\section{Introduction}

For \( n > k \geq 0 \), let \( M^n \) and \( Q^{n-k} \) be a smooth closed \( n \)-dimensional and a smooth \( (n-k) \)-dimensional manifold, respectively. We call a smooth map from \( M \) to \( Q \) a corank 1 map if the rank of its differential is not less than \( n - k - 1 \) at any point of \( M \). For a corank 1 map \( f: M \to Q \) let \( \Sigma \) denote the set of singular points in \( M \).

A basic example of a corank 1 map is a smooth map \( M \to Q \) with only Morse type singularities, that is a fold map. Note that the restriction \( f|_{\Sigma} \) is an immersion if \( f \) is a fold map.

Ando's h-principle [And04] states that there exists a fold map \( f: M \to Q \) such that the immersion \( f|_{\Sigma} \) is coorientable if and only if there exists a fiberwise epimorphism \( TM \oplus \epsilon \to TQ \), also see [Sae92]. Note that in the case of even \( k \) the immersion \( f|_{\Sigma} \) is always coorientable.

We call a corank 1 map \( f: M \to Q \) tame if the 1-dimensional cokernel bundle \( \text{coker} \ df|_{\Sigma} \) of the restriction \( df|_{\Sigma}: TM|_{\Sigma} \to f^*TQ \) is trivial. For example, every fold map is tame for \( k \equiv 0 \mod 2 \) [And04] and it is easy to construct not tame fold maps for odd \( k \leq n-3 \), even between orientable manifolds.

Ando's h-principle [And04] enables us to reduce the problem of the existence of tame fold maps (and more generally tame corank 1 maps) to the existence of \( n-k \) linearly independent sections of \( TM \oplus \epsilon \) if \( Q \) is stably parallelizable.

If such a partial framing exists, then clearly the Stiefel-Whitney classes \( w_i(TM) \) vanish for \( i \geq k+2 \). Hence we obtain the following easy

\textbf{Proposition 1.1.} Let \( n+1 = 2^D m \), where \( m > 1 \) is odd. There is no tame corank 1 map from \( \mathbb{R}P^n \) to any stably parallelizable \( Q^{n-k} \) if \( 2^D(m-1) \geq k+2 \).

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However, the tangential Stiefel-Whitney and Pontryagin classes of $\mathbb{R}P^{2^n-1}$ vanish, thus in order to obtain obstructions in this case, we need something else. By applying K-theory and following [Ati61], we obtain

**Proposition 1.2.** If $M^n$ admits a tame corank 1 map into a stably parallelizable $Q^{n-k}$, then $\gamma^i([TM] - [\epsilon^n]) = 0$ for $i \geq k + 2$.

Here $\gamma$ denotes the $\gamma$ operation in real $K$-theory, see [Ati61]. Proposition 1.2 can be useful if the higher tangential Stiefel-Whitney and Pontryagin classes of $M$ vanish. For example, we obtain

**Corollary 1.3.** Let $n \geq 4$, $2^n - 2^{\log_2 n} \geq k + 2$ and $Q$ stably parallelizable.

(1) $\mathbb{R}P^{2^n-1}$ admits no fold map into $Q^{2^n-1-k}$ if $k$ is even,

(2) $\mathbb{R}P^{2^n-1}$ admits no fold map with orientable singular set into $Q^{2^n-1-k}$ if $k$ is odd.

However, by using much sophisticated and deeper results of Atiyah, Bott and Shapiro [ABS64] and Steer [Ste67], which determine the geometric dimensions of the tangent bundles of the projective spaces, we have the stronger

**Proposition 1.4.** There exists a tame corank 1 map from the projective space $\mathbb{F}P^n$ into an $(n-k)$-dimensional stably parallelizable manifold if and only if $(n+1)d(F) - q(n+1,F) \leq k + 1$, where $d(F)$ denotes the dimension over $\mathbb{R}$ of the (skew) field $F \in \{\mathbb{R},\mathbb{C},\mathbb{H}\}$ and $q(n+1,F)$ denotes the Radon-Hurwitz number associated to $n+1$ and $F$.

## 2. Results

Let $M^n$ and $Q^{n-k}$ be a closed $n$-manifold and an $(n-k)$-manifold, respectively. For a finite CW-complex $X$, $\tilde{K}_{\mathbb{R}}(X)$ and $K_{\mathbb{R}}(X)$ denote the reduced and unreduced real $K$-rings of $X$, respectively, with $\tilde{K}_{\mathbb{R}}(X) \subseteq K_{\mathbb{R}}(X)$. Recall that for a finite CW-complex $X$ the geometric dimension $g.dim(x)$ of an element $x \in \tilde{K}_{\mathbb{R}}(X)$ is the least integer $k$ such that $x + k$ is a class of a genuine vector bundle over $X$ (see e.g. [Ati61]).

Similarly to [And04], we have

**Proposition 2.1.** The following are equivalent:

(1) $M$ admits a tame corank 1 map into $Q$,

(2) there is a fiberwise epimorphism $TM \oplus \epsilon^1 \to TQ$.

If $Q$ is stably parallelizable, then (1) and (2) hold if and only if $g.dim([TM] - [\epsilon^n]) \leq k + 1$.

For a finite CW-complex $X$, let $\lambda_t = \sum_{i=0}^{\infty} \lambda^i t^i$, where $\lambda^i$ are the exterior power operators (for details, see [Ati61]). Define $\gamma_t = \sum_{i=0}^{\infty} \gamma^i t^i$ to be the homomorphism $\lambda_{t/1-t}$ of $K_{\mathbb{R}}(X)$ into the multiplicative group of formal power series in $t$ with coefficients in $K_{\mathbb{R}}(X)$ and constant term 1. By the above proposition and [Ati61, Proposition 2.3], we immediately have
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Corollary 2.2. 1 If \( M \) admits a tame corank 1 map into a stably parallelizable \( Q \), then

1. \( w_i(TM) = 0 \) for \( i \geq k + 2 \),
2. \( p_i(TM) = 0 \) for \( 2i > k + 1 \),
3. \( \gamma^i([TM] - [e^n]) = 0 \) for \( i \geq k + 2 \).

Remark 2.3. Note that the conditions (1) and (2) may not give strong results in general: for example, all the positive degree Stiefel-Whitney and Pontryagin classes of \( \mathbb{R}P^{2n-1} \) vanish\(^2\), and if \( k + 1 \geq n/2 \), then condition (2) is satisfied trivially for any \( M \). In particular cases, though, condition (1) can still give strong results, e.g. all Stiefel-Whitney classes of \( \mathbb{R}P^{2n-2} \) of degree up to \( 2^n - 2 \) are non-zero.

For an integer \( s \) let \( 2^{R(s)} \) be the maximal power of 2 that divides \( s \), and define \( \kappa(n) = \max\{0 < s < 2^{n-1} : s - R(s) < 2^{n-1} - n\} \). By using Corollary 2.2 (3) and following a similar argument to [Ati61], we obtain the following:

Proposition 2.4. For \( n \geq 4 \), \( \mathbb{R}P^{2n-1} \) does not admit tame corank 1 map into any stably parallelizable \( Q^{2n-1-k} \) for \( k \leq \kappa(n) - 2 \).

Remark 2.5. Obviously \( s_0 = 2^{n-1} - 2^{\min\{r : r + 2^{r} > n\}} \) satisfies \( s_0 + n - R(s_0) < 2^{n-1} \), thus \( s_0 \leq \kappa(n) \) and we obtain that \( \mathbb{R}P^{2n-1} \) admits no fold map with orientable singular set into \( \mathbb{R}^{2^{n-1} + 2^{\min\{r : r + 2^{r} > n\}} + j} \) for \( n \geq 4 \) and \( j \geq 1 \). Also, since \( \min\{r : r + 2^{r} > n\} \leq \log_2 n \), the same conclusion holds in the case of the target \( \mathbb{R}^{2^{n-1} + 2^{\lceil\log_2 n\rceil} + j} \) for \( n \geq 4 \) and \( j \geq 1 \). For example, there exists neither a fold map from \( \mathbb{R}P^31 \) to \( \mathbb{R}^{21+2j} \) for \( 0 \leq j \leq 5 \) nor a fold map with orientable singular set from \( \mathbb{R}P^31 \) to \( \mathbb{R}^{22+3j} \) for \( 0 \leq j \leq 4 \).

Remark 2.6. However, we have stronger results about maps of the projective spaces that follow immediately from Proposition 2.1 and [Ste67], which determines the geometric dimensions of the tangent bundles of projective spaces in terms of Radon-Hurwitz numbers.

Proof of Proposition 2.1. (2) \( \implies \) (1): By [And04], if there is a \( TM \oplus \varepsilon^1 \to TQ \) epimorphism, then there is a fold map \( M \to Q \) with orientable singular set. (1) \( \implies \) (2): Assume that we have a tame corank 1 map \( f : M \to Q \). The bundle coker \( df|_\Sigma = (f^*TQ/f^*df(TM))|_\Sigma \) is considered as a subbundle of \( f^*TQ \) and it is trivial. Similarly to [And04, Proof of Lemma 3.1], let \( L : \varepsilon^1 \to TQ \) be an extension of the bundle monomorphism coker \( df|_\Sigma \to f^*TQ \to TQ \) as a bundle homomorphism covering \( f \). Then \( df + L \) is an epimorphism \( TM \oplus \varepsilon^1 \to TQ \).

Finally, if (1) or (2) holds and \( Q \) is stably parallelizable, then by the above, we have \( TM \oplus \varepsilon^1 \oplus \varepsilon^N \cong \zeta \oplus f^*TQ \oplus \varepsilon^N \cong \zeta \oplus \varepsilon^{N+n-k} \) for some \( N \gg 0 \) and a \((k+1)\)-dimensional bundle \( \zeta \). Thus \( g.dim([TM]-[e^n]) \leq k+1 \).

\(^1\)Compare with [Ati61, Proposition 3.2].

\(^2\)We have \( w(T\mathbb{R}P^{2^n-1}) = (1 + x)x^n = 1 \in \mathbb{Z}_2[x]/x^{2^n} = H^*(\mathbb{R}P^{2^n-1}; \mathbb{Z}_2) \), where \( x \) denotes the generator of \( H^*(\mathbb{R}P^{2^n-1}; \mathbb{Z}_2) \). The natural homomorphism \( H^*(\mathbb{R}P^{2^n-1}; \mathbb{Z}_2) \to H^*(\mathbb{R}P^{2^n-1}; \mathbb{Z}_2) \) is an isomorphism for all positive even \( s \). Our claim follows by applying the fact that \( p_i \equiv w_{2i}^2 \) mod 2.
If $Q$ is stably parallelizable and $g.\dim([TM]-[e^n]) \leq k + 1$, then $TM \oplus \epsilon^N \cong \zeta^{k+1} \oplus \epsilon^{N+n-k-1} \cong \zeta^{k+1} \oplus TQ \oplus \epsilon^{N-1}$ for some $N \gg 0$, and thus $TM \oplus \epsilon^1 \cong \zeta^{k+1} \oplus TQ$, which proves (2).

Proof of Proposition 2.4. Let $\varphi(n)$ denote the cardinality of the set \{0 < s \leq n : s \equiv 0, 1, 2, 4 \mod 8\}. By [Ati61, §5], $[\mathbb{R}P^n] - [e^n] = (n+1)x$ and $\gamma^i([\mathbb{R}P^n] - [e^n]) = 2^{i-1}x_i$, $i \geq 1$, where $x_i$ denotes the generator of $K_\mathbb{R}(\mathbb{R}P^n) = \mathbb{Z}_{2^{\varphi(n)}}$. Therefore $\gamma^i([\mathbb{R}P^n] - [e^n]) = 0$ if and only if $2^{\varphi(n)}$ divides $2^{i-1}(n+1)$. Let $r(n)$ denote the greatest integer $s$ for which $2^{s-1}(n+1)$ is not divisible by $2^{\varphi(n)}$. Then by Proposition 2.1 there is no tame corank 1 map of $\mathbb{R}P^{2^n-1}$ into $\mathbb{R}^{2^n-1-k}$ for $k \leq r(2^n - 1) - 2$. It is easy to see that $\varphi(2^n - 1) = 2^n - 1$ if $n \geq 3$. By a classical result of E. Kummer, the highest power $c(s)$ of 2 which divides $2^{\varphi(n)}$ can be obtained by counting the number of carries when $s$ and $2^n - s$ are added in base 2. For $s \leq 2^n - 1 - 1$, we claim that $c(s) = n - R(s)$, where $2^{R(s)}$ is the maximal power of 2 which divides $s$. Indeed, $2^n - 1 - s$ is obtained by negating the binary form of $s$ bitwise, hence $2^n - s$ is obtained by negating the binary form of $s$ bitwise from the $(n - 1)$st to the $R(s)$th binary position, where both of $s$ and $2^n - s$ have the digit 1, and after that position both have digits 0. Therefore when we add $s$ and $2^n - s$ in base 2, we have $n - R(s)$ carries. By the definition of $r(n)$ it follows that $r(2^n - 1)$ is the largest integer $s$ for which $s + n - R(s) < 2^{n-1}$.

When $n$ is not a power of 2, we have the following easy results for $\mathbb{R}P^{pn-1}$.

Proposition 2.7. Let $n = 2^Dm$, where $m > 1$ is odd. Then $\binom{n}{2^D}$ is odd. Hence $w_{2^D(m-1)}(\mathbb{R}P^{pn-1}) \neq 0$.

Proof. It is obvious from [Gla99], details are left to the reader. $

\section*{REFERENCES}


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