On the Dade -Tasaka correspondence between blocks of finite groups

渡辺アツミ (Atumi Watanabe)

熊本大学大学院自然科学研究科

Graduate School of Science and Technology, Department of Mathematics, Faculty of Science, Kumamoto University

1 Introduction

In this report we state a generalization of Tasaka's isotypy between blocks of finite groups obtained by the Dade character correspondence. Let $p$ be a prime and $(\mathcal{K}, \mathcal{O}, k)$ be a $p$-modular system such that $\mathcal{K}$ is a splitting field for all finite groups which we consider in this talk. Let $S$ denote $\mathcal{O}$ or $k$. For a finite abelian group $F$, we denote by $\bar{F}$ the character group of $F$ and by $\bar{F}_q$ the subgroup of $\bar{F}$ of order $q$ for $q \in \pi(F)$, where $\pi(F)$ is the set of all primes dividing $|F|$. Let $G$ be a finite group and $N$ be a normal subgroup of $G$. We denote by $\text{Irr}(G)$ the set of all ordinary irreducible characters of $G$ and $\text{Irr}^G(N)$ be the set of $G$-invariant irreducible characters of $N$. For $\phi \in \text{Irr}(N)$, we denote by $\text{Irr}(G|\phi)$ the set of irreducible characters $\chi$ of $G$ such that $\phi$ is a constituent of the restriction $\chi_N$ of $\chi$ to $N$.

**Hypothesis 1** $G$ is a finite group which is a normal subgroup of a finite group $E$ such that the factor group $F = E/G$ is a cyclic group of order $r$. $\lambda$ is a generator of $\bar{F}$. $E_0 = \{ x \in E \mid \bar{x}$ is a generator of $F \}$ where $\bar{x} = xG$. $E'$ is a subgroup of $E$ such that $E'G = E$, $G' = G \cap E'$ and $E_0 = E' \cap E_0$. Moreover $(E_0')^{f} \cap E_0'$ is the empty set, for all $\tau \in E - E'$.

Under the above hypothesis, in [2], E.C. Dade constructed a bijection between $\text{Irr}^E(G)$ and $\text{Irr}^{E'}(G')$ which is a generalization of the cyclic case of the Glauberman correspondence ([3] or, [6], Chap.13).

**Theorem 1** ([2], Theorem 6.8, Theorem 6.9) Assume Hypothesis 1 and $|F| \neq 1$. For each prime $q \in \pi(F)$, we choose some non-trivial character $\lambda_q \in \bar{F}_q$. There is a bijection

$$
\rho(E, G, E', G') : \text{Irr}^E(G) \rightarrow \text{Irr}^{E'}(G') \quad (\phi \mapsto \phi' = \phi_{(G')})
$$

which satisfies the following conditions. If $r$ is odd, then there are a unique integer $\epsilon_{\phi} = \pm 1$ and a unique bijection $\psi \mapsto \psi_{(E')}$ of $\text{Irr}(E|\phi)$ onto $\text{Irr}(E'|\phi')$ such that

$$(1.1) \quad \left( \prod_{q \in \pi(F)} (1 - \lambda_q) \cdot \psi \right)_{E'} = \epsilon_{\phi} \prod_{q \in \pi(F)} (1 - \lambda_q) \cdot \psi_{(E')},$$

for any $\psi \in \text{Irr}(E|\phi)$. If $r$ is even, and we choose $\epsilon_{\phi} = \pm 1$ arbitrarily, then there is a unique bijection $\psi \mapsto \psi_{(E')}$ of $\text{Irr}(E|\phi)$ onto $\text{Irr}(E'|\phi')$ such that (1.1) holds for all $\psi \in \text{Irr}(E|\phi)$. In both cases we have

$$(\lambda \psi)_{(E')} = \lambda \psi_{(E')}$$
for any \( \lambda \in \hat{F} \) and \( \psi \in \text{Irr}(E|\phi) \). Furthermore, the resulting bijection is independent of the choice of the non-trivial character \( \lambda_q \in \hat{F}_q \), for any \( q \in \pi(F) \).

Assume Hypothesis 1. We call \( \rho(E, G, E', G') \) the Dade correspondence, where \( \rho(E, G, E', G') \) denotes the identity map of \( \text{Irr}^E(G) \) when \( |F| = 1 \). Following the notations in [7], for \( \phi' \in \text{Irr}^{E'}(G) \), we set \( \phi'_G = \rho(E, G, E', G')^{-1}(\phi') \), and for \( \psi' \in \text{Irr}(E'|\phi') \), we set \( \psi'_E = \psi \) if \( \psi' = \psi|_{E'} \). From (1.1) \( \psi' \) is a constituent of \( (\lambda \psi|_{E'})^{E'} \) for some \( \lambda \in \hat{F} \), hence \( \phi'_G \) is a constituent of \( \phi_{G'} \). In particular if \( \phi \) is the trivial character of \( G \), then \( \phi_{G'} \) is the trivial character of \( G' \).

The Generalized Glauberman case: Let \( G \) and \( A \) be finite groups such that \( A \) is cyclic, \( A \) acts on \( G \) via automorphism and that \( (|C_G(A)|, |A|) = 1 \). We set \( E = G \times A \), \( G' = C_G(A) \) and \( E' = G' \times A \leq E \). By [2], Lemma 7.5, \( E, G, E' \) and \( G' \) satisfy Hypothesis 1. Moreover by [2], Proposition 7.8, if \( (|A|, |G|) = 1 \), then \( \rho(E, G, E', G') \) coincides with the Glauberman correspondence.

Theorem 2 (Horimoto[4]) Assume the generalized Glauberman case. Suppose that \( p \nmid |A| \) and that a Sylow \( p \)-subgroup of \( G \) is contained in \( G' \). Then there is an isotypy between \( b(G) \) and \( b(G') \) induced by the Dade correspondence where \( b(G) \) is the principal block of \( G \).

Isotypy is a concept introduced in [1].

Hypothesis 2 Assume Hypothesis 1. \( (p, r) = 1 \). \( b \) is an \( E \)-invariant block of \( G \) covered by \( r \) distinct blocks of \( E \).

Hypothesis 3 Assume Hypothesis 1. \( (p, r) = 1 \). \( b' \) is an \( E' \)-invariant block of \( G' \) covered by \( r \) distinct blocks of \( E' \).

Theorem 3 (Tasaka [7], Theorem 5.5) Assume Hypotheses 2 and 3, and \( r \) is a prime power. Moreover assume some \( \phi \in \text{Irr}(b) \), \( \phi_{(G')} \in \text{Irr}(b') \). If \( r \) is odd, or \( r = 2 \), or \( b \) is the principal block of \( G \), then there is an isotypy between \( b \) and \( b' \) induced by the Dade correspondence.

In this report we state that the arguments in [7] can be extended to the general case (see Theorem 8 below).

2 Dade correspondence and blocks

Let \( G \) be a finite group. We denote by \( G_0(\mathcal{K}G) \) the Grothendieck group of the group algebra \( \mathcal{K}G \). If \( L \) is a \( \mathcal{K}G \)-module, then let \([L]\) denote the element in \( G_0(\mathcal{K}G) \) determined by the isomorphism class of \( L \). For \( \phi \in \text{Irr}(G) \), we denote by \( \check{\phi} \). For a block \( b \) of \( G \), we denote by \( \text{Irr}(b) \) the set of irreducible characters belonging to \( b \), and by \( \mathcal{R}_\mathcal{K}(G, b) \) the additive group of generalized characters belonging to \( b \). For other notations, see [5] and [8].

Note that under the Hypothesis 2, any irreducible character in \( \text{Irr}(b) \) is \( E \)-invariant.
Theorem 4 (see [7], Proposition 3.5)  
(i) Assume Hypothesis 2. Then \( \{ \phi_{(G')} | \phi \in \text{Irr}(b) \} \) is contained in a block \( b_{(G')} \) of \( G' \). 
(ii) Assume Hypothesis 3. Then \( \{ \phi_{(G)} | \phi \in \text{Irr}(b') \} \) is contained in a block \( b'_{(G)} \) of \( G \).

Assume Hypothesis 2. We denote by \( \hat{b}_0 \) a block of \( E \) covering \( b \). For each \( \phi \in \text{Irr}(b) \), we denote \( \hat{\phi} \) by a unique extension of \( \phi \) which belongs to \( \hat{b}_0 \). For any \( i \in \mathbb{Z} \), we denote by \( \hat{b}_i \) the block of \( E \) which contains \( \lambda^i \hat{\phi} \) where \( \phi \in \text{Irr}(b) \).

Proposition 1 (see [7], Proposition 3.5, (3)) Assume Hypotheses 2 and 3, and assume \( b' = b_{(G')} \) using the notation in Theorem 4. Then there exists a block \( (\hat{b}_0)_{(E')} \) of \( E' \) such that \( \text{Irr}((\hat{b}_0)_{(E')}) = \{ (\hat{\phi})_{(E')} | \phi \in \text{Irr}(b) \} \). If \( r \) is odd, then \( (\hat{b}_0)_{(E')} \) is uniquely determined, and if \( r \) is even, we have exactly two choices for \( (\hat{b}_0)_{(E')} \).

With the notation in the above proposition, we denote by \( (\hat{b}_1)_{(E')} \) the block of \( E' \) containing \( \lambda^1(\hat{\phi})_{(E')} \) ( \( \phi \in \text{Irr}(b) \) ). Moreover, when \( r \) is even, we fix one of two \( (\hat{b}_0)_{(E')} \).

3 Local structure

Lemma 1 ([7], Lemma 3.3)) Assume \( p \nmid r \). For a block \( b \) of \( G \), \( b \) satisfies Hypothesis 2 if and only if there exists \( s \in E_0 \) such that \( C(s)b \) is invertible in \( Z(OEb) \).

Assume Hypothesis 2. By the above lemma and [7], Lemma 2.4, there exists an element \( s \in E_0 \) such that \( C(s)b \in Z(OEb)^{\times} \). Hence there exists a defect group \( D \) of \( b \) centralized by \( s \), and hence contained in \( G' \). Let \( P \leq D \). Then by [7], Lemma 3.9, \( C_E(P), C_G(P), C_{E'}(P) \) and \( C_{G'}(P) \) satisfy Hypothesis 1. Here we note \( F \equiv C_E(P)/C_G(P) \). Let \( e \in \text{Bl}(C_G(P), b) \). Then we see that \( Br_{OE}(C(s)b)e^* \in (Z(kC_E(P)e^*))^{\times} \). This implies that \( e \) is covered by \( r \) blocks of \( C_E(P') \). Similarly assume Hypothesis 3. Let \( D' \) be a defect group of \( b' \) and \( e' \in \text{Bl}(C_{G'}(P'), b') \) for a subgroup \( P' \) of \( D' \). Then \( e' \) is covered by \( r \) blocks of \( C_{E'}(P') \).

Theorem 5 (see [7], Proposition 3.11) Using the same notations as in Theorem 4 we have the following. 
(i) Assume Hypothesis 2. Let \( D \) be a defect group of \( b \) obtained in the above and let \( P \leq D \). Let \( e \in \text{Bl}(C_G(P), b) \). Then \( e_{(C_{G'}(P))} \in \text{Bl}(C_{G'}(P), b_{(G')}) \). In particular, \( b_{(G')} \) have a defect group containing \( D \).
(ii) Assume Hypothesis 3. Let \( D' \) be a defect group of \( b' \) and \( P' \leq D' \). Let \( e' \in \text{Bl}(C_{G'}(P'), b') \). Then \( e'_{(C_{G'}(P'))} \in \text{Bl}(C_{G'}(P'), b'_{(G')}) \). In particular, \( b'_{(G')} \) have a defect group containing \( D' \).

Assume Hypotheses 2 and 3, and \( b' = b_{(G')} \). The Dade correspondence \( \rho(E, G, E', G') \) gives a bijection between \( \text{Irr}(b) \) and \( \text{Irr}(b') \) by Theorem 4. By Theorem 5, \( b \) and \( b' \) have a common defect group \( D \). Let \( (D, b_D) \) be a maximal \( b \)-Brauer pair. For \( P \leq D \), let \( (P, b_P) \) be a \( b \)-Brauer pair contained in \( (D, b_D) \). We set
\[
(b_P)' = (b_P)_{(C_{G'}(P))}.
\]
By the above theorem \( (b_P)' \) is associated with \( b' \) and \( (D, (b_D)') \) is a maximal \( b' \)-Brauer pair. The following holds.

Theorem 6 (see [7], Theorem 5.2) Assume Hypotheses 2 and 3, and assume \( b' = b_{(G')} \). Then the Brauer categories \( B_G(b) \) and \( B_{G'}(b') \) are equivalent.
4 Perfect isometry and isotypy

Assume Hypotheses 2 and 3, and \( b' = b(G') \) using the notations in Theorem 4. With the notations in the previous section, we put

\[
b_i = \sum_{l=0}^{r-1} (b_i)_{(E')} \hat{b}_{l+i} \quad (\forall i \in \mathbb{Z}).
\]

Then \((b_i)^2 = b_i\) and \(b_i \in (OGbb')^E\) for each \(i\). For each prime \( q \in \pi(F)\), let \( \lambda_q \in \hat{F}_q \) be a non-trivial character as in Theorem 1. Set \( l = |\pi(F)| \). Moreover we set for \( t \quad (1 \leq t \leq l) \) distinct primes \( q_1, q_2, \ldots, q_t \in \pi(F) \)

\[
\lambda_{q_1} \cdots \lambda_{q_t} = \lambda^{m_{\{q_1, \ldots, q_t\}}} \quad (m_{\{q_1, \ldots, q_t\}} \in \mathbb{Z})
\]

where \( \lambda \) is a generator of \( \hat{F} \). Then we have

\[
\prod_{q \in \pi(F)} (1 - \lambda_q) = 1 + \sum_{t=1}^{l} (-1)^t \sum_{\{q_1, \ldots, q_t\} \subseteq \pi(F)} \lambda^{m_{\{q_1, \ldots, q_t\}}}
\]

where \( \{q_1, \ldots, q_t\} \) runs over the set of \( t \)-element subsets of \( \pi(F) \).

Proposition 2 (see [7], Proposition 4.4) With the above notations we have

\[
[b_0\mathcal{K}G] + \sum_{t=1}^{l} (-1)^t \sum_{\{q_1, \ldots, q_t\} \subseteq \pi(F)} [b_{m_{\{q_1, \ldots, q_t\}}} \mathcal{K}G]
\]

\[
= \sum_{\phi \in \text{Irr}(b)} \epsilon_{\phi} [L_{\phi\mathcal{G}}(G') \otimes_{\mathbb{K}} L_{\phi}]
\]

in \( G_0(\mathcal{K}(G' \times G)) \).

From the above proposition and [1], Proposition 1.2, we have the following.

Theorem 7 (see [7], Theorem 4.5) Assume Hypotheses 2 and 3, and that \( b' = b(G') \). Set \( \mu = \sum_{\phi \in \text{Irr}(b)} \epsilon_{\phi} \phi(G') \). Then \( \mu \) induces a perfect isometry \( R_\mu : \mathcal{R}_G(G, b) \rightarrow \mathcal{R}_G(G', b') \) which satisfies \( R_\mu(\phi) = \epsilon_{\phi}(G') \).

Let \( D \) be a common defect group of \( b \) and \( b' \). For \( P \leq D \), \( R^P \) be the perfect isometry between \( \mathcal{R}_G(C_G(P), b_P) \) and \( \mathcal{R}_G(C'_G(P), (b_P)_{(C_G(P))}) \) obtained by the Dade correspondence.

Theorem 8 (see [7], Theorem 5.5) Assume Hypotheses 2 and 3, and assume \( b' = b(G') \). Then \( b \) and \( b' \) are isotypic with the local system \( (R^P)_{P(cyclic) \leq D} \).

Example Suppose \( p = 5 \). Let \( G = Sz(2^{2n+1}) \), the Suzuki group, \( A = \langle \sigma \rangle \) where \( \sigma \) is the Frobenius automorphism of \( G \) with respect to \( GF(2^{2n+1})/GF(2) \). Set \( G' = Sz(2) = C_G(A), E = G \times A, E' = G' \times A. \) Suppose that \( 5 \nmid 2n+1 \). Then \( (2n+1, |G'|) = 1 \). Moreover a Sylow 5-subgroup of \( G \) has order 5. By the above theorem, the Dade correspondence gives an isotypy between \( b(G) \) and \( b(G') \). Moreover, if \( 5 \mid (2^{2n+1} + 2^{n+1} + 1) \), then \( b(G) \) and \( b(G') \) are splendidly Morita equivalent.
References


