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Lifting of equivalences and perfect isometries between blocks of finite groups with Appendices on perfect isometries by Masao Kiyota and on blocks with elementary abelian defect group of order 9 by Atumi Watanabe

Department of Mathematics and Informatics,
Graduate School of Science, Chiba University
e-mail koshitan@math.s.chiba-u.ac.jp
千葉大学 大学院理学研究科

Shigeo Koshitani 越谷 重夫

1. Introduction and notation.

In modular representation theory of finite groups, it has to be important and meaningful to investigate structure of p-blocks (block algebras) of finite groups G, where p is a prime number.

Notation 1.1. Throughout this note we use the following notation and terminology, which should be standard. We denote by G always a finite group, and let p be a prime. Then, a triple $(K, \mathcal{O}, k)$ is so-called a p-modular system, which is big enough for all finitely many finite groups which we are looking at, including G. Namely, $\mathcal{O}$ is a complete descrete valuation ring, K is the quotient field of $\mathcal{O}$, K and $\mathcal{O}$ have characteristic zero, and k is the residue field $\mathcal{O}/\mathrm{rad}(\mathcal{O})$ of $\mathcal{O}$ such that k has characteristic p. We mean by "big enough" above that K and k are both splitting fields for the finite groups mentioned above. Let A be a block of $\mathcal{O}G$ (and sometimes of kG) with a defect group P. Then, we denote by $e = e(A)$ the (so-called) *inertial index* of A, that is, e is defined as $e := |N_G(P, a)/P{\cdot}C_G(a)|$, where a is a root of A, in other words, a is a p-block of $P{\cdot}C_G(P)$ such that the block induction a^G is defined and it is equal to A, see [13, Chap.5, p.348]. It is noted that for the case where P is a Sylow p-subgroup of G, then $e = e(A) = |N_G(P)/P{\cdot}C_G(P)|$. We denote by mod-$\mathcal{O}G$ and by mod-A the categories of finitely generated right $\mathcal{O}G$-lattices and finitely generated right $\mathcal{O}G$-lattices belonging to A, respectively. We write $B_0(\mathcal{O}G)$ for the principal block algebra of $\mathcal{O}G$. We denote by C_n a cyclic group of order n for a positive integer n. For notation and terminology we shall not explain precisely, see the books of [13].

Setup 1.2. Throughout this section our situation is the following: Namely, $\tilde{G}$ and $\tilde{H}$ are finite groups which have a common p-subgroup $\tilde{P}$, and hence $\tilde{P} \subseteq \tilde{G} \cap \tilde{H}$. Assume that G is a normal subgroup of $\tilde{G}$ and H is a normal subgroup of $\tilde{H}$ such that G and H have a common p-subgroup P, and hence $P \subseteq G \cap H$, and moreover that $\tilde{G}/G \cong \tilde{H}/H$. We are interested in a question/problem such as *lifting* some relations that happen downstairs between G and H to those upstairs between $\tilde{G}$ and $\tilde{H}$. The author believes that this has to be a quite natual and interesting (and even fundamental) question/problem. Let us look at the situation more closely. If the factor groups $\tilde{G}/G$ and $\tilde{H}/H$ (which are isomorphic as the above) are p'-groups, then we have a well-known theory, so-called "Clifford Theory". Thus, roughly speaking, there may be a big chance to be able to lift the relations happening downstairs to upstairs.

Hence, we might be interested in the other cases. Namely, we may want to look at the cases where the indices $|\tilde{G}/G| = |\tilde{H}/H|$ are divisible by p. So, as a first step, looking at the case where $|\tilde{G}/G| = |\tilde{H}/H|$ is just p, should be a nice starting point, from the author's point of view. Therefore, from now on, we assume this. Namely, $\tilde{G}/G \cong \tilde{H}/H \cong C_p$.

Questions 1.3. Our main concern in this short note is the following:

If there is a kind of *nice* equivalence between mod-kG and mod-kH, can we lift it to a *nice* equivalence between mod-$k\tilde{G}$ and mod-$k\tilde{H}$?

More exactly, we should say the following: Let A be a block algebra of $\mathcal{O}G$ which is $\tilde{G}$-stable (invariant) (and hence there is a unique block algebra $\tilde{A}$ of $\mathcal{O}\tilde{G}$ which covers A since the factor group $\tilde{G}/G$ is a p-group). Similarly, let B be a block algebra of $\mathcal{O}H$ which is $\tilde{H}$-stable

(invariant) and is covered by a unique block algebra $\tilde{B}$ of $\mathcal{O}\tilde{H}$. In addition, we assume that A and B have a common defect group P, and that $\tilde{A}$ and $\tilde{B}$ have a common defect group $\tilde{P}$ such that P is normal in $\tilde{P}$ with $\tilde{P}/P \cong \tilde{G}/G \cong \tilde{H}/H \cong C_p$.
(*) If there is a kind of *nice* equivalence between mod-A and mod-B, can we lift it to a kind of *nice* equivalence between mod-$\tilde{A}$ and mod-$\tilde{B}$?

1.4.Theorem (Holloway-Koshitani-Kunugi [8]). *We keep the notation G, $\tilde{G}$, H, $\tilde{H}$, A, $\tilde{A}$, B, $\tilde{B}$ just as in* **1.3**. *In addition, we assume that, first of all, $\tilde{H} = N_{\tilde{G}}(\tilde{P})$ and that $H = N_G(P) = \tilde{H} \cap G$, and also that P is a cyclic* **Sylow** *p-subgroup of order p^n for an integer $n \geqslant 2$ (that is, A and $\tilde{A}$ are both* **full defect** *blocks), and that $\tilde{P} = P \rtimes C_p \cong M_{n+1}(p)$, which is a non-abelian metacyclic p-group that has a cyclic subgroup of index p, see* [6, p.190]. *Since the defect group P of A and B is cyclic, it is well-known that A and B are splendid Rickard equivalent, so that in particular there is a perfect isometry $I : \mathbb{Z}\mathrm{Irr}(A) \to \mathbb{Z}\mathrm{Irr}(B)$ between A and B.*

Then, there is an isometry

$$\tilde{I} : \mathbb{Z}\mathrm{Irr}(\tilde{A}) \to \mathbb{Z}\mathrm{Irr}(\tilde{B})$$

between $\tilde{A}$ and $\tilde{B}$ such that $\tilde{I}$ satisfies Separability Condition (2) *in* **2.1**, *and that $\tilde{I}$ preserves heights of irreducible ordinary charcters. Furthermore, we know that $k_0(\tilde{A}) = pe + p(p^{n-1} - 1)/e$, $k_1(\tilde{A}) = p^{n-2}(p-1)/e$, $k(\tilde{A}) = pe + (p^n + p^{n-1} - p^{n-2} - p)/e$, and $\ell(\tilde{A}) = e$, where $k_i(\tilde{A})$ is the number of all elements in $\mathrm{Irr}(\tilde{A})$ whose heights are i, and e is the inertial index of $\tilde{A}$, and it turns out that a result of Hendren* [7, Theorem 5.21] *is generalized in a sense.*

1.5.Remark. Of course in **1.4.**Theorem one might expect that the isometry $\tilde{I}$ between $\tilde{A}$ and $\tilde{B}$ should be *perfect*. That is, Condition(3) in **2.1.**Definition is missing in **1.4.**Theorem above, unfortunately.

1.6.Remark. The above result **1.4.**Theorem is a partial answer to Rouquier's Conjecture, though the result is just in a very specific situation. For more precise and detailed explanation on *Rouquier's Conjecture*, see [8, Conjecture 4.1].

Speaking of lifting an equivalence between two block algebras, the following two examples also might be interesting at least for the author. Actually, much more general statements (claims) are proved such as for an arbitrary prime and much bigger defect groups.

1.7.Example. Assume that $p = 3$, $G = \mathrm{SL}_2(4^3)$, $A = B_0(\mathcal{O}G)$, P is a Sylow 3-subgroup of G, $H = N_G(P)$, and $B = B_0(\mathcal{O}H)$. Moreover, set $Q = \mathrm{Gal}(4^3/4) \cong C_3$ where 4^3 and 4 respectively are finite fields of 64 elements and 4 elements, $\tilde{G} = G \rtimes Q$ where Q acts on G canonically, $\tilde{P} = P \rtimes Q$ and finally $\tilde{H} = N_{\tilde{G}}(\tilde{P})$. Then, we have the following:

(i) Downstairs between A and B

(1) $P = C_9$ and $H = C_{63} \rtimes C_2 = (P \rtimes C_2)C_G(P)$.

(2) The block algebras A and B have the **same** Brauer trees

$m = 4$

with multiplicity $m = 4$. Actually, there exists a Puig equivalence

$$(\mathcal{E}) \ : \ \text{mod-}A \xrightarrow{\approx} \text{mod-}B.$$

between A and B. Recall that a Puig equivalence is stronger than a Morita equivalence.

(ii) Upstairs between $\tilde{A}$ and $\tilde{B}$

(1) $\tilde{P} \cong M_3(3)$, the extra-special group of order $27 = 3^3$ with exponent $9 = 3^2$, and $\tilde{H} = (\tilde{P} \rtimes C_2)\cdot C_{\tilde{G}}(\tilde{P})$.

(2) The Puig equivalence $(\mathcal{E})$ occurring between A and B lifts to a Puig equivalnce

$$(\tilde{\mathcal{E}}) \ : \ \text{mod-}\tilde{A} \xrightarrow{\approx} \text{mod-}\tilde{B}.$$

between $\tilde{A}$ and $\tilde{B}$.

1.8.Remark. **1.7.Example** was motivated by a result in a Master Thesis written by Maeda [12].

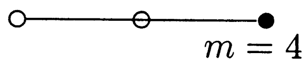
Now, let us go to a second example, which is a similar case as in **1.7.Example** in some sense, but on the other hand it is much different from **1.7.Example** if we look at them and compare those carefully.

1.9.Example (Holloway-Koshitani-Kunugi [8, Example 4.3] and Maeda [12]). Assume that $p = 3$, $G = \mathrm{SL}_2(2^3)$, $A = B_0(\mathcal{O}G)$, P is a Sylow 3-subgroup of G, $H = N_G(P)$, and $B = B_0(\mathcal{O}H)$. Moreover, set $Q = \mathrm{Gal}(2^3/2) \cong C_3$ where 2^3 and 2 respectively are finite fields of 8 elements and 2 elements, $\tilde{G} = G \rtimes Q$ where Q acts on G canonically, $\tilde{P} = P \rtimes Q$ and finally $\tilde{H} = N_{\tilde{G}}(\tilde{P})$. Then, we have the following:

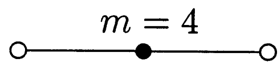
(i) Downstairs between A and B

(1) $P = C_9$ and $H = P \rtimes C_2$.

(2) The block algebras A and B respectively have the following different Brauer trees



A



B

with multiplicity $m = 4$. It is well-known that there is a derived equivalence (a splendid Rickard equivalence) between A and B, see J. Rickard [14]. In fact, hence, there exists a perfect isometry

$$I \ : \ \mathbb{Z}\mathrm{Irr}(A) \longrightarrow \mathbb{Z}\mathrm{Irr}(B)$$

between A and B.

(ii) Upstairs between $\tilde{A}$ and $\tilde{B}$

(1) $\tilde{P} \cong M_3(3)$, the extra-special group of order $27 = 3^3$ with exponent $9 = 3^2$, and $\tilde{H} = \tilde{P} \rtimes C_2$.

(2) The perfect isometry I occurring between A and B lifts to a perfect isometry

$$\tilde{I} \ : \ \mathbb{Z}\mathrm{Irr}(\tilde{A}) \longrightarrow \mathbb{Z}\mathrm{Irr}(\tilde{B})$$

between $\tilde{A}$ and $\tilde{B}$.

1.10.Remark. As it has already been mentioned above, these two exmaples **1.7** and **1.9** can be discussed in much more general situation. They will be presented in a coming paper, hopefully.

2. Appendix on perfect isometries by Masao Kiyota

College of Liberal Arts and Sciences
Tokyo Medical and Dental University
東京医科歯科大学 教養部
Masao Kiyota 清田正夫

In §2, which is an appendix, we shall give a result obtained around mid 1990's. It is on *Perfect Isometries* due to M.Broué, see [3, 1.4.Définition]. First, we shall recall a definition of *Perfect Isometries*. As you can see below our result is useful and convenient when we want to check the integrality condition once we have been able to check the separability condition.

2.1.Definition [3, 1.4.Définition]. Let G and H be finite groups. Let $(K, \mathcal{O}, k)$ be a p-modular system which is big enough, see **1.1.Notation**. For ordinary characters χ and ψ of G we denote by $(\chi, \psi)_G$ the inner product of χ and ψ in G. Let e and f respectively be (non-zero) idempotents in $\mathcal{O}G$ and $\mathcal{O}H$. We write $\mathrm{Irr}(G, e) = \mathrm{Irr}_K(G, e)$ for the set of all irreducible ordinary (K-) characters of G which are not killed by e. Thus we denote by $\mathbb{Z}\mathrm{Irr}(G, e)$ the set of all generalized (virtual) characters of G which are not killed by e. We say that an element $g \in G$ is p-regular if g is a p'-element, namely if $p \nmid |g|$; and we say that an element $g \in G$ is p-singular if $p \,\Big|\, |g|$. We use a notation $(\alpha, \beta)'_G$ which is defined by

$$(\alpha, \beta)'_G = \frac{1}{|G|} \sum_{g \in G_{p'}} \alpha(g)\beta(g^{-1})$$

for K-valued class functions α, β on $G_{p'}$, where $G_{p'}$ is the set of all p'-elements of G, see [13, Chap.3, p.237].

Now, we say that there exists a *perfect isometry* I from $\mathcal{O}Ge$ to $\mathcal{O}Hf$ if I is a bijective $\mathbb{Z}$-linear map

$$I \;:\; \mathbb{Z}\mathrm{Irr}(G, e) \longrightarrow \mathbb{Z}\mathrm{Irr}(H, f)$$

which satisfies the next conditions:

(1) I is an isometry, namely, $(\chi,\chi)_G = \Big(I(\chi), I(\chi)\Big)_H$ for any $\chi \in \mathrm{Irr}(G,e)$, and hence $I(\chi)$ or $-I(\chi)$ is in $\mathrm{Irr}(H)$ for any χ. Set $\mu := \mu_I : \; G \times H \longrightarrow K$ which is defined by

$$\mu(g,h) := \sum_{\chi \in \mathrm{Irr}(G,e)} \chi(g)\cdot\Big(I(\chi)\Big)(h^{-1}).$$

Then μ satisfies the following two conditions.

(2) (Separation Condition)
If $\mu(g,h) \neq 0$ for $g \in G$ and $h \in H$, then "$p \nmid |g|$ and $p \nmid |h|$" or "$p \mid |g|$ and $p \mid |h|$".

(3) (Integrality Condition)
μ is the same as in (1). For any $g \in G$ and $h \in H$, it holds

$$\frac{\mu(g,h)}{|C_G(g)|} \in \mathcal{O} \quad \text{and} \quad \frac{\mu(g,h)}{|C_H(h)|} \in \mathcal{O}.$$

2.2.Theorem (Kiyota, around 1995, see Kiyota [11, Remark 1.3]). *We keep the notation given in* **2.1.Definition**. *Assume that μ satisfies* (2)Separability Condition. *Then, in order to check* (3)Integrality Condition, *it is sufficient to check* (3) *only for any p-singular elements $g \in G$ and $h \in H$, namely, for any $g \in G$ with $p \mid |g|$ and any $h \in H$ with $p \mid |h|$.*

Proof. It is enough to check Condition(3) for a p'-element $g \in G$ and a p'-element $h \in H$.

Fix a p'-element $h \in H$. Define a function $\psi : G \to K$ by $\psi(g) := \mu(g,h)$. Clearly, ψ is a K-valued class function on G. Moreover, Separability Condition(2) implies that $\psi(g) = \mu(g,h) = 0$ for any p-singular element $g \in G$. Thus, it follows by e.g. [13, Chap.3, Theorem 6.15(i)] that we can write

$$\psi = \sum_{i=1}^{\ell} c_i \Phi_i \qquad \text{for elements } c_i \in K,$$

where $\Phi_1, \cdots, \Phi_\ell$ are all $\mathcal{O}$-characters of G induced by projective indecomposable $\mathcal{O}G$-modules. Since $\Big(I(\chi)\Big)(h^{-1}) \in \mathcal{O}$ by [13, Chap.3, p.189], it holds that ψ is an $\mathcal{O}$-linear combination of elements in $\mathrm{Irr}(G,e)$, that is, we can write

$$\psi = \sum_{\chi\in\mathrm{Irr}(G,e)} a_\chi \chi \qquad \text{for } a_\chi \in \mathcal{O}.$$

Now, in general as is well-known (due to R.Brauer), for any irreducible Brauer character $\varphi_i \in \mathrm{IBr}(G,e)$, define a function $\theta_i : G \to K$ by $\theta_i(g) := \varphi_i(g_{p'})$ for any $g \in G$, where $g_{p'}$ is the p'-part of g. Then, by [13, Chap.3, Lemma 6.13], we have

$$\theta_i = \sum_{\chi\in\mathrm{Irr}(G)} m_\chi^{(i)}\cdot\chi \qquad \text{for } m_\chi^{(i)} \in \mathbb{Z}.$$

Then,

$$\begin{aligned}(\psi,\varphi_i)'_G &= (\psi,\theta_i)_G \qquad\qquad \text{since } \psi(g)=0 \ \text{ for } g\in G-G_{p'}\\ &= \Big(\psi, \sum_{\chi\in\mathrm{Irr}(G)} m_\chi^{(i)}\cdot\chi\Big)_G\\ &= \Big(\sum_{\chi\in\mathrm{Irr}(G,e)} a_\chi\chi, \sum_{\chi\in\mathrm{Irr}(G)} m_\chi^{(i)}\cdot\chi\Big)_G\\ &= \sum_\chi a_\chi m_\chi^{(i)} \quad \in \mathcal{O}\end{aligned}$$

since $a_\chi \in \mathcal{O}$ and $m_\chi^{(i)} \in \mathbb{Z}$. This means that $(\psi,\varphi_i)'_G \in \mathcal{O}$.

On the other hand, since $(\Phi_j,\varphi_i)'_G = \delta_{ji}$ (Kronecker's delta) by [13, Chap.3, Theorem 6.10(i)], it holds that

$$(\psi,\varphi_i)'_G = \Big(\sum_{j=1}^{\ell} c_j\Phi_j, \varphi_i\Big)'_G = \sum_j c_j(\Phi_j,\varphi_i)'_G = c_i.$$

These yield that $c_i \in \mathcal{O}$ for any i.

Now, take any p'-element $g \in G$. Then, recall that

$$\frac{\Phi_i(g)}{|C_G(g)|} \in \mathcal{O} \quad \text{for any } i$$

by [13, Chap.3, Theorem 6.10(ii)]. Hence, it follows that

$$\frac{\mu(g,h)}{|C_G(g)|} = \frac{\psi(g)}{|C_G(g)|} = \frac{\sum_i c_i\Phi_i(g)}{|C_G(g)|} = \sum_i c_i\cdot\frac{\Phi_i(g)}{|C_G(g)|} \in \mathcal{O}.$$

We are done. ■.

3. Appendix on blocks with elementary abelian defect group of order 9 by Atumi Watanabe

Department of Mathematics
Graduate School of Science and Technology
Faculty of Science
Kumamoto University
熊本大学 大学院自然科学研究科
Atumi Watanabe 渡辺 アツミ

In §3, which is an appendix, we shall give a result obtained around early 1980's. It is on 3-blocks of finite groups with an elementary abelian defect group of order 9 where we treated with the case that the inertial quotient of the 3-block is a semi-dihedral group of order 16, which was not completed in a paper of Kiyota [9].

In a paper of Kiyota [9] he proves that, if B is an arbitrary 3-block with an elementary abelian defect group D of order 9, then he completely determines the numbers $k(B)$ of irreducible ordinary characters and $\ell(B)$ of irreducible Brauer characters, for almost of all cases, except the cases where the inertial quotient $E(B)$ is a cyclic group of order 8, is a quaternion group of order 8, and is a semi-dihedral group of order 16. Actually, in [9, a footnote on page 34] Kiyota says that "After this paper was written, A.Watanabe proved that in case $e(B) = 16$ the values of $k(B)$ and $\ell(B)$ in the table 1 are true for any B."

3.1.Notation/Definition. Throughout this section we use the following notation. Actually in principle and essentially we follow Kiyota's paper [9] as long as possible.

Here G is always a finite group and p is a prime number. We denote by a triple $(K, \mathcal{O}, k)$ a p-modular system which is big enough. Namely, $\mathcal{O}$ is a complete descrete valuation ring, K is the quotient field of $\mathcal{O}$, K and $\mathcal{O}$ have characteristic zero, and k is the residue field $\mathcal{O}/\mathrm{rad}(\mathcal{O})$ of $\mathcal{O}$ such that k has characteristic p, and K and k are splitting fields for all the finite groups we are dealing with. For $g, h \in G$ we define

$g^h := h^{-1}gh$. For subsets X and Y of G we write $Y \subseteq_G X$ if there is an element $g \in G$ such that $g^{-1}Yg \subseteq X$.

Let B be a p-block of G with a defect group D. We write $1_B := e_B$ for the block idempotent of B in kG. We denote by $\mathrm{Irr}(B)$ and $\mathrm{IBr}(B)$, respectively, the sets of all irreducible *ordinary* and *Brauer* characters of G belonging to B. We write $k(B)$ and $\ell(B)$ for the numbers of these sets, respectively, that is to say, $k(B) = |\mathrm{Irr}(B)|$ and $\ell(B) = |\mathrm{IBr}(B)|$. We let b be a root of B in $D{\cdot}C_G(D)$, namely, b is a p-block of $D{\cdot}C_G(D)$ with $b^G = B$ (block induction in the sense of R.Brauer). We set $T(b) := N_G(D,b) := \{g \in N_G(D) \mid b^g = b\}$, that is, $T(b) = N_G(D,b)$ is the inertial group of b in $N_G(D)$. Then, we set $E(B) := T(b)/D{\cdot}C_G(D) = N_G(D,b)/D{\cdot}C_G(D)$, and $e(B) := |E(B)|$. We call $E(B)$ and $e(B)$, respectively, the inertial quotient and the inertial index of B. Recall that $E(B)$ is a subquotient p'-group of $\mathrm{Aut}(D)$. We write $\mathrm{Cl}(G)$ and $\mathrm{Cl}(G_{p'})$ respectively for the sets of all *conjugacy* and *p'-conjugacy* classes of G. For a p-subgroup Q of G we denote by $\mathrm{Cl}(G|Q)$ and $\mathrm{Cl}(G_{p'}|Q)$ respectively for the sets of all *conjugacy* and *p'-conjugacy* classes of G that have Q as their defect group. For $C \in \mathrm{Cl}(G)$, we define $\widehat{C}$ by $\widehat{C} := \sum_{g \in C} g \in kG$. We write C_n for the cyclic group of order n for a positive integer n.

The following lemma is probably well-known. But actually it is useful to get our main result in §3.

3.2.Lemma. *Let Q be a normal p-subgroup of G, and set $\bar{G} := G/Q$. Assume that B is a p-block of G with a defect group D, and hence $Q \subseteq D$. Set $\bar{D} := D/Q$, $N := N_G(D)$ and $\bar{N} := N/Q$, and hence $\bar{N} = N_{\bar{G}}(\bar{D})$. Define a k-algebra-epimorphism*

$$\mu_Q^G \,:\, kG \twoheadrightarrow k\bar{G}$$

which is induced by the canonical group-epimorphism $G \twoheadrightarrow \bar{G}$. Similarly, we define μ_Q^N. Let Br_D^G be the (usual) Brauer homomorphism with respect to $(G, D, N_G(D))$. Namely,

$$\mathrm{Br}_D^G \,:\, Z(kG) \to Z(kN)$$

and similar for $\mathrm{Br}_{\bar{D}}^{\bar{G}}$. Then, we get the following:

(i) $\mu_Q^N \circ \mathrm{Br}_D^G(1_B) = \mathrm{Br}_{\bar{D}}^{\bar{G}} \circ \mu_Q^G(1_B)$.

(ii) *Let* $\{\bar{B}_1, \cdots, \bar{B}_m\}$ *be the set of all* p*-blocks of* $\bar{G}$ *which are dominated by* B *for an integer* m. *Suppose that* $\bar{B}_i$ *has* $\bar{D}$ *as its defect group for any* $i = 1, \cdots, m$. *Hence, we can define the Brauer correspondents* $\bar{b}_i$ *of* $\bar{B}_i$ *for each* i. *That is,* $1_{\bar{b}_i} = \mathrm{Br}_{\bar{D}}^{\bar{G}}(1_{\bar{B}_i})$ *for* $i = 1, \cdots, m$. *Then, it holds that* $\{\bar{b}_1, \cdots, \bar{b}_m\}$ *is the set of all* p*-blocks of* $\bar{N}$ *wihch are dominated by* β*, where* β *is the Brauer correspondent of* B *in* N*, that is,* $1_\beta = \mathrm{Br}_D^G(1_B)$.

Namely, the following diagram is commutative:

$$\begin{array}{ccc} kG & \xrightarrow{\mu_Q^G} & k\bar{G} \\ {\scriptstyle \mathrm{Br}_D^G}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Br}_{\bar{D}}^{\bar{G}}} \\ kN & \xrightarrow[\mu_Q^N]{} & k\bar{N} \end{array}$$

Proof. (i) For $C \in \mathrm{Cl}(G)$ we denote by $\bar{C}$ a conjugacy class of $\bar{G}$ such that $gQ \in \bar{C}$ for $g \in C$. Now, take $C \in \mathrm{Cl}(G_{p'}|D)$ such that $C \subseteq C_G(Q)$. Then, it follows from [13, Chap.5, Lemmas 2.14 and 8.9(ii)] that $\bar{C} \in \mathrm{Cl}(\bar{G}_{p'}|\bar{D})$, so that $\bar{C} \cap C_{\bar{G}}(\bar{D}) \in \mathrm{Cl}(\bar{N}_{p'}|\bar{D})$. Clearly, $\emptyset \neq \overline{C \cap C_G(D)} \subseteq \bar{C} \cap C_{\bar{G}}(\bar{D})$, and hence $\overline{C \cap C_G(D)} = \bar{C} \cap C_{\bar{G}}(\bar{D})$ since both sets are conjugacy classes of $\bar{N}$. This yields that $\mu_Q^N \circ \mathrm{Br}_D^G(\widehat{C}) = \mathrm{Br}_{\bar{D}}^{\bar{G}} \circ \mu_Q^G(\widehat{C})$ by [13, Chap.5, Theorem 3.5(i), Lemmas 2.14 and 8.9(ii)–(iii)]. Now it is well-known that the block idempotent 1_B can be written

$$1_B \;=\; \sum_C \alpha_C \widehat{C} \quad \text{for } \alpha_C \in k$$

where C runs through all p'-conjugacy classes of G such that $C \subseteq C_G(Q)$ and $\delta(C) \subseteq_G D$ where $\delta(C)$ is a defect group of C, see [13, Chap.3, Theorem 6.22] and [13, Chap.5, Lemma 1.7(iv) and Theorem 2.8(ii)]. Since $\mathrm{Br}_D^G(\widehat{C}) = 0$ if $\delta(C) \not\subseteq_G D$ (see [13, Chap.5, Exercise 2.4]), we finally get (i).

(ii) This follows by (i). ■.

3.3.Lemma. *Suppose that* B *is a* 3*-block of* B *with a defect group* $D = C_3 \times C_3$ *such that* B *is of type* E_4*, namely,* $E(B) \cong C_2 \times C_2$.

In addition, let b be a root of B, namely, b is a 3-block of $C_G(D)$ with $b^G = B$ (block induction). Then, it holds that

$$\ell(B) = \ell(b^{T(b)}) = \ell(b^{N_G(D)}).$$

Proof. Let x and y be generators of D, namely, $D = \langle x \rangle \times \langle y \rangle \cong C_3 \times C_3$. As in [9, line -4, p.38] we can assume that the $T(b)$-orbits of D are

(1) $\{1\},\ \{x, x^{-1}\},\ \{y, y^{-1}\},\ \{xy, xy^{-1}, x^{-1}y, x^{-1}y^{-1}\}.$

Let $C^*_G(x)$ be the extended centralizer of x in G and set $L := C^*_G(x)$. That is,

$$L \ := \ C^*_G(x) \ := \ \{g \in G \,|\, x^g = x \text{ or } x^g = x^{-1}\}.$$

We can define two block inductions $b_x := b^{C_G(x)}$ and $b^*_x := b^L = (b_x)^L$ since $C_G(D) \subseteq C_G(x) \subseteq L$. Clearly, b^*_x has D as its defect group and b is a root of b^*_x in $C_L(D)$, so that $N_L(D, b) = T(b) \cap L = T(b) = N_G(D, b)$. Thus, $E(b^*_x) = N_L(D, b)/C_L(b) = E(B) \cong C_2 \times C_2$. This means that b^*_x has the same defect group D, the same root b as B, and that b^*_x is of type E_4. It follows from [9, lines $5 \sim 7$, p.39] and [9, Proposition (2E)] that

(2) $\ell(B) \ = \ \ell(b^*_x).$

Let c^*_x be the Brauer correspondent of b^*_x in $N_L(D)$, and set $\bar{L} := L/\langle x \rangle$. In addition, let $\{\bar{b}_1, \cdots, \bar{b}_m\}$ be the set of all 3-blocks of $\bar{L}$ which are dominated by b^*_x for some integer m. Set $\bar{D} := D/\langle x \rangle$, and hence $\bar{D} \cong C_3$.

Next, we claim that $\bar{b}_i$ has $\bar{D}$ as its defect group for any i. Suppose that $\bar{D}$ is not a defect group of $\bar{b}_1$. Then, by [13, Chap.5, Theorem 8.7(ii)], $\bar{b}_1$ has defect zero, so that we can set $\mathrm{Irr}(\bar{b}_1) =: \{\bar{\chi}\} \subseteq \mathrm{Irr}(b^*_x)$. Since $\bar{b}_1$ has defect zero, $\nu_3(\bar{\chi}(1)) = \nu_3(|L/\langle x \rangle|) = \nu_3(|L|) - 1 = \nu_3(|L|) - 2 + 1$. This means that $\bar{\chi}$ has height one as an irreducible character of L in b^*_x. On the other hand, since b^*_x has defect two, it follows from a result of Brauer-Feit [5, IV Theorem 4.18] that every irreducible ordinary character in b^*_x has height zero, a contradiction.

Thus, $\bar{b}_1, \cdots, \bar{b}_m$ all have $\bar{D}$ as their defect group. Hence we can define their Brauer correspondents with respect to $(\bar{L}, \bar{D}, \bar{N})$, where $\bar{N} := N_L(D)/\langle x\rangle = N_{\bar{L}}(\bar{D})$. So, let $\bar{c}_i$ be the Brauer correspondent of $\bar{b}_i$ in $\bar{N}$, namely, $\bar{c}_i$ is a 3-block of $\bar{N}$ with a defect group $\bar{D}$. Let c_x^* be the Brauer correspondent of b_x^* in $N_L(D)$, that is, c_x^* is a 3-block of $N_L(D)$. Then, it follows from **3.2.Lemma** that $\bar{c}_1, \cdots, \bar{c}_m$ are all 3-blocks of $\bar{N}$ that are dominated by c_x^*. Obviously, $\bar{b}_i$ and $\bar{c}_i$ have the same defect group $\bar{D} \cong C_3$, and they have the same inertial quotient. Hence we know by a result of Dade [4] that

$$\ell(\bar{b}_i) \;=\; \ell(\bar{c}_i) \quad \text{for } i = 1, \cdots, m. \tag{3}$$

Hence,

$$\ell(b_x^*) \;=\; \sum_{i=1}^{m} \ell(\bar{b}_i) \;=\; \sum_{i=1}^{m} \ell(\bar{c}_i) \;=\; \ell(c_x^*) \tag{4}$$

by the definition of "being dominated". Recall that the 3-blocks c_x^* and $b^{T(b)}$ have the same root b, and that $T(b) \subseteq N_L(D)$. Hence it follows from results of Clifford and of Fong-Reynolds [13, Chap.5, Theorem 5.10] that $\ell(c_x^*) = \ell(b^{T(b)}) = \ell(b^{N_G(D)})$. Therefore, we finally have

$$\ell(B) \;=\; \ell(b_x^*) \;=\; \ell(c_x^*) \;=\; \ell(b^{T(b)}) \;=\; \ell(b^{N_G(D)}).$$

This completes the proof. ■

3.4.Theorem (A.Watanabe, 1984). *Suppose that B is an arbitrary 3-block of G with a defect group $D \cong C_3 \times C_3$ such that the inertial quotient $E(B)$ of B is the semi-dihedral group SD_{16} of order 16. Then it holds that $k(B) = 9$ and $\ell(B) = 7$.*

Proof. Let x and y be generators of D, that is, $D := \langle x\rangle \times \langle y\rangle \cong C_3 \times C_3$. Let b be a root of B in $C_G(D)$. Then, we can write

$$E := E(B) \;:=\; T(b)/C_G(D) \;=\; \langle \sigma, \tau \mid \sigma^8 = \tau^2 = 1,\ \tau\sigma\tau = \sigma^3\rangle.$$

In fact, we can set

$$\sigma \;:=\; \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \tau \;:=\; \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

in $\mathrm{GL}_2(\mathbb{F}_3)$ such that the actions of σ and τ on D are given by

$$x^\sigma = xy^{-1}, \quad y^\sigma = xy, \quad x^\tau = x^{-1}, \quad y^\tau = y.$$

Clearly, E-orbits of D are $\{1\}$ and $D - \{1\}$. Set $b_x := b^{C_G(x)}$ (block induction). Then, by a result of Brauer [13, Chap.5, Theorems 9.4 and 9.10],

$$k(B) \ = \ \ell(B) + \ell(b_x).$$

Now, we want to claim that $\ell(b_x) = 2$. We easily know that

$$(T(b) \cap C_G(x))/C_G(D) = \left\langle \sigma^4\tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle \cong C_2.$$

Hence we get by [9, Propositions (2B) and (2C)] that $\ell(b_x) = 2$. Thus,

$$k(B) \ = \ \ell(B) + 2. \tag{5}$$

Now, let $L := C^*_G(x)$ be the extended centralizer of x in G, see the proof of **3.3.Lemma**. Obviously, $|L : C_G(x)| = 2$ since $\sigma^4 \in L - C_G(x)$, so that $L = N_G(\langle x \rangle)$ since $\mathrm{Aut}(\langle x \rangle) \cong C_2$. Then, we can define a block induction $b^*_x := b^L$, so that b^*_x is a *unique* 3-block of L covering b_x. Clearly, D is a defect group of b^*_x, and b is a root of b^*_x in $C_L(D)$. We easily know that $(T(b) \cap L)/C_G(D) = E(b^*_x) = \langle \sigma^4 \rangle \times \langle \tau \rangle \ \cong \ C_2 \times C_2$, so that b^*_x is of type E_4. Hence, it follows from [9, Proposition (2E)] that $\ell(b^*_x) = 4$ or 1. Now, let $\lambda \in \mathrm{Irr}(b)$ be the canonical character of B, and hence it is the canonical characater of b_x and b^*_x, too. Since $\mathrm{H}^2(SD_{16}, \mathbb{C}^\times) = 1$ (see [9, Proof of Corollary (2J)]) and since $E \cong SD_{16}$, we know that λ extends to $T(b)$, and hence to $T(b) \cap L$. Now, we can set $\mathrm{IBr}(b_x) := \{\varphi^x_1, \varphi^x_2\}$. Clearly, we can define $b^{T(b)\cap L}$ (block induction).

Note that λ is irreducible even as a Brauer character of $C_G(D)$, see [13, Chap.5, line 15 p.365]. Namely, we can consider $\lambda \in \mathrm{IBr}(b)$. That is, $\lambda \in \mathrm{IBr}(b)$ extends to $T(b) \cap L$. Since $(T(b) \cap L)/C_G(D) \cong C_2 \times C_2$ which is a $3'$-group, we know that $\ell(b^{T(b)\cap L}) = |(T(b) \cap L)/C_G(D)| = 4$, see [1, Theorem 15.1(1),(5), p.106] and [10, (6.17)Corollary].

Hence, by **3.3.Lemma**, we have $\ell(b^*_x) = \ell(b^{T(b)\cap L}) = 4$. This means that *Subcase*(a) in [9, p.39] occurs, see [9, Proposition (2E)]. This yields that the extended centralizer L of x in G fixes both of φ_1^x and φ_2^x by the action of conjugation. This implies

$$d^{x^{-1}}_{\chi,\varphi_j^x} = d^x_{\chi,\varphi_j^x} \text{ for } j = 1,2 \text{ and for any } \chi \in \mathrm{Irr}(B),$$

where d^x_{χ,φ_j^x} and $d^{x^{-1}}_{\chi,\varphi_j^x}$ are the generalized 3-decomposition numbers with respect to x and x^{-1}, respectively (note $C_G(x^{-1}) = C_G(x)$, so that it makes sense). In genenral, we know

$$d^{x^{-1}}_{\chi,\varphi_j^x} = \overline{d^x_{\chi,\varphi_j^x}} \quad \text{(complex conjugate)}$$

by the definition of generalized decomposition numbers. Thus, we have

$$(6) \qquad d^x_{\chi,\varphi_j^x} \in \mathbb{Z} \text{ for } j = 1,2 \text{ and for any } \chi \in \mathrm{Irr}(B),$$

see [9, line 7, p.39]. Now, let $\bar{b}_x$ be a unique block of $C_G(x)/\langle x\rangle$ dominated by b_x, see [13, Chap.5, Theorem 8.11]. Set $\bar{D} := D/\langle x\rangle \cong C_3$. We know by [13, Chap.5, Theorem 8.10] that $\bar{D}$ is a defect group of $\bar{b}_x$. Obviously, $\ell(\bar{b}_x) = \ell(b_x) = 2$. Hence a result of Dade [4] says that the Cartan matrix $C_{\bar{b}_x}$ of $\bar{b}_x$ is of the form $C_{\bar{b}_x} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. So that the Cartan matrix C_{b_x} of b_x is of the form $C_{b_x} = \begin{pmatrix} 6 & 3 \\ 3 & 6 \end{pmatrix}$ by [13, Chap.5, Theorem 8.11]. Then, by Brauer's 2nd main theorem [13, Chap.5, Theorem 4.2] and [13, Chap.5, Theorem 4.11], it holds that

$$(7) \qquad \sum_{\chi\in\mathrm{Irr}(B)} d^x_{\chi,\varphi_j^x} \cdot \overline{d^x_{\chi,\varphi_{j'}^x}} = \begin{cases} 6, & \text{if } j = j' \\ 3, & \text{if } j \neq j'. \end{cases}$$

On the other hand, by [9, Lemma (1D)], it holds $k(B) = 3, 6$ or 9. If $k(B) = 3$, then it follows from (1) that $\ell(B) = 1$, and hence $D \cong C_3$ by a result of Brandt [2, p.513] (see [9, Lemma (1E)]), a contradiction. This yields that $k(B) = 6$ or 9. Set $\mathbf{d}_j^x = \left(d^x_{\chi,\varphi_j^x}\right)_{\chi\in\mathrm{Irr}(B)}$ for $j = 1, 2$.

Then, it follows by elementary calculations using (6) and (7) that

$$(\mathbf{d}_1^x, \mathbf{d}_2^x) = \begin{pmatrix} \varepsilon_1 & 0 \\ \varepsilon_2 & 0 \\ \varepsilon_3 & 0 \\ \varepsilon_4 & \varepsilon_4 \\ \varepsilon_5 & \varepsilon_5 \\ \varepsilon_6 & \varepsilon_6 \\ 0 & \varepsilon_7 \\ 0 & \varepsilon_8 \\ 0 & \varepsilon_9 \end{pmatrix}, \qquad \text{where } \varepsilon_i \in \{\pm 1\}.$$

Therefore we eventually have $k(B) = 9$, so that $\ell(B) = 7$ by (5). We are done. ■

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