating family* and corresponding graphlike Legendrian unfolding a *non-degenarate graphlike Legendrian unfolding.* The second condition is equivalent to the condition that $\pi_2 \circ \overline{\pi} \circ i$ is a submersion at any point $p \in L$. Our situation is dropping the second condition. We can reduce more strict form of graphlike generating families as follows: Let $\mathcal{F}$ be a graphlike Morse family of hypersurfaces. By the implicit function theorem, there exists a Morse family of functions $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ such that $\langle \mathcal{F}(q, x, t) \rangle_{\mathcal{E}_{(q,x,t)}} =$

$\langle F(q,x)-t\rangle_{\mathcal{E}_{(q,x,t)}}$. Therefore $F(q,x)-t$ is a graphlike generating family of $\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$. In this case,

$$\Sigma_*(\mathcal{F}) = \{(q,x,F(q,x)) \in (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}),0) \mid (q,x) \in C(F)\}$$

and $\mathcal{L}_{\mathcal{F}} : (\Sigma_*(\mathcal{F}),0) \to J^1(\mathbb{R}^n,\mathbb{R})$ is given by

$$\mathcal{L}_{\mathcal{F}}(q,x,F(q,x)) = (L(F)(q,x),F(q,x)) \in J^1(\mathbb{R}^n,\mathbb{R}) \equiv T^*\mathbb{R}^n \times \mathbb{R}.$$

Define a map $\mathfrak{L}_F : (C(F),0) \to J^1(\mathbb{R}^n,\mathbb{R})$ by

$$\mathfrak{L}_F(q,x) = \left(x, F(q,x), \frac{\partial F}{\partial x_1}(q,x), \dots, \frac{\partial F}{\partial x_n}(q,x)\right),$$

then we have $\mathfrak{L}_F(C(F)) = \mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$. We call $W(\mathfrak{L}_F) = \overline{\pi}(\mathfrak{L}_F(C(F)))$ the *graphlike wave fronts* of the graphlike Legendrian unfolding $\mathfrak{L}_F$. We simply call F a *generating family* of the graphlike Legendrian unfolding $\mathfrak{L}_F$.

For any Morse family of functions F, we denote that $\overline{F}(q,x,t) = F(q,x) - t$. Since $\overline{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}),0) \to (\mathbb{R},0)$ is a big Morse family, we can use all the definitions of equivalence relations in section 3. Moreover, we can translate the propositions and theorems into corresponding assertions in terms of graphlike Legendrian unfoldings.

5 Modified Graphlike Legendrian unfoldings

Let $F : (\mathbb{R}^k \times \mathbb{R}^n,0) \to (\mathbb{R},0)$ be a Morse family of functions. We consider the following graphlike generating family $\mathcal{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}),0) \to (\mathbb{R},0)$ given by $\mathcal{F}(q,x,t) = F(q,x) - \varphi(t)$, where $\varphi : (\mathbb{R},0) \to (\mathbb{R},0)$ is a diffeomorphism germ. We denote $\varphi^{-1} \circ F(q,x)$ by $F_\varphi(q,x)$. Since the definition of a graphlike generating family, $\overline{F}_\varphi(q,x,t) = F_\varphi(q,x) - t$ is same as $\mathcal{F}(q,x,t)$. We call $\overline{F}_\varphi$ a *modified graphlike Legendrian unfolding* of $\overline{F}$ and φ. We clarify relationships between the functions F and F_φ. By a direct caluculation, we have the following proposition.

Proposition 5.1 *Let $F : (\mathbb{R}^k \times \mathbb{R}^n,0) \to (\mathbb{R},0)$ be a function germ and $\varphi : (\mathbb{R},0) \to (\mathbb{R},0)$ be a diffeomorphism germ. Then we have*

(1) $C(F) = C(F_\varphi)$,

(2) $\mathcal{B}_F = \mathcal{B}_{F_\varphi}$,

(3) *F is a Morse family of functions if and only if F_φ is a Morse family of functions.*

Suppose that F is a Morse family of functions and $\varphi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ is a diffeomorphism germ. Then we can construct two Lagrangian submanifold germs $L(F)(C(F))$ and $L(F_\varphi)(C(F_\varphi))$ by proposition 5.1. It is easy to see that the caustic $C_{C(F)}$ of $L(F)(C(F))$ coinsides with the caustic $C_{C(F_\varphi)}$ of $L(F_\varphi)(C(F_\varphi))$. We call $L(F_\varphi)(C(F_\varphi))$ an *induced Lagrangian submanifold germ* of F and φ.

We also give a relationship between the functions F and F_φ with respect to versality (cf. [4, 17, 18]).

Theorem 5.2 *Let $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a function germ and $\varphi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ be a diffeomorphism germ. F is a $\mathcal{R}^+$-versal unfolding of $f = F|_{\mathbb{R}^k \times \{0\}}$ if and only if F_φ is a $\mathcal{R}^+$-versal unfolding of $f_\varphi = F_\varphi|_{\mathbb{R}^k \times \{0\}}$.*

6 Relationship between equivalence relations

We consider a relationship of the equivalence relations between Lagrangian submanifold germs and induced graphlike Legendrian unfoldings, that is, between Morse families of functions and big Morse families of graphlike Legendrian unfoldings. As a consequence, we give a relationship between caustics and graphlike wave fronts.

Proposition 6.1 *Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be Morse families of functions. If Lagrangian submanifold germs $L(F)(C(F))$ and $L(G)(C(G))$ are Lagrangian equivalent, then the induced graphlike Legendrian unfoldings $\mathfrak{L}_F(C(F))$ and $\mathfrak{L}_G(C(G))$ are $S.P^+$-Legendrian equivalent.*

The above proposition asserts that the Lagrangian equivalence is stronger than the $S.P^+$-Legendrian equivalence. The $S.P^+$-Legendrian equivalence relation among graphlike Legendrian unfoldings preserves both the diffeomorphism types of bifurcations for families of small fronts and caustics. On the other hand, if we observe the real caustics of rays, we cannot observe the structure of wave front propagations. In this sense, there are hidden structure behind the picture of real caustics. By the above proposition, the Lagrangian equivalence preserve not only the diffeomorphism type of caustics, but also the hidden geometric structure of wave front propagations.

Conversely we have the following theorem [14].

Theorem 6.2 *Suppose that $L(F)(C(F))$ and $L(G)(C(G))$ are Lagrange stable. Then Lagrangian submanifold germs $L(F)(C(F))$ and $L(G)(C(G))$ are Lagrangian equivalent if and only if graphlike wave fronts $W(\mathfrak{L}_F)$ and $W(\mathfrak{L}_G)$ are $S.P^+$-diffeomorphic.*

7 Caustics of submanifolds in Euclidean space

As an application of the previous sections, we give a relationship between caustics of a submanifold and of the canal hypersurface of the submanifold in Euclidean space.

Let $\boldsymbol{x} : U \to \mathbb{R}^n$ be an embedding, where U is an open subset in $\mathbb{R}^r$. We denote the codimension of U in $\mathbb{R}^n$ by s $(= n - r)$. In order to consider the caustics (evolutes), we use the distance squared function germ of $\boldsymbol{x}$,

$$D : (U \times \mathbb{R}^n, (u_0, \boldsymbol{v}_0)) \to \mathbb{R}_+;\ D(u, \boldsymbol{v}) = ||\boldsymbol{x}(u) - \boldsymbol{v}||^2,$$

where $\mathbb{R}_+$ is the set of positive real numbers. We consider the case when $\boldsymbol{v}$ does not belong to the image of $\boldsymbol{x}$, so that we adopt $\mathbb{R}_+$ here. We can show that the distance squared function germ of $\boldsymbol{x}$ is a Morse family of functions, and hence we have a Lagrangian submanifold germ $L(D)(C(D))$.

We now consider a diffeomorphism germ $\varphi : (\mathbb{R}_+, t_0) \to (\mathbb{R}_+, t_1)$ which is given by $\varphi(t) = t^2$. We consider a modified graphlike Legendrian unfolding of $\overline{D}$ and φ;

$$\overline{D}_\varphi : (U \times \mathbb{R}^n \times \mathbb{R}_+, (u_0, \boldsymbol{v}_0, t_0)) \to (\mathbb{R}_+, 0).$$

Remark that we will consider $t_0 = t_0' + \alpha$ later, since we consider a relationship between caustics of a submanifold and of a canal hypersurface of the submanifold.

By a straightforward calculation, we have the following proposition.

Proposition 7.1 $\overline{D}_\varphi(u, \boldsymbol{v}, t) = (\partial \overline{D}_\varphi / \partial u_i)(u, \boldsymbol{v}, t) = 0, (i = 1, \dots, r)$ *if and only if there exist real numbers* $\lambda_1, \cdots, \lambda_s$ *such that* $\boldsymbol{v} = \boldsymbol{x}(u) - \lambda_1 \boldsymbol{n}_1(u) - \cdots - \lambda_s \boldsymbol{n}_s(u)$ *and* $t = \sqrt{\lambda_1^2 + \cdots + \lambda_s^2}$.

On the other hand, a canal hypersurface $\boldsymbol{y} : U \times S^{s-1} \to \mathbb{R}^n$ of $\boldsymbol{x} : U \to \mathbb{R}^n$ is defined by

$$\boldsymbol{y}(u, \mu_1, \dots, \mu_s) = \boldsymbol{x}(u) + \alpha \cdot \sum_{i=1}^{s} \mu_i \boldsymbol{n}_i(u),$$

where $\{\boldsymbol{x}_{u_1}, \dots, \boldsymbol{x}_{u_r}, \boldsymbol{n}_1, \dots, \boldsymbol{n}_s\}$ is a frame field of $\mathbb{R}^n$ along $\boldsymbol{x}(U)$. Remark that there exists a positive real number A such that $\boldsymbol{y}$ is a regular hypersurface for $0 < \alpha < A$. We write that $\boldsymbol{e}(u, \mu) = \sum_{i=1}^{s} \mu_i \boldsymbol{n}_i(u)$. Then the normal of the canal hypersurface at $\boldsymbol{y}(u, \mu)$ is given by $\boldsymbol{e}(u, \mu)$.

We also consider the distance squared function germ of $\boldsymbol{y}$,

$$\widetilde{D} : (U \times S^{s-1} \times \mathbb{R}^n, (u_0, \mu_0, \boldsymbol{w}_0)) \to \mathbb{R}_+;\ \widetilde{D}(u, \mu, \boldsymbol{w}) = ||\boldsymbol{y}(u, \mu) - \boldsymbol{w}||^2.$$

We have already shown that the distance squared function germ of hypersurfaces is a Morse family of functions in [14]. We also have a Lagrangian submanifold germ $L(\widetilde{D})(C(\widetilde{D}))$.

Let $\psi : (\mathbb{R}_+; t'_0) \to (\mathbb{R}_+, t'_1)$ be a diffeomorphism germ which is given by $\psi(t) = t^2$. We also consider a modified graphlike Legendrian unfolding of $\overline{\widetilde{D}}$ and ψ;

$$\overline{\widetilde{D}}_\psi : (U \times S^{s-1} \times \mathbb{R}^n \times \mathbb{R}_+, (u_0, \mu_0, w_0, t'_0)) \to (\mathbb{R}_+, 0).$$

Also by a straightforward calculation, we have the following proposition.

Proposition 7.2 $\overline{\widetilde{D}}_\psi(u, \mu, w, t') = (\partial\overline{\widetilde{D}}_\psi/\partial u_i)(u, \mu, w, t') = (\partial\overline{\widetilde{D}}_\psi/\partial \mu_j)(u, \mu, w, t') = 0, (i = 1, \dots, r, j = 1, \dots, s-1)$ *if and only if there exists a real number* a *such that* $w = x(u) + (\alpha - a)e(u, \mu)$ *and* $t' = \sqrt{a^2}$.

Here we take a local coordinate $(\mu_1, \dots, \mu_{s-1})$ of S^{s-1}. We may suppose that $\alpha - a \geq \alpha$, *i.e.*, $a \leq 0$.

Proposition 7.3 *Under the above notations, graphlike wave front germs* $W(\overline{D}_\varphi)$ *and* $W(\overline{\widetilde{D}}_\psi)$ *are* $S.P^+$*-diffeomorphism.*

Then we can show the following theorems [15].

Theorem 7.4 *Caustics* $C_{C(D)}$ *coinsides with* $C_{C(\tilde{D})}$.

Theorem 7.5 *If* $L(D)(C(D))$ *and* $L(\widetilde{D})(C(\widetilde{D}))$ *are Lagrange stable, then the induced Lagrangian submanifold germs of* D *and* φ*, and of* $\widetilde{D}$ *and* ψ *are Lagrangian equivalent, so that caustics* $C_{C(D)}$ *and* $C_{C(\tilde{D})}$ *are diffeomorphic.*

Remark 7.6 For a curve in $\mathbb{R}^3$, under the condition that its curveture dose not vanish, caustics of the curve and of a canal surface of the curve in $\mathbb{R}^3$ are the same by a direct caluculation. However, it is very hard to caluculate directly for the case of higher codimensional submanifolds.

Remark 7.7 The analogous results to the above theorems are approved in various situations. For example, submanifolds in Euclidean sphere and submanifolds in hyperbolic space or de-Sitter space in Minkowski space. In the case of Euclidean sphere, we may consider a height function as a Morse family of functions. In [11, 12, 13], we consider caustics (evolutes) of hypersurface in hyperbolic or de-Sitter space by using timelike or spacelike height functions. We can apply the method in this paper to such situations. For submanifolds in Euclidean sphere, we may take the local diffeomorphism germ $\varphi(t) = \cos t$. Besides, for submanifolds in hyperbolic space or de-Sitter space in Minkowski space, we may take the local diffeomorphism germs $\varphi(t) = \cosh t$ or $\sinh t$.

Remark 7.8 We can consider relationship between contact with submanifolds and contact with canal hypersurfaces, for instance, see in [10, 13, 14].

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