数理解析研究所講究録
第 1707 巻 2010 年 149-152

γ OPERATIONS IN K-THEORY AND EXISTENCE OF SINGULAR MAPS

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ABSTRACT. We introduce new obstructions to the existence of fold maps with orientable cokernel bundle by relating K-theory and the γ operation of Grothendieck [Ati61] to the h-principle of Ando [And04]. We compute these obstructions for fold maps of the projective spaces.

1. INTRODUCTION

For $n > k \geq 0$, let M^n and Q^{n-k} be a smooth closed n-dimensional and a smooth $(n-k)$-dimensional manifold, respectively. We call a smooth map from M to Q a *corank* 1 *map* if the rank of its differential is not less than $n-k-1$ at any point of M. For a corank 1 map $f\colon M \to Q$ let Σ denote the set of singular points in M.

A basic example of a corank 1 map is a smooth map $M \to Q$ with only Morse type singularities, that is a *fold map.* Note that the restriction $f|_\Sigma$ is an immersion if f is a fold map.

Ando's h-priciple [And04] states that there exists a fold map $f\colon M \to Q$ such that the immerison $f|_\Sigma$ is coorientable if and only if there exists a fiberwise epimorphism $TM \oplus \varepsilon^1 \to TQ$, also see [Sae92]. Note that in the case of even k the immersion $f|_\Sigma$ is always coorientable.

We call a corank 1 map $f\colon M \to Q$ *tame* if the 1-dimensional cokernel bundle $\operatorname{coker} df|_\Sigma$ of the restriction $df|_\Sigma\colon TM|_\Sigma \to f^*TQ$ is trivial. For example, every fold map is tame for $k \equiv 0 \mod 2$ [And04] and it is easy to construct not tame fold maps for odd $k \leq n-3$, even between orientable manifolds.

Ando's h-priciple [And04] enables us to reduce the problem of the existence of tame fold maps (and more generally tame corank 1 maps) to the existence of $n-k$ linearly independent sections of $TM \oplus \varepsilon^1$ if Q is stably parallelizable.

If such a partial framing exists, then clearly the Stiefel-Whitney classes $w_i(TM)$ vanish for $i \geq k+2$. Hence we obtain the following easy

Proposition 1.1. *Let $n+1 = 2^D m$, where $m > 1$ is odd. There is no tame corank 1 map from $\mathbb{R}P^n$ to any stably parallelizable Q^{n-k} if $2^D(m-1) \geq k+2$.*

Key words and phrases. Singular map, K-theory, fold map, geometric dimension.
2010 *Mathematics Subject Classification.* Primary 57R45; Secondary 57R25, 55S25, 19L64.
The first author has been partially supported by Magyary Zoltán Postdoctoral Fellowship and OTKA grant NK81203.
The second author has been supported by OTKA grant NK81203.

However, the tangential Stiefel-Whitney and Pontryagin classes of $\mathbb{R}P^{2^n-1}$ vanish, thus in order to obtain obstructions in this case, we need something else. By applying K-theory and following [Ati61], we obtain

Proposition 1.2. *If M^n admits a tame corank 1 map into a stably parallelizable Q^{n-k}, then $\gamma^i([TM]-[\varepsilon^n])=0$ for $i\geq k+2$.*

Here γ denotes the γ operation in real K-theory, see [Ati61]. Proposition 1.2 can be useful if the higher tangential Stiefel-Whitney and Pontryagin classes of M vanish. For example, we obtain

Corollary 1.3. *Let $n\geq 4$, $2^{n-1}-2^{\lceil\log_2 n\rceil}\geq k+2$ and Q stably parallelizable.*

(1) *$\mathbb{R}P^{2^n-1}$ admits no fold map into Q^{2^n-1-k} if k is even,*
(2) *$\mathbb{R}P^{2^n-1}$ admits no fold map with orientable singular set into Q^{2^n-1-k} if k is odd.*

However, by using much sophisticated and deeper results of Atiyah, Bott and Shapiro [ABS64] and Steer [Ste67], which determine the geometric dimensions of the tangent bundles of the projective spaces, we have the stronger

Proposition 1.4. *There exists a tame corank 1 map from the projective space $\mathbb{F}P^n$ into an $(n-k)$-dimensional stably parallelizable manifold if and only if $(n+1)d(\mathbb{F})-q(n+1,\mathbb{F})\leq k+1$, where $d(\mathbb{F})$ denotes the dimension over $\mathbb{R}$ of the (skew) field $\mathbb{F}\in\{\mathbb{R},\mathbb{C},\mathbb{H}\}$ and $q(n+1,\mathbb{F})$ denotes the Radon-Hurwitz number associated to $n+1$ and $\mathbb{F}$.*

2. Results

Let M^n and Q^{n-k} be a closed n-manifold and an $(n-k)$-manifold, respectively. For a finite CW-complex X, $\widetilde{K}_{\mathbb{R}}(X)$ and $K_{\mathbb{R}}(X)$ denote the reduced and unreduced real K-rings of X, respectively, with $\widetilde{K}_{\mathbb{R}}(X)\subseteq K_{\mathbb{R}}(X)$. Recall that for a finite CW-complex X the geometric dimension $g.dim(x)$ of an element $x\in\widetilde{K}_{\mathbb{R}}(X)$ is the least integer k such that $x+k$ is a class of a genuine vector bundle over X (see e.g. [Ati61]).

Similarly to [And04], we have

Proposition 2.1. *The following are equivalent:*

(1) *M admits a tame corank 1 map into Q,*
(2) *there is a fiberwise epimorphism $TM\oplus\varepsilon^1\to TQ$.*

If Q is stably parallelizable, then (1) and (2) hold if and only if $g.dim([TM]-[\varepsilon^n])\leq k+1$.

For a finite CW-complex X, let $\lambda_t=\sum_{i=0}^{\infty}\lambda^i t^i$, where λ^i are the exterior power operators (for details, see [Ati61]). Define $\gamma_t=\sum_{i=0}^{\infty}\gamma^i t^i$ to be the homomorphism $\lambda_{t/1-t}$ of $K_{\mathbb{R}}(X)$ into the multiplicative group of formal power series in t with coefficients in $K_{\mathbb{R}}(X)$ and constant term 1. By the above proposition and [Ati61, Proposition 2.3], we immediately have

Corollary 2.2. [1] *If M admits a tame corank 1 map into a stably parallelizable Q, then*

(1) $w_i(TM) = 0$ *for* $i \geq k+2$,
(2) $p_i(TM) = 0$ *for* $2i > k+1$,
(3) $\gamma^i([TM] - [\varepsilon^n]) = 0$ *for* $i \geq k+2$.

Remark 2.3. Note that the conditions (1) and (2) may not give strong results in general: for example, all the positive degree Stiefel-Whitney and Pontryagin classes of $\mathbb{R}P^{2^n-1}$ vanish[2], and if $k+1 \geq n/2$, then condition (2) is satisfied trivially for any M. In particular cases, though, condition (1) can still give strong results, e.g. all Stiefel-Whitney classes of $\mathbb{R}P^{2^n-2}$ of degree up to $2^n - 2$ are non-zero.

For an integer s let $2^{R(s)}$ be the maximal power of 2 that divides s, and define $\kappa(n) = \max\{0 < s < 2^{n-1} : s - R(s) < 2^{n-1} - n\}$. By using Corollary 2.2 (3) and following a similar argument to [Ati61], we obtain the following:

Proposition 2.4. *For $n \geq 4$, $\mathbb{R}P^{2^n-1}$ does not admit tame corank 1 map into any stably parallelizable Q^{2^n-1-k} for $k \leq \kappa(n) - 2$.*

Remark 2.5. Obviously $s_0 = 2^{n-1} - 2^{\min\{r:r+2^r>n\}}$ satisfies $s_0 + n - R(s_0) < 2^{n-1}$, thus $s_0 \leq \kappa(n)$ and we obtain that $\mathbb{R}P^{2^n-1}$ admits no fold map with orientable singular set into $\mathbb{R}^{2^{n-1}+2^{\min\{r:r+2^r>n\}}+j}$ for $n \geq 4$ and $j \geq 1$. Also, since $\min\{r : r+2^r > n\} \leq \lceil \log_2 n \rceil$, the same conclusion holds in the case of the target $\mathbb{R}^{2^{n-1}+2^{\lceil \log_2 n \rceil}+j}$ for $n \geq 4$ and $j \geq 1$. For example, there exists neither a fold map from $\mathbb{R}P^{31}$ to $\mathbb{R}^{21+2j}$ for $0 \leq j \leq 5$ nor a fold map with orientable singular set from $\mathbb{R}P^{31}$ to $\mathbb{R}^{22+2j}$ for $0 \leq j \leq 4$.

Remark 2.6. However, we have stronger results about maps of the projective spaces that follow immediately from Proposition 2.1 and [Ste67], which determines the geometric dimensions of the tangent bundles of projective spaces in terms of Radon-Hurwitz numbers.

Proof of Proposition 2.1. (2) $\Longrightarrow$ (1): By [And04], if there is a $TM \oplus \varepsilon^1 \to TQ$ epimorphism, then there is a fold map $M \to Q$ with orientable singular set. (1) $\Longrightarrow$ (2): Assume that we have a tame corank 1 map $f\colon M \to Q$. The bundle $\operatorname{coker} df|_\Sigma = (f^*TQ/f^*df(TM))|_\Sigma$ is considered as a subbundle of f^*TQ and it is trivial. Similarly to [And04, Proof of Lemma 3.1], let $L\colon \varepsilon^1 \to TQ$ be an extension of the bundle monomorphism $\operatorname{coker} df|_\Sigma \to f^*TQ \to TQ$ as a bundle homomorphism covering f. Then $df + L$ is an epimorphism $TM \oplus \varepsilon^1 \to TQ$.

Finally, if (1) or (2) holds and Q is stably parallelizable, then by the above, we have $TM \oplus \varepsilon^1 \oplus \varepsilon^N \cong \zeta \oplus f^*TQ \oplus \varepsilon^N \cong \zeta \oplus \varepsilon^{N+n-k}$ for some $N \gg 0$ and a $(k+1)$-dimensional bundle ζ. Thus $g.dim([TM] - [\varepsilon^n]) \leq k+1$.

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[1] Compare with [Ati61, Proposition 3.2].

[2] We have $w(T\mathbb{R}P^{2^n-1}) = (1+x)^{2^n} = 1 \in \mathbb{Z}_2[x]/x^{2^n} = H^*(\mathbb{R}P^{2^n-1}; \mathbb{Z}_2)$, where x denotes the generator of $H^1(\mathbb{R}P^{2^n-1}; \mathbb{Z}_2)$. The natural homomorphism $H^s(\mathbb{R}P^{2^n-1}; \mathbb{Z}) \to H^s(\mathbb{R}P^{2^n-1}; \mathbb{Z}_2)$ is an isomorphism for all positive even s. Our claim follows by applying the fact that $p_i \equiv w_{2i}^2$ mod 2.

If Q is stably parallelizable and $g.dim([TM]-[\varepsilon^n]) \leq k+1$, then $TM \oplus \varepsilon^N \cong \zeta^{k+1} \oplus \varepsilon^{N+n-k-1} \cong \zeta^{k+1} \oplus TQ \oplus \varepsilon^{N-1}$ for some $N \gg 0$, and thus $TM \oplus \varepsilon^1 \cong \zeta^{k+1} \oplus TQ$, which proves (2). □

Proof of Proposition 2.4. Let $\varphi(n)$ denote the cardinality of the set $\{0 < s \leq n : s \equiv 0,1,2,4 \mod 8\}$. By [Ati61, §5], $[T\mathbb{R}P^n]-[\varepsilon^n] = (n+1)x$ and $\gamma^i([T\mathbb{R}P^n]-[\varepsilon^n]) = 2^{i-1}\binom{n+1}{i}x$, $i \geq 1$, where x denotes the generator of $\widetilde{K}_{\mathbb{R}}(\mathbb{R}P^n) = \mathbb{Z}_{2^{\varphi(n)}}$. Therefore $\gamma^i([T\mathbb{R}P^n]-[\varepsilon^n]) = 0$ if and only if $2^{\varphi(n)}$ divides $2^{i-1}\binom{n+1}{i}$. Let $r(n)$ denote the greatest integer s for which $2^{s-1}\binom{n+1}{s}$ is not divisible by $2^{\varphi(n)}$. Then by Proposition 2.1 there is no tame corank 1 map of $\mathbb{R}P^{2^n-1}$ into $\mathbb{R}^{2^n-1-k}$ for $k \leq r(2^n-1)-2$. It is easy to see that $\varphi(2^n-1) = 2^{n-1}-1$ if $n \geq 3$. By a classical result of E. Kummer, the highest power $c(s)$ of 2 which divides $\binom{2^n}{s}$ can be obtained by counting the number of carries when s and 2^n-s are added in base 2. For $s \leq 2^{n-1}-1$, we claim that $c(s) = n - R(s)$, where $2^{R(s)}$ is the maximal power of 2 which divides s. Indeed, 2^n-1-s is obtained by negating the binary form of s bitwise, hence 2^n-s is obtained by negating the binary form of s bitwise from the $(n-1)$st to the $R(s)$th binary position, where both of s and 2^n-s have the digit 1, and after that position both have digits 0. Therefore when we add s and 2^n-s in base 2, we have $n-R(s)$ carries. By the definition of $r(n)$ it follows that $r(2^n-1)$ is the largest integer s for which $s+n-R(s) < 2^{n-1}$. □

When n is not a power of 2, we have the following easy results for $\mathbb{R}P^{n-1}$.

Proposition 2.7. *Let $n = 2^D m$, where $m > 1$ is odd . Then $\binom{n}{2^D}$ is odd. Hence $w_{2^D(m-1)}(T\mathbb{R}P^{n-1}) \neq 0$.*

Proof. It is obvious from [Gla99], details are left to the reader. □

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